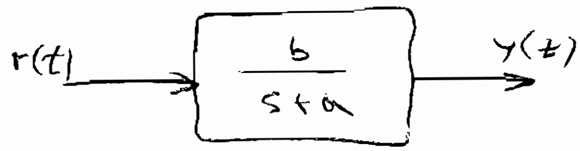
ELE 457 Fall '09

Chapter 4

First-order System



$a > 0$ for stability

if $r(t)$ = step of height H, what is $\lim_{t\to\infty} y(t)$?

$$Y(s) = \left[\frac{b}{s+a}\right]\left[\frac{H}{s}\right]$$

$$y_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} \frac{H \cdot b}{s+a} = \frac{H \cdot b}{a}$$

In fact,

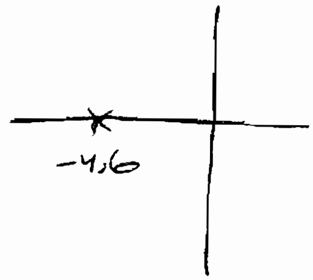
$$y(t) = H \cdot \frac{b}{a}\left(1 - e^{-at}\right)$$

How long does it take for $y(t) \to y_{ss}$?

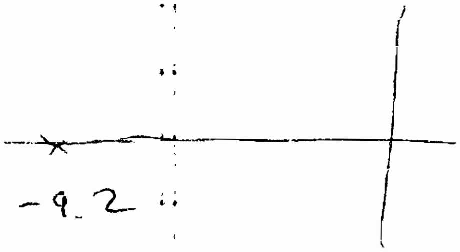
Define T_s as the 1% settling time

$$y(T_s) = .99\,\frac{H \cdot b}{a}$$

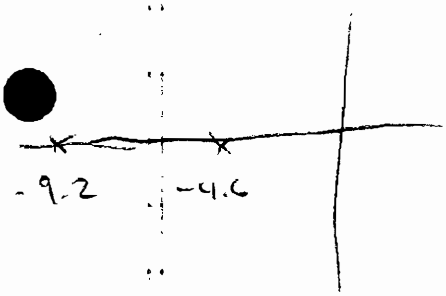
Pole Plots



$T_s = 1$ sec

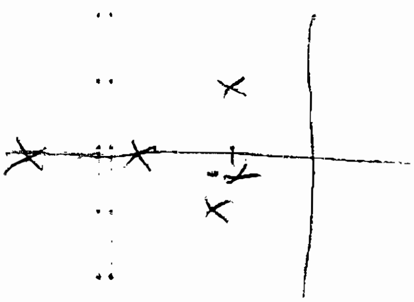


$T_s = \frac{1}{2}$ sec



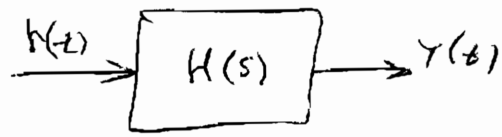
$T_s \simeq 1$ sec

Approximation: T_s is given by real part of slowest pole



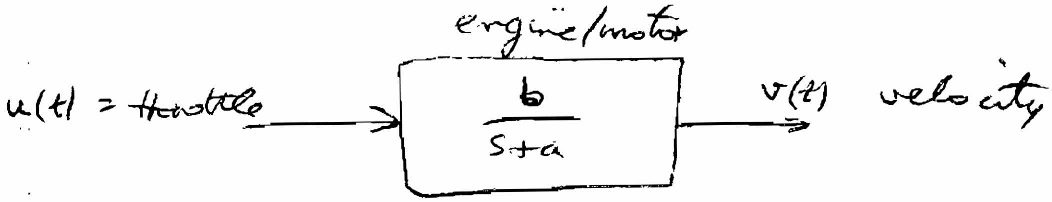
$$T_s \approx \frac{4.6}{\alpha}$$

DC gain

