ELE 457 Feedback Control System: System Modeling

Fall 2009

How to use MATLAB

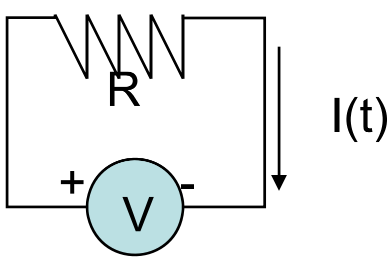
- Two commands can help you
  - HELP <function name>
  - LOOKFOR <keyword>

Review of Last Lecture

- Control system
- Feedback control
- Desired characteristics of control system

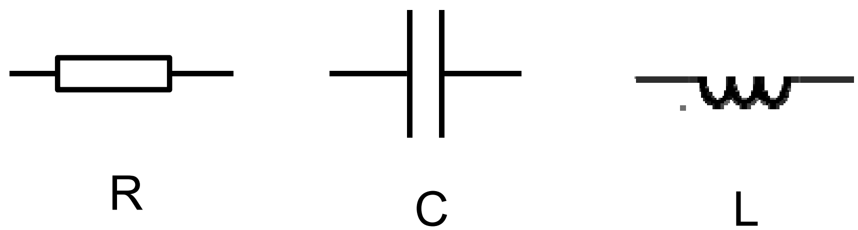
System Modeling

- Using mathematical equations to describe the relationship between the input and output of a physical system
- System modeling is the first step to analyze and design a control system
  - It has been stated that the development of the models of the physical systems involved is from 80-90 percent of the effort required in control system analysis and design.



$I(t) = V(t)/R$

How to describe a system mathematically?

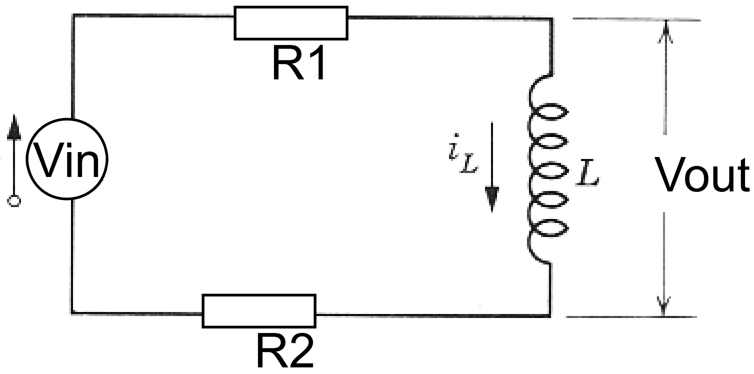


Resistor: $v(t) = i(t)R$ R is **resistance** (ohm)

Capacitor: $v(t) = \frac{1}{C}\int_{\tau=0}^{t} i(\tau)d\tau + v(0)$ C is **Capacitance** (Farad)

Inductor: $v(t) = L\frac{di(t)}{dt}$ *L* is the **inductance** (Henry)

How to describe a system mathematically?

