ELE 457 Feedback Control System: System Modeling

Fall 2009

Review of Last Lecture

- System Modeling: Differential Equation
- Complex variable and S-plane
- Laplace Transform
- Theorem of Laplace Transform

Theorem of Laplace Transform

- Linear

$$L[f_1(t)] = F_1(s);$$

$$L[f_2(t)] = F_2(s)$$

$$L[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$$

- Scale

$$L[f(at)] = \frac{1}{a}F(\frac{s}{a})$$

Theorem of Laplace Transform

- Frequency shift

$$L[e^{-at} f(t)] = F(s+a)$$

- Shifting

$$L[f(t-t_0)u(t-t_0)] = e^{-t_0 s} F(s)$$

Example: $F(s) = \dfrac{\omega}{(s+a)^2 + \omega^2}$

Give $\mathrm{L}[e^{-t}] = \dfrac{1}{s+1}$, $L[e^{-t+3}u(t-3)]$? $\mathrm{L}[e^{-t+3}] = ?$

what about $L[e^{-(t+3)}]$? $L[e^{-(t+3)}u(t+3)]$?

Theorem of Laplace Transform

- Integral

$$L[\int_0^t f(\tau)d\tau] = \frac{F(s)}{s}$$

- Convolution

$$L^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)d\tau$$

Time-domain convolution becomes the frequency-domain multiplication!!!

Theorem of Laplace Transform

- Derivative

$$L[\frac{df(t)}{dt}] = sF(s) - f(0^-)$$

If signals f(t) is discontinuous at t=0

Time-domain differentiation becomes multiplication by frequency variable s!!!

- Nth-order Derivative

$$L[\frac{d^n f(t)}{dt^n}] = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f^{'}(0^-) \ldots - f^{(n-1)}(0^-)$$

- When finding transfer function, zero initial condition must be assumed.

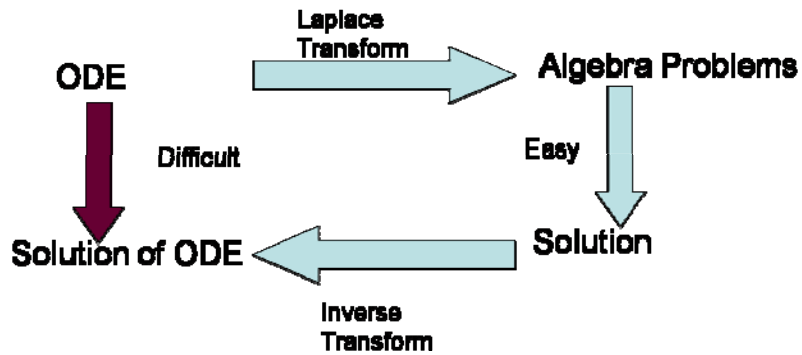
- If the initial value is considered, the initial value can be considered as a impulse input.

Inverse Laplace Transform

- General rational function

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0} \quad m<n$$

How to obtain the inverse Laplace transform from F(s)?



Inverse LT: Example 1

$$F(s)=\frac{12s+45}{s^2+5s}=\frac{9(s+5)+3s}{s(s+5)}=\frac{9}{s}+\frac{3}{s+5}$$

$$f(t)=9+3e^{-5t},t\geq 0$$

$$or\ \ f(t)=[9+3e^{-5t}]u(t)$$

Inverse Laplace Transform

- General rational function

$$F(s)=\frac{N(s)}{D(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0} \quad m<n$$

If D(s) does not have repeated roots

partial fraction expansion

$$F(s)=\frac{N(s)}{\prod_{i=0}^{n}(s-p_i)}=\frac{k_1}{(s-p_1)}+\frac{k_2}{(s-p_2)}+\ldots+\frac{k_n}{(s-p_n)}$$

