1 Fourier Transform Properties

Definition of a Fourier transform:

$$\mathcal{F}(w(t)) = W(f) = \int_{-\infty}^{\infty} w(t)e^{-j2\pi ft}dt \tag{1}$$

1.1 Linearity:

$$a_1w_1(t) + a_2w_2(t) \overset{FT}{\leftrightarrow} a_1W_1(f) + a_2W_2(f))$$

$$\begin{aligned}
\mathcal{F}(a_1w_1(t) + a_2w_2(t)) &= \int_{-\infty}^{\infty} (a_1w_1(t) + a_2w_2(t))e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} (a_1w_1(t))e^{-j2\pi ft}dt + (a_2w_2(t))e^{-j2\pi ft}dt \\
&= a_1\int_{-\infty}^{\infty} w_1(t)e^{-j2\pi ft}dt + a_2\int_{-\infty}^{\infty} w_2(t)e^{-j2\pi ft}dt \\
&= a_1W_1(f) + a_2W_2(f)) \qquad (2)
\end{aligned}$$

1.2 Time Delay:

$$w(t - T_d) \overset{FT}{\leftrightarrow} W(f)e^{-j2\pi fT_d}(where \omega = 2\pi f)$$

$$\mathcal{F}(w(t - T_d)) = \int_{-\infty}^{\infty} w(t - T_d)e^{-j2\pi ft}dt \tag{3}$$

Let $x = t - T_d \Rightarrow t = x + T_d, dx = dt$

$lowerlimit : t = -\infty \Rightarrow x = -\infty$

$upperlimit : t = \infty \Rightarrow x = \infty$

$$\begin{aligned}
\mathcal{F}(w(t - T_d)) &= \int_{-\infty}^{\infty} w(x)e^{-j2\pi f(x+T_d)}dx \\
&= \int_{-\infty}^{\infty} w(x)e^{-j2\pi fx}e^{-j2\pi fT_d}dx \\
&= e^{-j2\pi fT_d}\int_{-\infty}^{\infty} w(x)e^{-j2\pi fx}dx \\
&= e^{-j2\pi fT_d}W(f) \qquad (4)
\end{aligned}$$

1.3 Scale Change:

$$w(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|}W(\frac{f}{a})$$

$$\mathcal{F}(w(at)) = \int_{-\infty}^{\infty} w(at)e^{-j2\pi ft}dt \tag{5}$$

$$\text{Let } x = at, t = \frac{x}{a}, dx = \frac{dt}{a}$$

$If\ a > 0$

$$lowerlimit : t = -\infty \Rightarrow x = -\infty$$
$$upperlimit : t = \infty \Rightarrow x = \infty$$

$$\begin{aligned} \mathcal{F}(w(at)) &= \int_{-\infty}^{\infty} w(x) e^{-j2\pi f \frac{x}{a}} \frac{dx}{a} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} w(x) e^{-j2\pi \frac{f}{a} x} dx \\ &= \frac{1}{a} W(\frac{f}{a}) \end{aligned} \tag{6}$$

$If\ a < 0$

$$lowerlimit : t = -\infty \Rightarrow x = \infty$$
$$upperlimit : t = \infty \Rightarrow x = -\infty$$

$$\begin{aligned} \mathcal{F}(w(at)) &= \int_{\infty}^{-\infty} w(x) e^{-j2\pi f \frac{x}{a}} \frac{dx}{a} \\ &= -\int_{-\infty}^{\infty} w(x) e^{-j2\pi \frac{f}{a} x} \frac{dx}{a} \\ &= -\frac{1}{a} \int_{-\infty}^{\infty} w(x) e^{-j2\pi \frac{f}{a} x} dx \\ &= -\frac{1}{a} W(\frac{f}{a}) \end{aligned} \tag{7}$$

By combining the above two equations we get,

$$w(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|} W(\frac{f}{a})$$

1.4 Conjugation:

$$w^*(t) \overset{FT}{\leftrightarrow} W^*(-f)$$

$$\mathcal{F}(w^*(t)) = \int_{-\infty}^{\infty} w^*(t) e^{-j2\pi f t} dt \tag{8}$$

By the taking the conjugate of the equation twice, we get the following

$$\mathcal{F}(w^*(t)) = ((\int_{-\infty}^{\infty} w^*(t) e^{-j2\pi f t} dt)^*)^* \tag{9}$$

By conjuagating the equation once, we get.

$$
\begin{aligned}
&= (\int_{-\infty}^{\infty} (w^*(t))^* (e^{-j2\pi ft})^* dt)^* \\
&= (\int_{-\infty}^{\infty} w(t)(e^{j2\pi ft}) dt)^* \\
&= (\int_{-\infty}^{\infty} w(t)(e^{-j2\pi(-f)t}) dt)^* \\
&= W^*(-f) \qquad (10)
\end{aligned}
$$