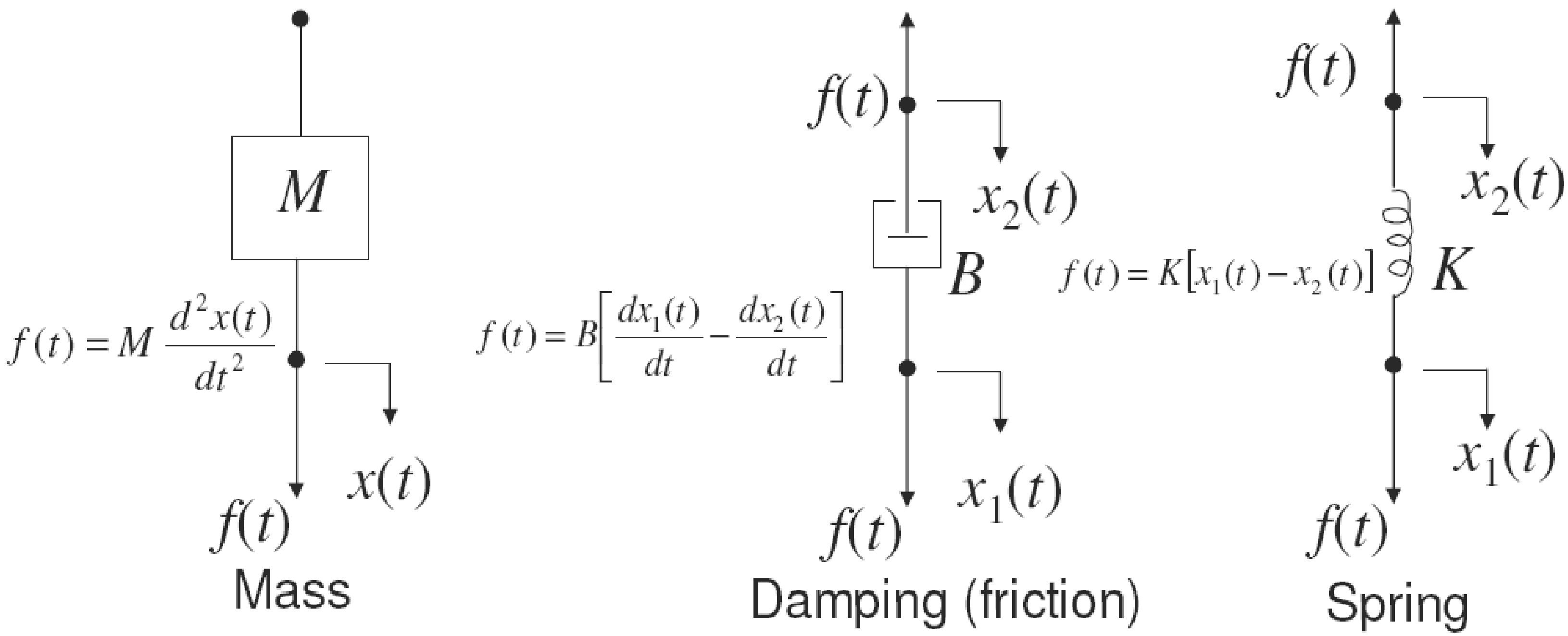


Kirchhoff's law:

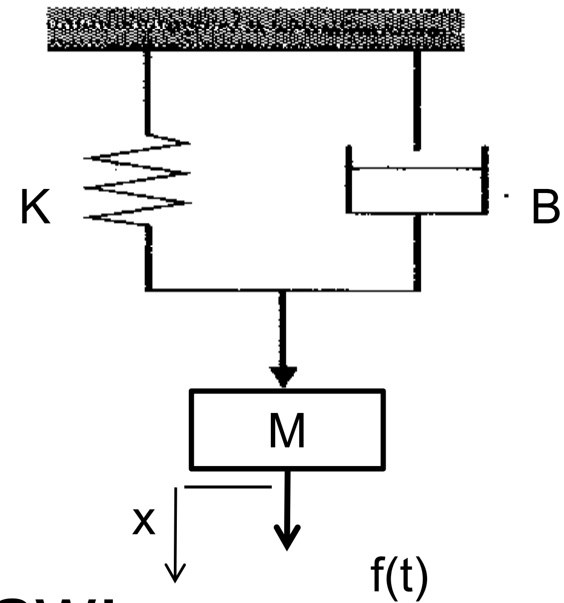
$$V_{in}(t) = (R1 + R2)i(t) + L\frac{di(t)}{dt}$$

$$V_{out}(t) = L\frac{di(t)}{dt}$$

How to describe a system mathematically?



How to describe a system mathematically?



- Newton's law:

$$f(t) - Kx(t) - B\frac{dx(t)}{dt} = M\frac{d^2x}{dt^2}$$

Differential Equations

- Linear ordinary differential equations (ODE)
  - A wide range of systems in engineering are modeled mathematically by differential equations

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

— Ordinary differential equation is an equality involving a function and its derivatives.