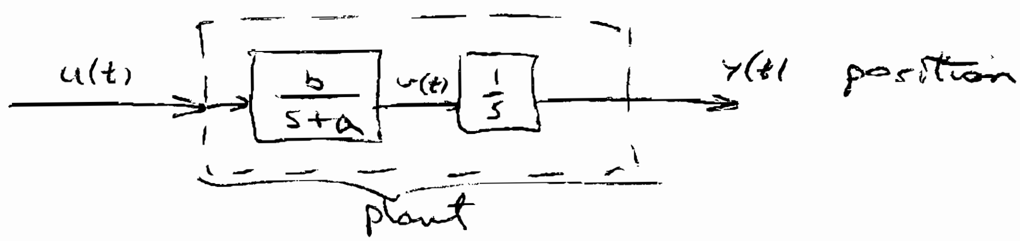


If $H(s)$ is stable,
DC Gain $= H(0)$

Why is a first-order system useful?



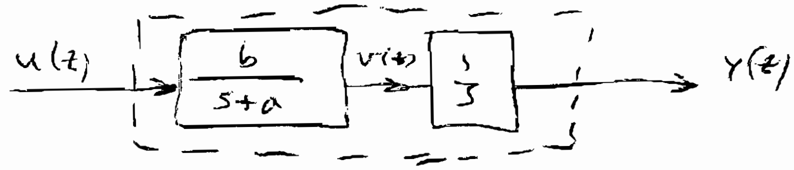
Position System



For a position control system, the plant is usually second order.

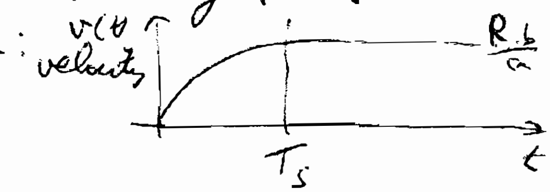
Where do the poles of a CL (proportional) control system go? (Ans: could be complex for large enough k).

Recall the model of a motor-driven system for position control:



Q: What happens if u(t) is a step input of height R?

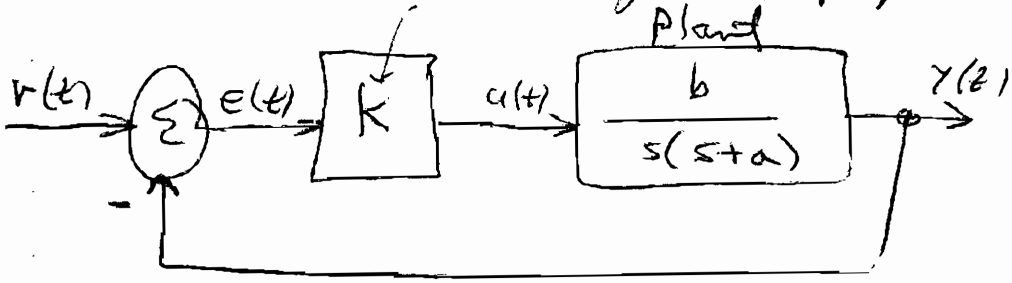
A:



Not good for position control!

Consider a feedback control system

control gain ("proportional control")



b, a > 0 are fixed.

We can choose any value for K.

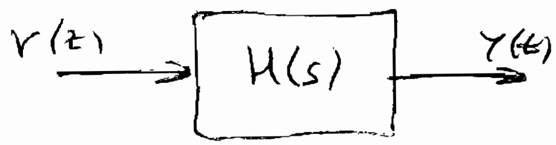
- $e(t) = r(t) - y(t)$ = error signal
- Plant input $u(t) = K \cdot e(t)$. Plant input is <u>proportional</u> to the error signal
- This is a negative unity feedback control system

Q: What happens if r(t) is a step input of height R?

