Routh-Hurwitz Stability Criterion

a linear system is stable iff all of its poles are in the left half plane.

Given an nth-order polynomial

$$\bar{a}(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

note: subscripts in opposite order from book

how can we tell if all its roots have negative real parts?

Observe: by dividing through by a_0, we obtain the following polynomial, which has the same roots as $\bar{a}(s)$:

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$$

Example: let $a(s) = (s-p_1)(s-p_2)(s-p_3)$ with $p_i < 0$ (real poles in LHP)

$$(s-p_1)(s-p_2) = s^2 - (p_1+p_2)s + p_1 p_2$$

$$s - p_3$$

(n=3) $$a(s) = s^3 - (p_1+p_2+p_3)s^2 + (p_1p_2 + p_1p_3 + p_2p_3)s - p_1p_2p_3$$

$a_1 = -$ sum of roots

$a_2 =$ sum of products of roots taken two at a time

$(-1)^n$ $a_3 = -$ product of all three roots

$$\overbrace{\text{If all } p_i < 0}^{A} \text{ then } \overbrace{a_1, a_2, a_3 \text{ are all positive.}}^{B}$$

If A then B is logically equivalent to
If $\underset{\text{"not"}}{\sim}$B then ~A

~B not all a_i are positive
~A not all p_i in LHP

[Note that: if B then A is not logically equivalent.]

<u>Result</u>: If any coefficient $a_i \leq 0$, the system is not stable.

e.g.

$$\longrightarrow \boxed{\frac{s^2+2s+4}{s^4+3s^2+2s-1}} \longrightarrow$$

This system is unstable
$a_1 = 0, \ a_4 < 0$

To be stable, all coefficients must be positive, but not all polynomials with positive coefficients are stable.

How can we tell if a polynomial with positive coefficients is stable?

Ans: Form a Routh Array

Example: $s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5$

Routh Array

$$\begin{array}{l|lll} & 1 & a_2 & a_4 \\ 3 & a_1 & a_3 & a_5 \\ & b_1 & b_2 & 0 \\ 2 \text{ nonzero} & c_1 & c_2 & \\ & d & 0 & \\ 1 \text{ nonzero} & e & & \end{array}$$

start with first coeff, then every other coeff (row $1, a_2, a_4$)

start with second, then every other (row a_1, a_3, a_5)

add zeros as needed

determinant

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_2 \\ a_1 & a_2 \end{vmatrix}, \quad b_2 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}, \quad c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & 0 \end{vmatrix}$$

$$d = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$e = -\frac{1}{d}\begin{vmatrix} c_1 & c_2 \\ d & 0 \end{vmatrix}$$

<u>Result</u>: The number of RHP poles is equal to the number of sign changes in the first column of the Routh Array.

$\therefore$ Polynomial is stable (Hurwitz) if all numbers in first column are > 0.

Example: $s^4 + 3s^3 + 2s^2 + s + 1$

$$\begin{array}{|lll} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5/3 & 1 & \\ -4/5 & 0 & \\ 1 & & \end{array}$$

Thus this polynomial must have two RHP roots.

Matlab: >> roots ([1 3 2 1 1])

gives -2.2056
-1
0.1028 ± j.6655 ← here they are

What happens if a number in the first column is zero? - zero is not positive or negative
- can't divide by zero to complete the array

Solution: Consider the polynomial with reciprocal roots. Reverse the order of the coefficients and proceed as usual.

Example: $s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

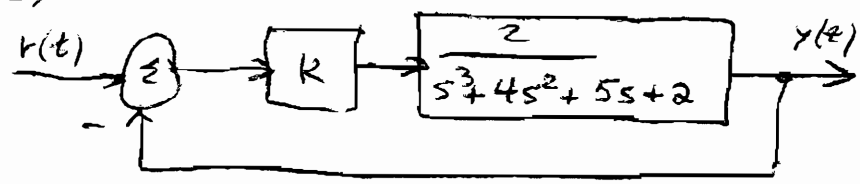
$$\begin{array}{|lll} 1 & 2 & 11 \\ 2 & 4 & 10 \\ 0 & & \end{array}$$

$$\begin{array}{l|lll} & 10 & 4 & 2 \\ & 11 & 2 & 1 \\ & 24/11 & 12/11 & 0 \\ \text{two} \rightarrow & -7/2 & 1 & \\ \text{RHP roots} \rightarrow & 1.7143 & 0 & \\ & 1 & & \end{array}$$

Confirmed by >> roots([1 2 2 4 11 10])

Read Examples 6.6 and 6.7

Consider



For what values of K is this system stable?

CLTF: $\dfrac{2K}{s^3+4s^2+5s+2+2K}$

Routh Array

$$\begin{array}{|ll} 1 & 5 \\ 4 & 2+2K \\ \frac{9-K}{2} & 0 \\ 2+2K & \end{array}$$

$-\dfrac{2+2K-20}{4} = \dfrac{9-K}{2}$

$\frac{9-K}{2}$ ⟶ $K < 9$ to get positive #

$2+2K$ ⟶ $K > -1$

Note: for K near 9, system response will be very oscillatory.