Differential Equations

- We only consider linear time-invariant (LTI) system in this class
- Nonlinear differential equations can be analyzed by linearization

How to solve differential equation?

Example: $$\frac{dy(t)}{dt} + ky(t) = 1 \quad and \quad y(0) = 0$$

Step 1: Find the general homogeneous solution (involving solving the characteristic equation)

$$\frac{dy(t)}{dt} + ky(t) = 0$$
$$y(t) = e^{zt}$$
$$zy(t) + ky(t) = 0$$
$$Characteristic\ equation: z + k = 0 \quad z = -k$$
$$\tilde{y}(t) = c_1 e^{-kt} + c_2$$

How to solve differential equation?

- Step 2: Find the non-homogeneous solution

$$\because \frac{dy(t)}{dt} + ky(t) = 1 \quad and \quad \tilde{y}(t) = c_1 e^{-kt} + c_2$$

$$\therefore -c_1 k e^{-kt} + k(c_1 e^{-kt} + c_2) = 1$$

$$kc_2 = 1$$

$$c_2 = \frac{1}{k}$$

$$\tilde{y}(t) = c_1 e^{-kt} + \frac{1}{k}$$

How to solve differential equation?

Step 3: To solve the initial value problem

$$\because y(0) = 0$$

$$\therefore \tilde{y}(0) = c_1 e^{-k0} + \frac{1}{k} = 0$$

$$c_{1+} \frac{1}{k} = 0$$

$$c_1 = -\frac{1}{k}$$

$$\therefore y(t) = -\frac{1}{k} e^{-kt} + \frac{1}{k}$$

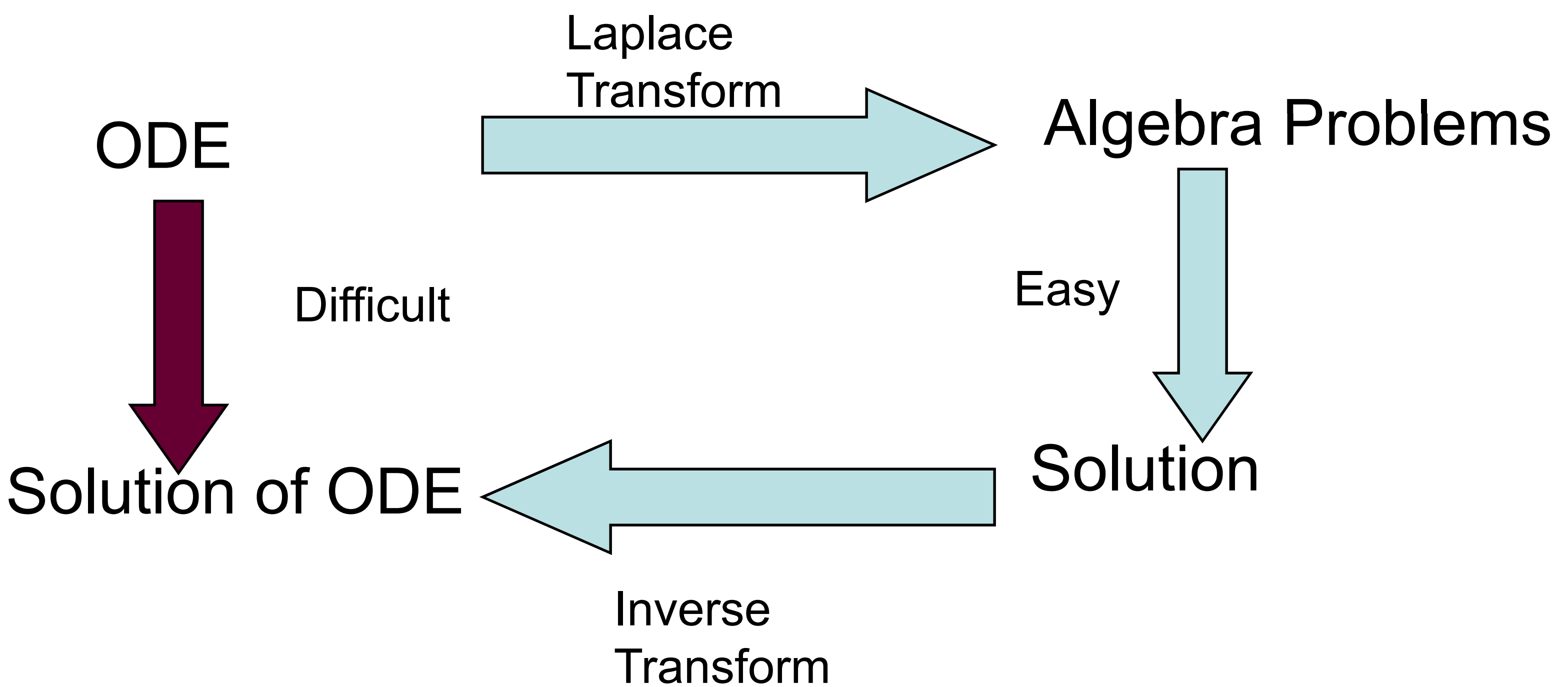
How to solve differential equation?