From Residual Theorem

$$k_j=(s-p_j)F(s)\Big|_{s=p_j}$$

$$f(t)=k_1e^{p_1 t}+k_2e^{p_2 t}+\ldots+k_ne^{p_n t}$$

Inverse Laplace Transform

If D(s) have repeated roots

$$F(s)=\frac{N(s)}{(s-p)^r(s-c)}=\frac{k_{p1}}{(s-p)}+\frac{k_{p2}}{(s-p)^2}+\ldots+\frac{k_{pr}}{(s-p)^r}+\frac{k_c}{s-c}$$

$$k_{pi}=\frac{1}{(r-i)!}\frac{d^{r-i}}{ds^{r-i}}[(s-p)^r F(s)]\Big|_{s=p}$$

Example: Inverse LT

$$F(s) = \frac{2s+3}{s^3+2s^2+s}$$

$$F(s) = \frac{3}{s} + \frac{-3}{s+1} + \frac{-1}{(s+1)^2}$$

$$f(t) = 3 - 3e^{-t} - te^{-t} \qquad \mathrm{t} \geq 0$$

$$or \quad f(t) = [3 - 3e^{-t} - te^{-t}]u(t)$$

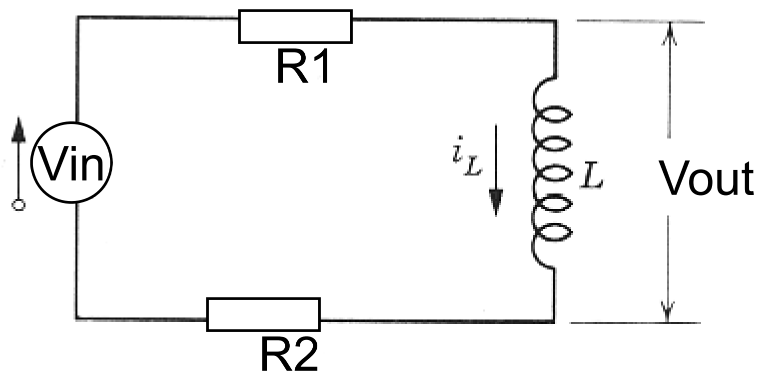
Example: Inverse LT

$$F(s) = \frac{2s+1}{s^2+2s+10}$$

Hint:

$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

How to describe a system mathematically?



Kirchhoff's law:

$$V_{in}(t) = (R1 + R2)i(t) + L\frac{di(t)}{dt}$$

$$V_{out}(t) = L\frac{di(t)}{dt}$$

Transfer Function

$$V_{in}(t) = (R1 + R2)i(t) + L\frac{di(t)}{dt}$$

$$V_{out}(t) = L\frac{di(t)}{dt}$$

LT:

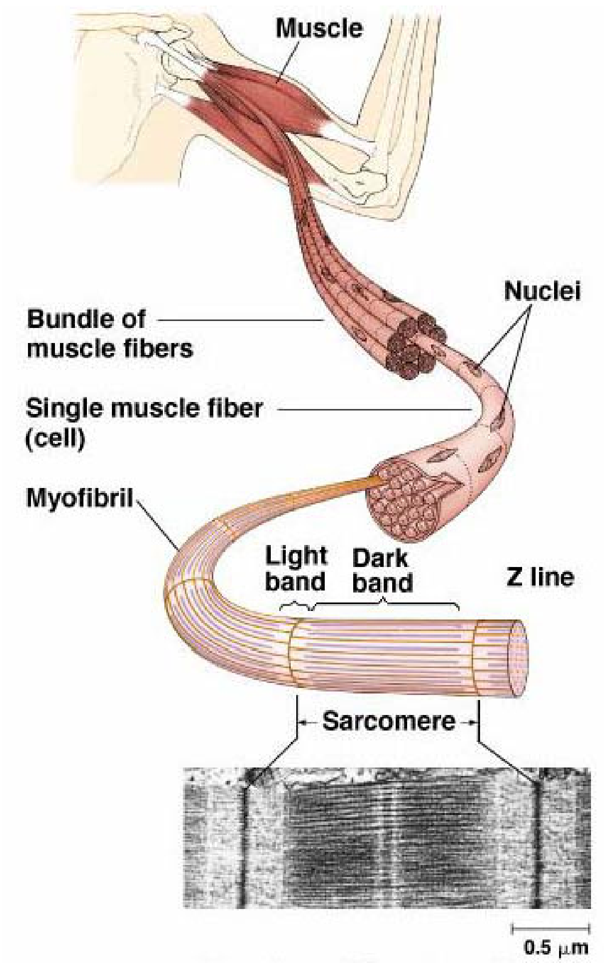
$$V_{in}(s) = (R1 + R2)I(S) + LSI(s)$$
$$V_{out}(s) = LSI(s)$$

