Property : To show that $J_n(\beta) = (-1)^n J_{-n}(\beta)$

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{j(\beta\sin x - nx)}\,dx$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi} e^{j(\beta\sin x - nx)}\,dx$$

Replace x by $-x$ in the first term

$$\frac{1}{2\pi}\int_{-\pi}^{0} e^{j(\beta\sin x - nx)}\,dx = \frac{1}{2\pi}\int_{\pi}^{0} e^{-j(\beta\sin x - nx)}\,(-dx)$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{-j(\beta\sin x - nx)}\,dx$$

Thus

$$J_n(\beta) = \frac{1}{2\pi}\left[\int_{0}^{\pi} e^{-j(\beta\sin x - nx)} + e^{j(\beta\sin x - nx)}\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(\beta\sin x - nx)\,dx \quad \text{---(1)}$$

Now replace x by $\pi - x$
dx by $-dx$

$$J_n(\beta) = \frac{1}{\pi}\int_{0}^{\pi} \cos(\beta\sin x - n\pi + nx)\,dx$$

$$J_n(\beta) = \frac{1}{\pi}\int_0^{\pi} [\cos(\beta \sin x + nx) \cos n\pi + \sin n\pi \sin(\beta \sin x \pm nx)] \, dx$$

$$\sin n\pi = 0$$

$$\cos n\pi = (-1)^n$$

$$J_n(\beta) = \frac{(-1)^n}{\pi}\int_0^{\pi} \cos(\beta \sin x + nx) \, dx \quad -(2)$$

~~Thus~~ From (1)

$$J_{-n}(\beta) = \frac{1}{\pi}\int_0^{\pi} \cos(\beta \sin x + nx) \, dx$$

Thus

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

i.e. if n is even

$$J_n(\beta) = J_{-n}(\beta) \qquad \text{even } n$$

$$J_n(\beta) = -J_{-n}(\beta) \qquad \text{odd } n$$