A: First we need to find the transfer function of the CL system from r(t) to y(t) ["G over 1 plus G"]

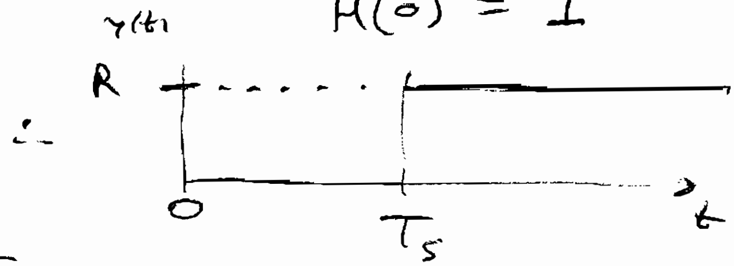
CLTF $H(s) = \dfrac{\dfrac{k \cdot b}{s(s+a)}}{1 + \dfrac{k \cdot b}{s(s+a)}} = \dfrac{k \cdot b}{s(s+a) + k \cdot b}$

$$= \frac{k \cdot b}{s^2 + as + k \cdot b}$$



What is the dc gain of this system?

$H(0) = 1$



Q: What happens between zero and T_s seconds?

A: Depends on the pole locations (roots of $s^2 + as + k \cdot b$)

CL pole locations: $-\dfrac{a}{2} \pm \dfrac{\sqrt{a^2 - 4kb}}{2}$

or

$$-\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - kb}$$

Note: if k is negative, $\sqrt{\left(\frac{a}{2}\right)^2 \underbrace{-kb}_{>0}} > \dfrac{a}{2}$

and one pole is in the RHP. Thus, choose $\underline{k > 0}$

Case I (Overdamped)

If $k \cdot b < \left(\frac{a}{2}\right)^2$, the poles are at

$-\frac{a}{2} \pm$ a positive real number smaller than $\frac{a}{2}$

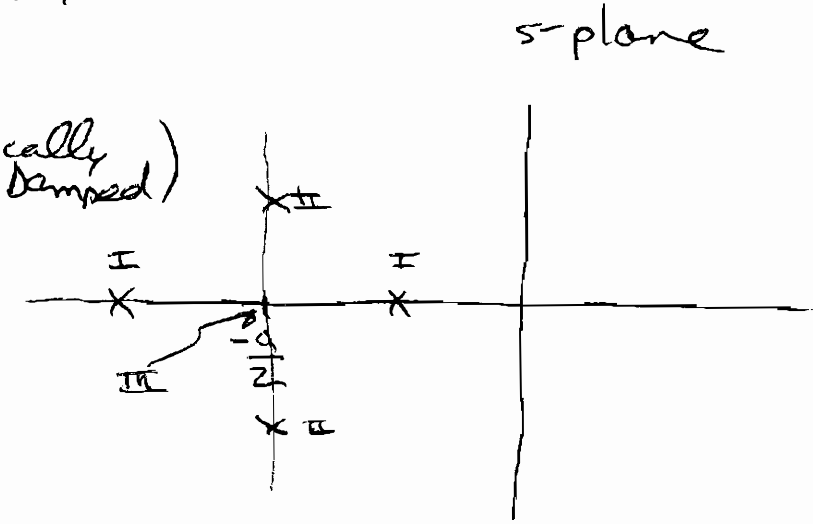
Case II (Underdamped)

If $k \cdot b > \left(\frac{a}{2}\right)^2$, the poles are at

$-\frac{a}{2} \pm j \cdot$ number

Case III (Critically Damped)

If $k \cdot b = \left(\frac{a}{2}\right)^2$, the poles are both at $-\frac{a}{2}$



A motor-driven control system with proportional control can behave like any of these systems, I, II, III, depending on the value of K that is used.

Recall CLTF $H(s) = \dfrac{k \cdot b}{s^2 + as + k \cdot b}$

rewrite this in standard form as:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Note: $\omega_n^2 = k \cdot b$

$$\zeta = \frac{a}{2\omega_n} = \frac{a}{2\sqrt{k \cdot b}}$$

ζ is called the <u>damping ratio</u>
ω_n " " " natural (or undamped) freq.