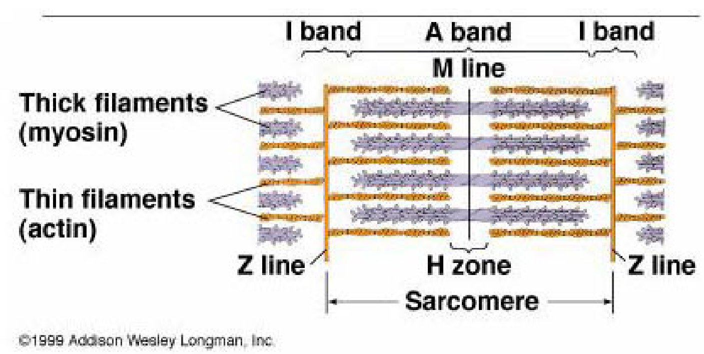
TF: $\Rightarrow$ $\therefore \dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{LS}{(R1 + R2 + LS)}$

A transfer function can be written only for the case in which the system model is a linear time-invariant differential equation and the system initial condition are ignored.

Mechanical Modeling

- Mechanical System: Skeletal Muscle

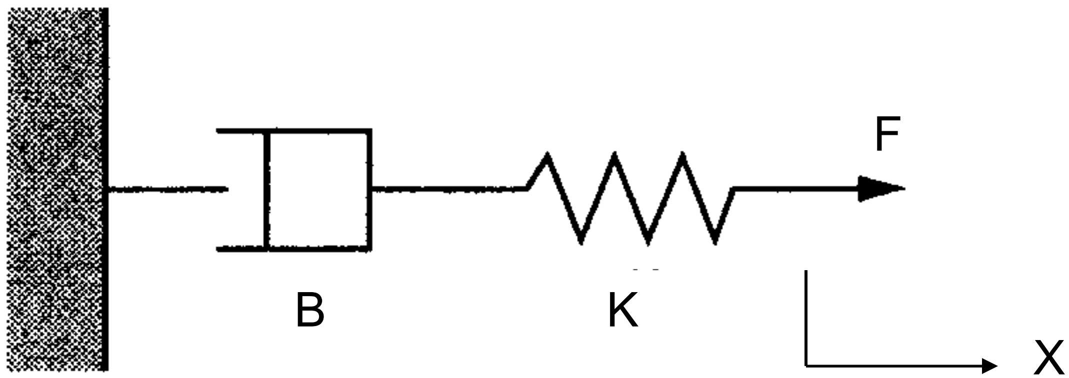




Viscoelastic Theory

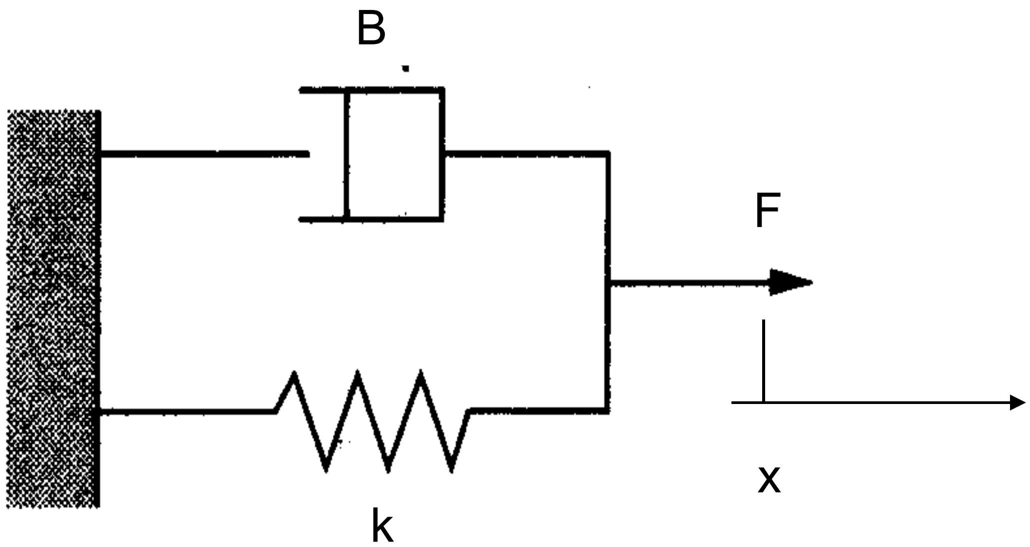
- Generalize materials
- Deformation properties
- With types of elements:
  - Linear elastic elements = springs
  - Viscous elements = dashpots

Combinations of Elements #1
Maxwell Model



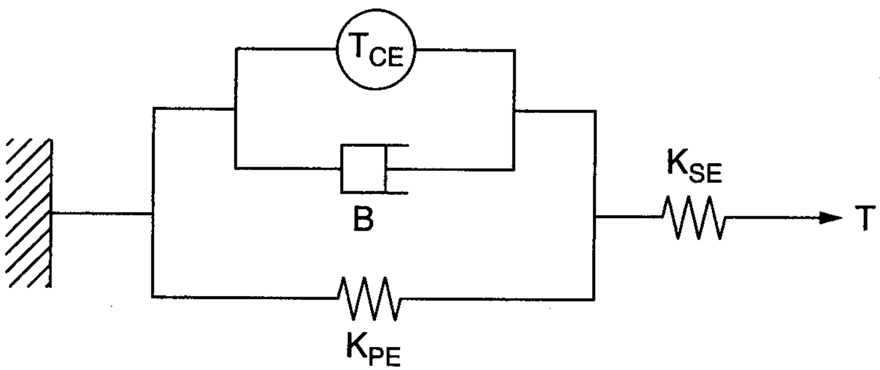
$$\begin{cases} B\dot{x}_1(t) = kx_2(t) = F(t) \\ x_1(t) + x_2(t) = x(t) \end{cases}$$