- What about

$$\frac{d^3 y(t)}{dt^3} + 6\frac{d^2 y(t)}{dt^2} + 9\frac{dy(t)}{dt} + 10 = \sin(t) + t$$

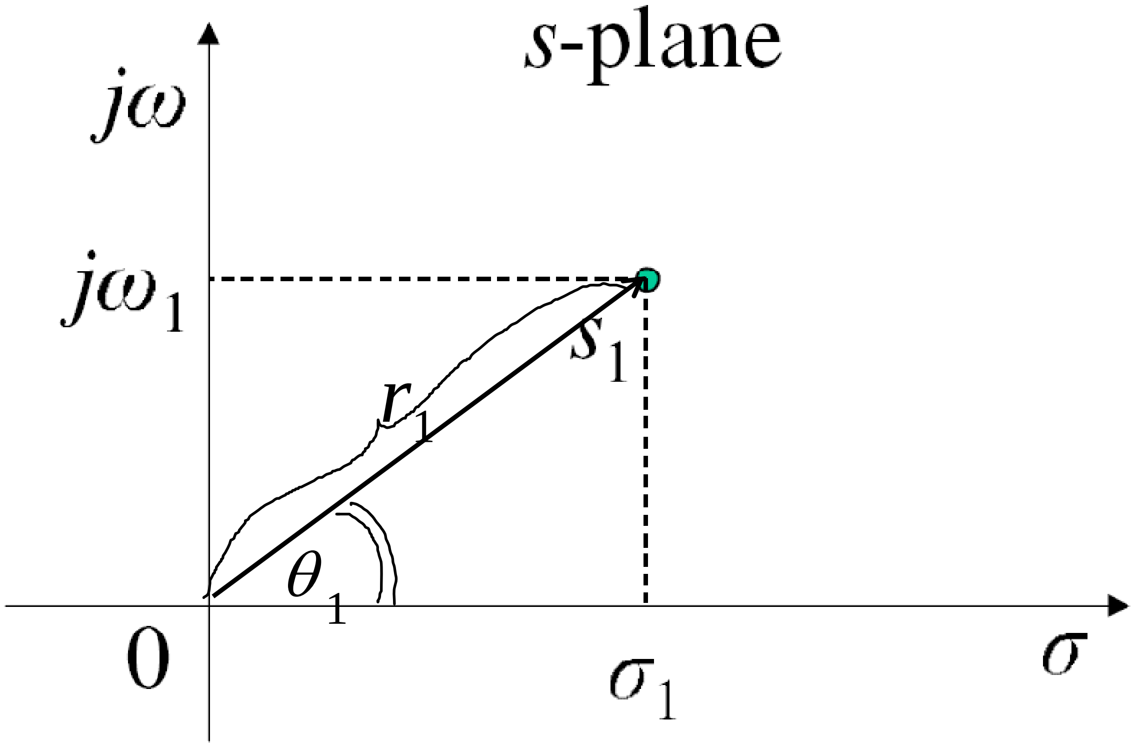
Another Approach

- Laplace Transform



Compare Variable

- Complex variables and s-plane



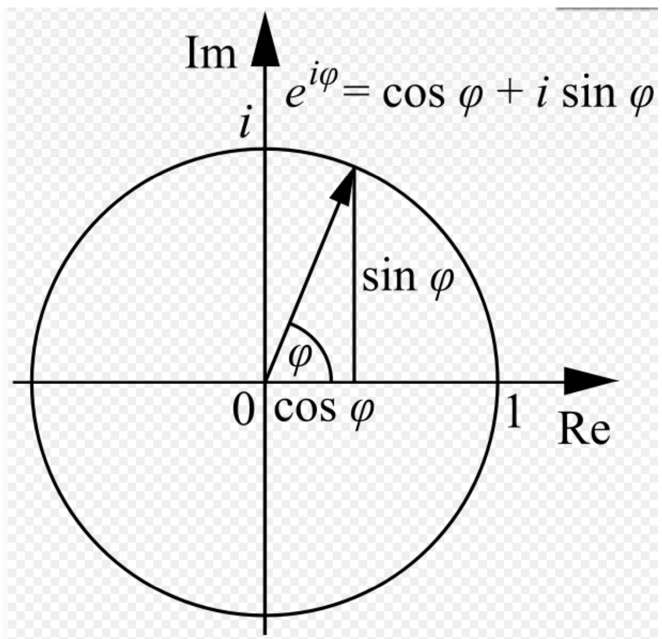
$$s_1 = \sigma_1 + j\omega_1$$