<u>Case I</u> $(\zeta > 1)$ overdamped

$$k \cdot b < \left(\frac{a}{2}\right)^2 \Rightarrow \sqrt{k \cdot b} < \frac{a}{2}$$

$$\zeta = \frac{(a/2)}{\sqrt{k \cdot b}} > 1$$

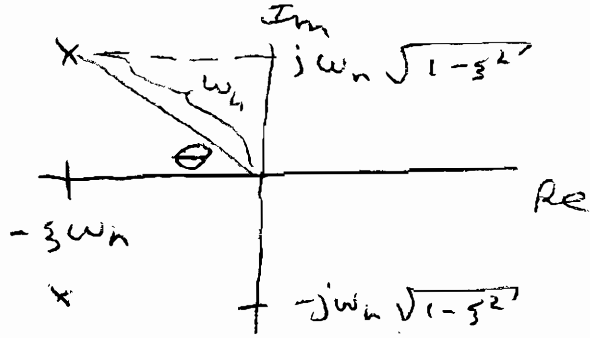
<u>Case II</u>

$$k \cdot b > \left(\frac{a}{2}\right)^2 \Rightarrow \zeta < 1 \quad \text{(underdamped)}$$

<u>Case III</u>

$$k \cdot b = \left(\frac{a}{2}\right)^2 \Rightarrow \zeta = 1 \quad \text{(critically damped)}$$

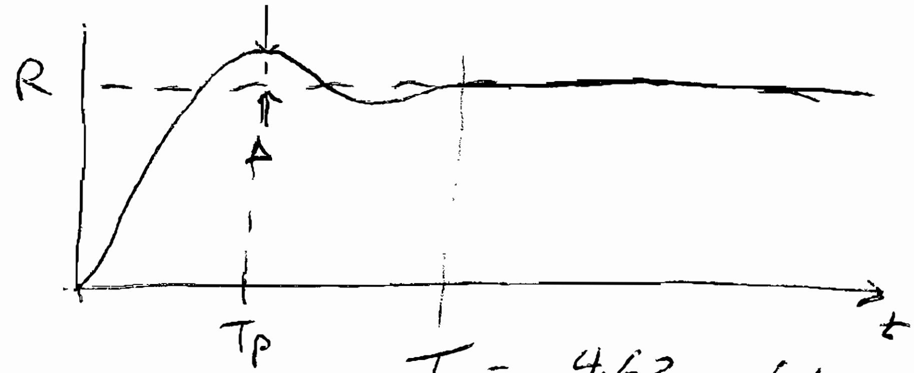
For $\zeta < 1$, the poles of $H(s)$ are at:



$\theta = \cos^{-1}(\zeta)$ as $\zeta \to 0$, $\theta \to 90°$
(poles $\to$ imag axis)

as $\zeta \to 1$, $\theta \to 0°$
(poles $\to$ real axis)

Height R step response



$$T_s = \frac{4.62}{\xi\omega_n} \quad \left(\text{depends on } \xi \;\&\; \omega_n\right)$$

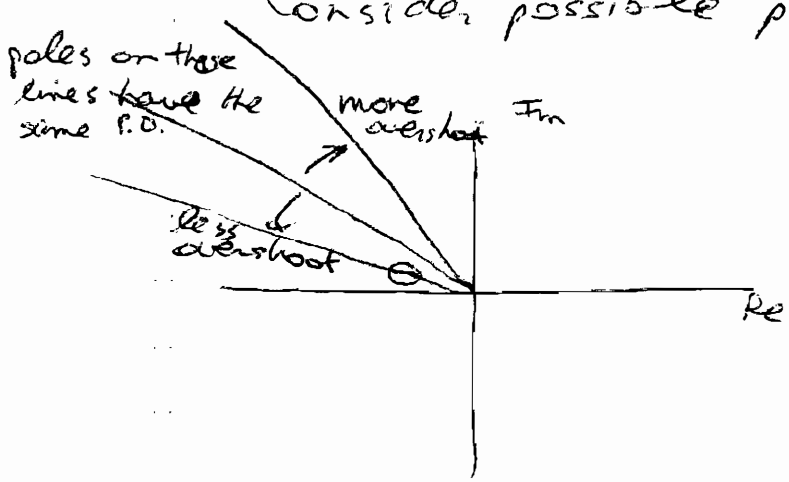
$$\text{Percent Overshoot (P.O.)} = \frac{\Delta}{R} \times 100$$