Combinations of Elements #2



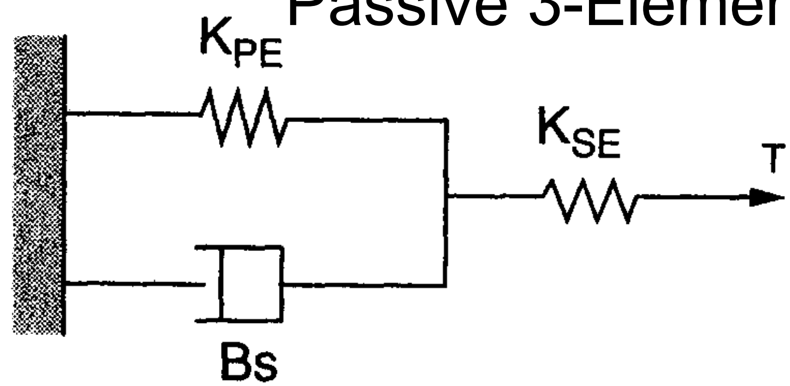
Hill’s Muscle Model

Active 4- element



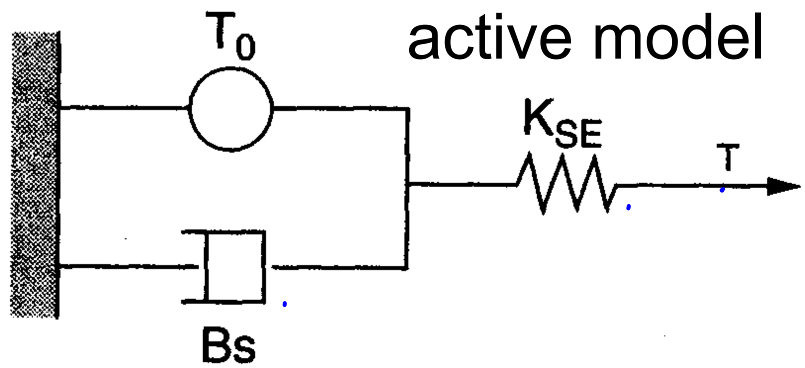
Kevin’s model

Passive 3-Element

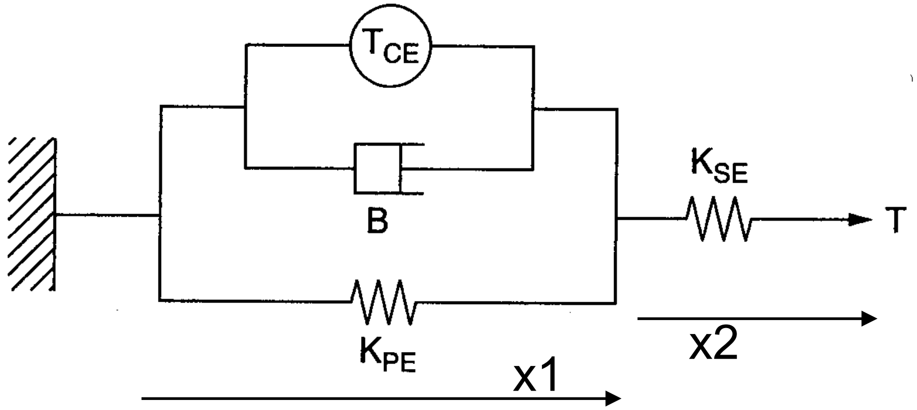


- CE: Contractile element
- SE: Elastic element
- PE: Passive element
- B: Fluid caused damping
- Exertion Force: T-output

Simplified active model



Hill's Model



$$\begin{cases} x_1 + x_2 = x \Rightarrow X_1 + X_2 = X \\ T = k_{SE} x_2 \Rightarrow X_2 = \dfrac{T}{k_{SE}} \\ -T = T_{CE} - B\dot{x}_1 - k_{PE} x_1 \Rightarrow T_{CE} = -T + (Bs + k_{PE}) X_1 \end{cases}$$