$$s_1 = r_1 e^{j\theta_1}$$

$$\sigma_1 = r_1 \cos(\theta_1)$$

$$\omega_1 = r_1 \sin(\theta_1)$$

- Euler’s Identity



$$e^{j\pi} + 1 = 0$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

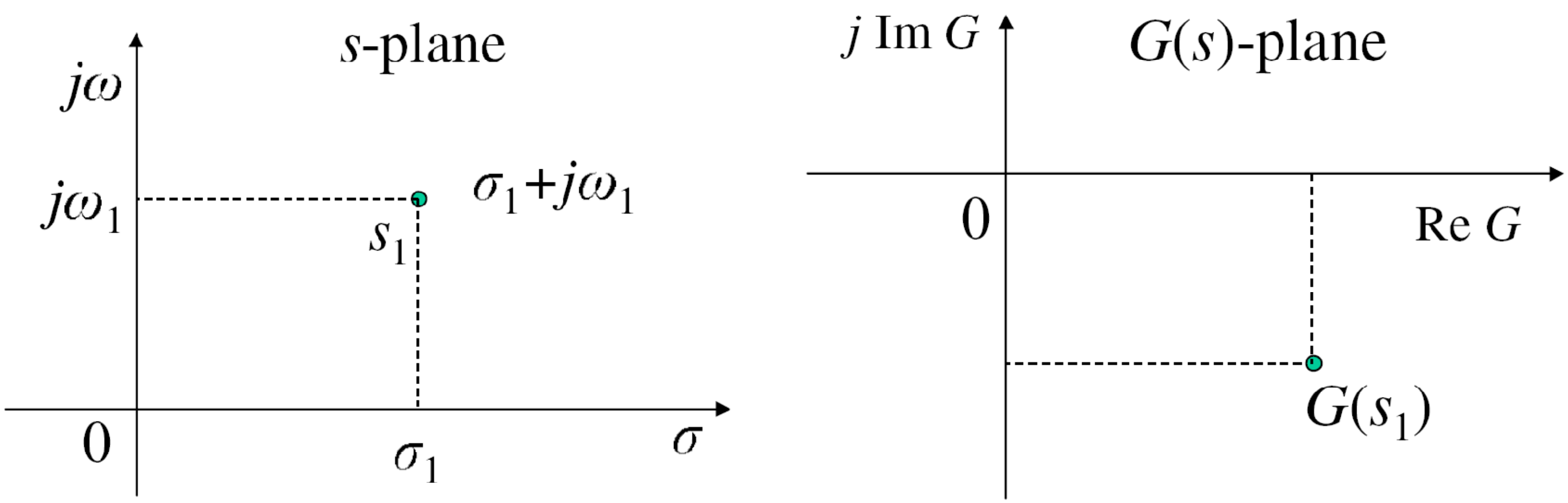
http://en.wikipedia.org/wiki/Euler's_identity