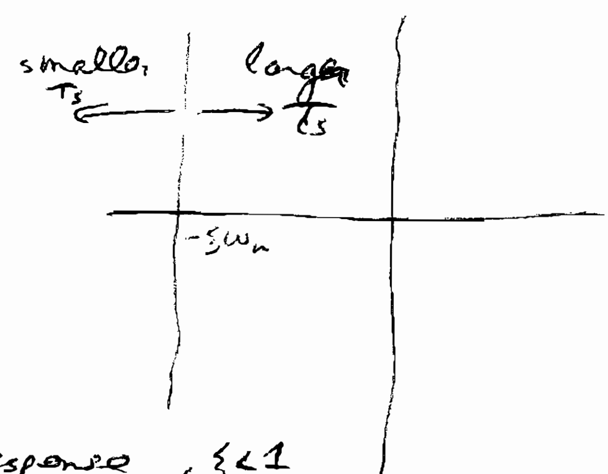
$$= 100\, e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \quad \left(\text{depends only on } \xi\right)$$

$$\left(\begin{aligned} T_p &= \text{time to first peak} \\ &= \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \quad \text{depends only on } \xi \end{aligned} \right)$$

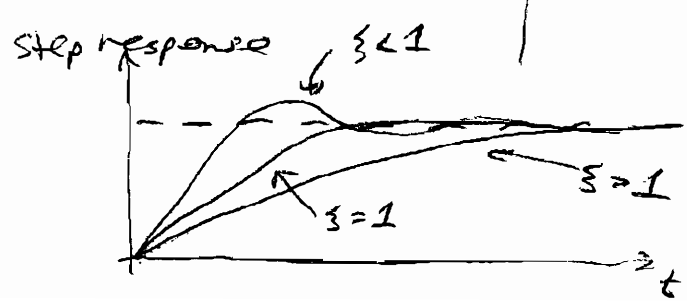
we will not use this much

Consider possible pole locations:

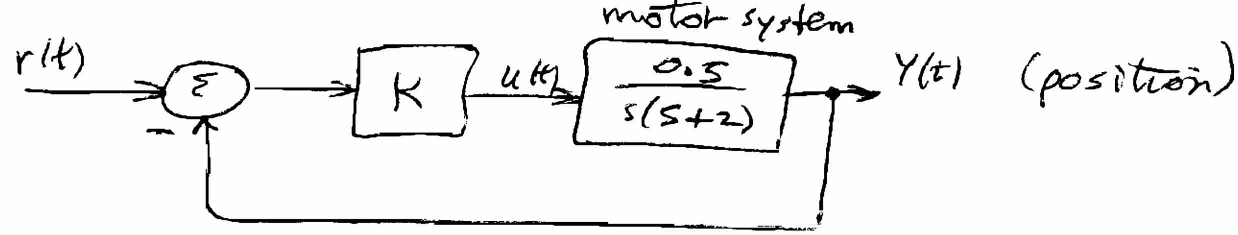




$$\theta = \cos^{-1}(\xi)$$



Example 4.2 (Book, pg 127)



CLTF is $\dfrac{0.5k}{s^2+2s+0.5K}$

Problem specifies critical damping (fastest response with no overshoot)
$\boxed{\zeta = 1}$

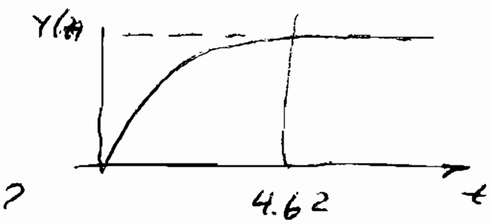
Coefficient of s is $2 = 2\zeta\omega_n$

$\therefore \boxed{\omega_n = 1}$

Then $\omega_n^2 = 1 = 0.5K \Rightarrow \boxed{K=2}$

"4.62 times faster"