$$T_{CE} = -T + (Bs + k_{PE})(X - X_2)$$

$$T_{CE} = -T + (Bs + k_{PE})(X - \frac{T}{k_{SE}})$$

$$T_{CE} = (Bs + k_{PE})X - (1 + \frac{Bs + k_{PE}}{k_{SE}})T$$

Hill's Model

$$T_{CE} = (Bs + k_{PE})X - (1 + \frac{Bs + k_{PE}}{k_{SE}})T$$

- Discussion

(1) Isometric muscle contraction: X=0

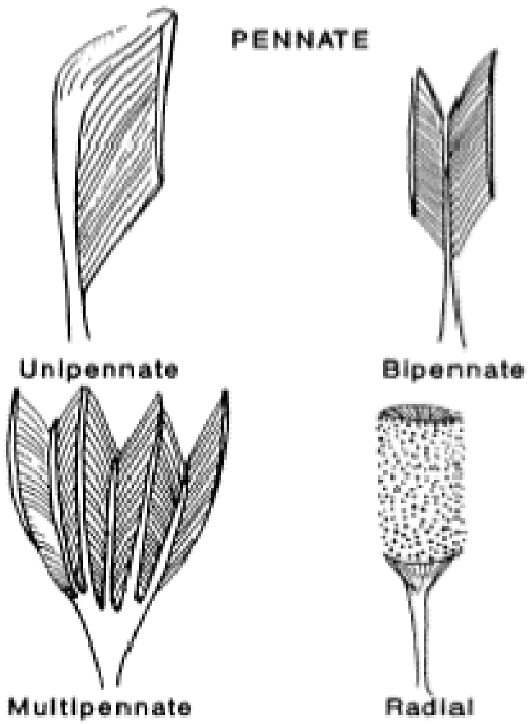
(2) Stress relaxation in muscle: Tce=0; Passively stretch muscle

(3) Creep in muscles: Tce=0; Adding a constant passive muscle external force

(4) Isotonic muscle contraction: T is a constant.