S-Plane

- In mathematics and engineering, the **S plane** is the name for the complex plane on which Laplace transforms are graphed.
- It is a mathematical domain where, instead of viewing processes in the time domain modelled with time-based functions, they are viewed as equations in the frequency domain. It is used as a graphical analysis tool in engineering and physics.

Function of Complex Variable

G(s) is a function of complex variable



Complex Variables

- A function G(s) of the complex variable is called analytic function in a region of the s-plane if the function and all its derivatives exist in the region.

- The singularities of a function are the points in the s-plane at which the function or its derivatives do not exist.
  - Singularities are extremely important in complex analysis, where they characterize the possible behaviors of analytic functions.

Complex Variable

- Zeros of a function
  - If a function G(s) is analytic at $s=s_i$, it is said to have a zero of order r at $s=s_i$ if the limit

$$\lim_{s \to s_i} (s - s_i)^{-r} G(s)$$

has a finite, nonzero value.

- Poles of a function
  - If a function G(s) is analytic in the neighborhood of s_i, it is said to have a pole of order r at $s=s_i$ if the limit

$$\lim_{s \to s_i} (s - s_i)^{r} G(s)$$

has a finite, nonzero value.

Laplace Transform

- The Laplace transform of a function f(t) is defined as

- The inverse Laplace transform of a function f(t) is defined as

NOTE:

- The Laplace transform of a function f(t) is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

The unilateral transform is always intended in this class unless otherwise stated.

$$F(s) = L[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

Bilateral transform was told in ELE314

Examples

$$f(t) = \delta(t)$$
$$f(t) = u(t)$$
$$f(t) = e^{-at}$$
$$f(t) = \sin(at)$$
$$f(t) = \cos(at)$$
$$t \geq 0$$

Laplace Transform

- Important table

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$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

Laplace Transform

$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\dfrac{C-aB}{\omega}\right)\sin\omega t\right]$	$\dfrac{Bs+C}{(s+a)^2+\omega^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{s+\alpha-\beta j}+\dfrac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{(s+\alpha-\beta j)^2}+\dfrac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\dfrac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\dfrac{s+c}{(s+a)(s+b)}$
$\dfrac{e^{-at}-e^{-bt}}{b-a}$	$\dfrac{1}{(s+a)(s+b)}$

Example 1

$$Given\ f(t) = 9 + 3e^{-5t}, t \geq 0$$

$$L[f(t)] = \ ?$$

Theorem of Laplace Transform

- Final value theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \quad \text{or} \quad f(+\infty) = \lim_{s \to 0} sF(s)$$

$$\text{if} \lim_{t \to \infty} f(t) \text{exists}$$

Example:

$$f(t) = \sin(\omega t) \xrightarrow{L} F(s) = \frac{\omega}{s^2 + \omega^2}$$

Theorem of Laplace Transform

- Initial value theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \quad or \quad f(0) = \lim_{s \to \infty} sF(s)$$

$$if \ \lim_{s \to \infty} sF(s) \ exists$$

Example: $f(t) = \sin(\omega t) \xrightarrow{L} F(s) = \dfrac{\omega}{s^2 + \omega^2}$