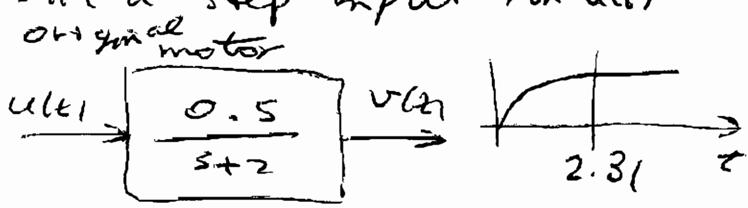
$T_s = \dfrac{4.62}{\zeta\omega_n} = 4.62$ sec



Suppose we want $T_s = 1$ sec?

Book suggests (pg 125) use a faster motor.
Better answer: use feedback control to get a faster system using the ~~same~~ original motor.

~~How~~ How fast does the original motor speed up with a step input in u(t)?



2.31 sec to reach steady state speed.

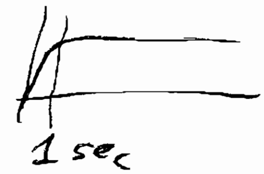
A motor that is 4.62× faster will reach steady-state speed in $\frac{2.31}{4.62} = \underset{\text{new } T_s \text{ value}}{\underline{0.5 \text{ sec}}}$

The faster motor pole will be at $\frac{4.62}{p} = 0.5$ sec

or $p = 9.24$

fast motor

$u(t) \rightarrow \boxed{\dfrac{0.5}{s+9.24}} \rightarrow v(t)$

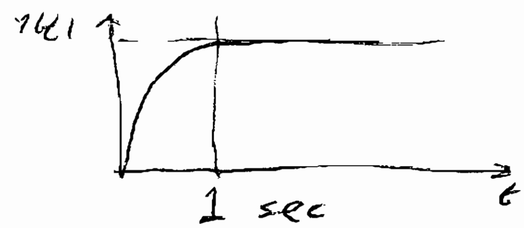


With this fast motor,

CLTF is $\dfrac{.5K}{s^2 + \underbrace{9.24}s + 0.5K}$

let $\boxed{\zeta = 1}$; $2\zeta\omega_n = 9.24 \Rightarrow \omega_n = \frac{9.24}{2} = \boxed{4.62}$

$\omega_n^2 = 21.34 = 0.5K \Rightarrow \boxed{K = 42.68}$

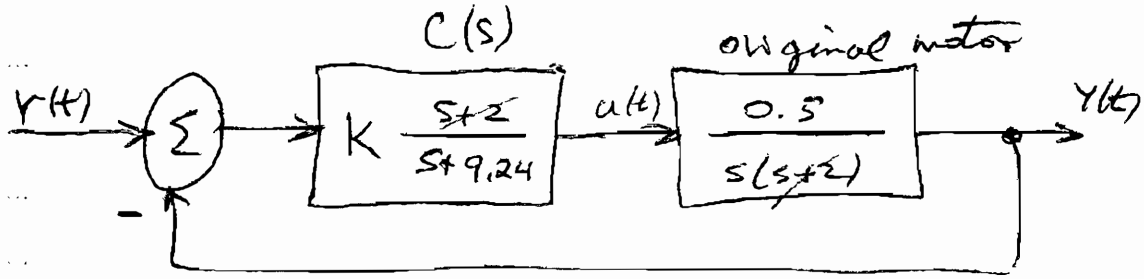


original motor $\boxed{\dfrac{0.5}{s+2}}$

fast motor $\boxed{\dfrac{0.5}{s+9.24}}$

Can we get a 1-second settling time (no overshoot) with the original motor?

Yes, by a well-known trick. Use a compensator to cancel the slow plant pole and replace it with a faster one.



CLTF is $\frac{.5K}{s^2 + 9.24s + .5K}$. From previous calculation, set $\boxed{K = 42.68}$ done!

Pole-zero cancellation in LHP is okay.
" " in RHP is NOT okay.

If pole of C(s) is too fast the plant input may be too large.