Improved Hill's Model

Zajac 1986



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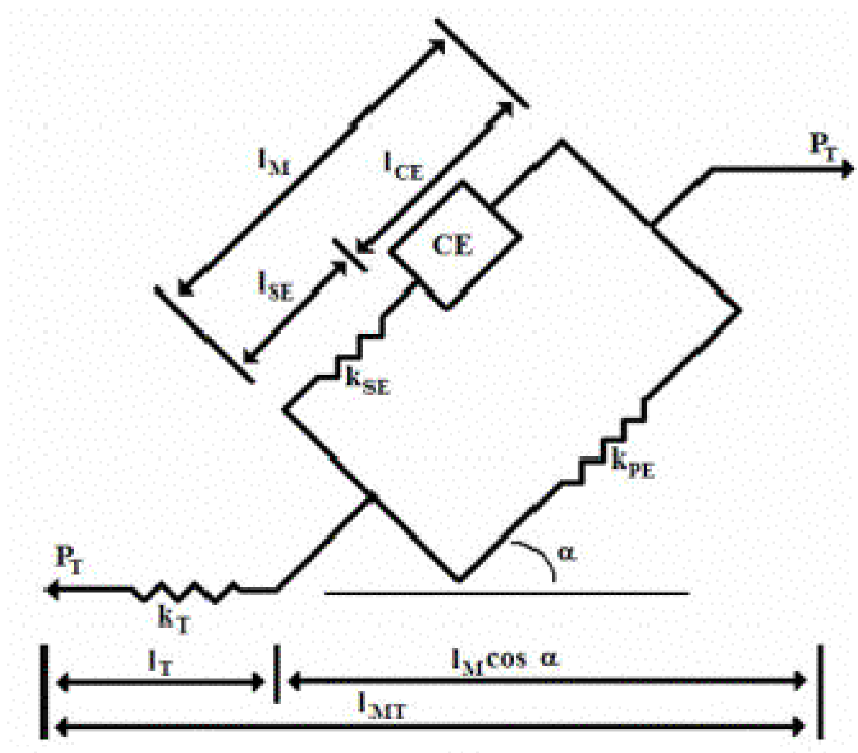
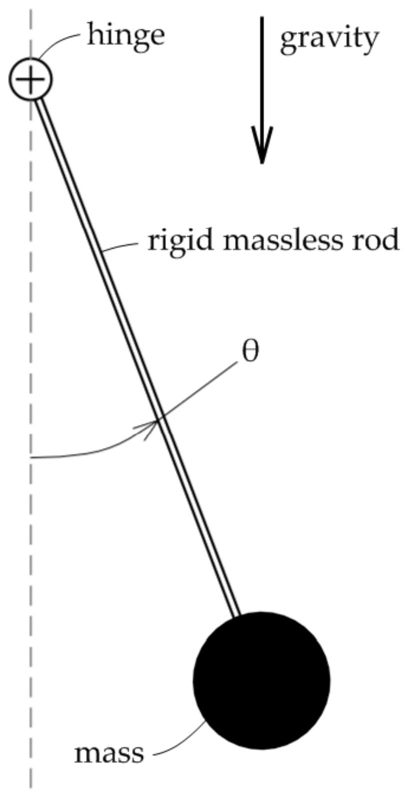


b

Nonlinear differential equations



$$ML\frac{d^2\theta(t)}{dt^2} = Mg\sin\theta(t)$$

Small θ

$$ML\frac{d^2\theta(t)}{dt^2} = Mg\theta(t)$$

http://en.wikipedia.org/wiki/Nonlinearity

Block Diagram

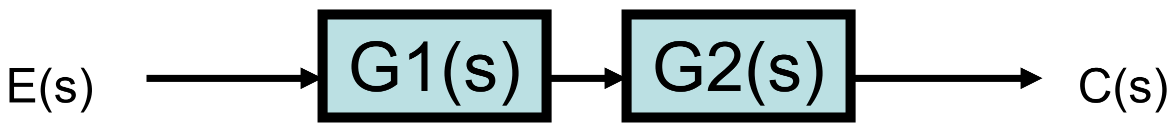
- E(s): Laplace transform of input variable
- C(s): Laplace transform of output variable
- G(s): System transfer function

C(s)=?

Block Diagram

- The transfer function relationship C(s)=G(s)E(s) can be graphically denoted through a block diagram

Series connection

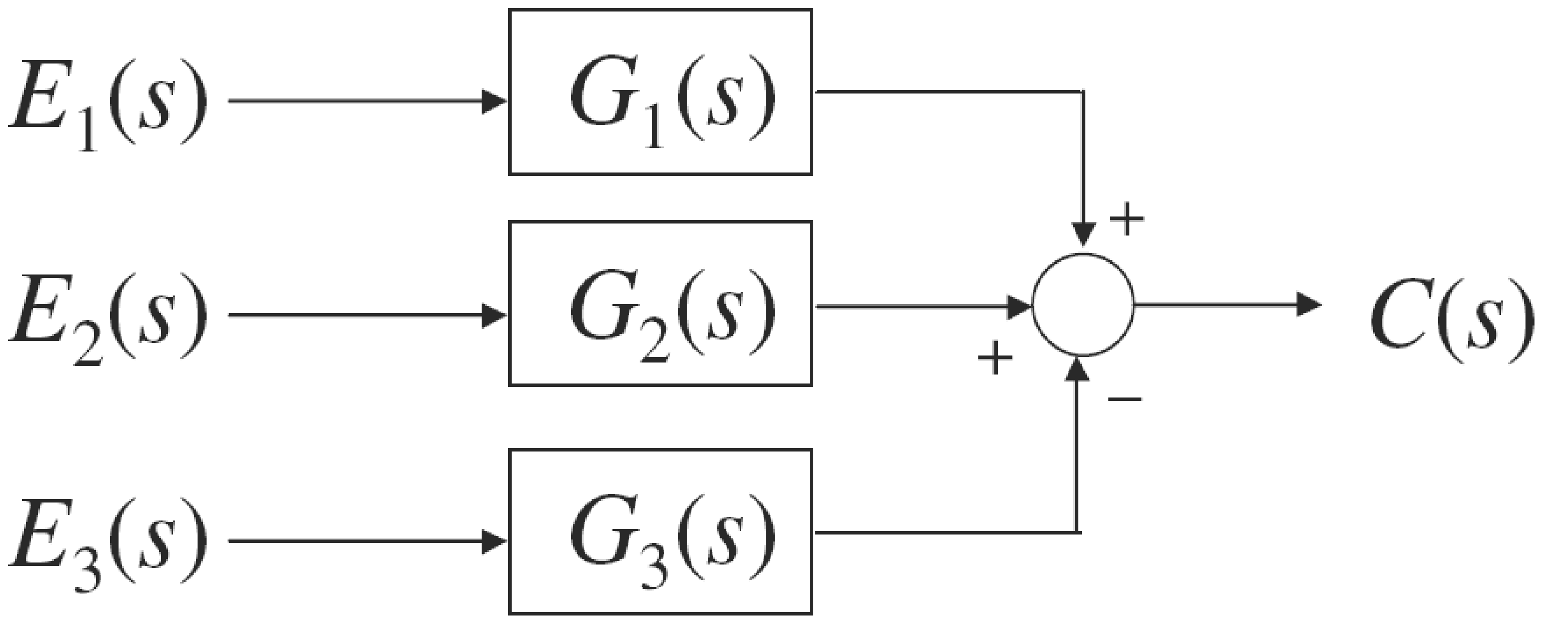


$C(s) = G1(s)G2(s)E(s)$

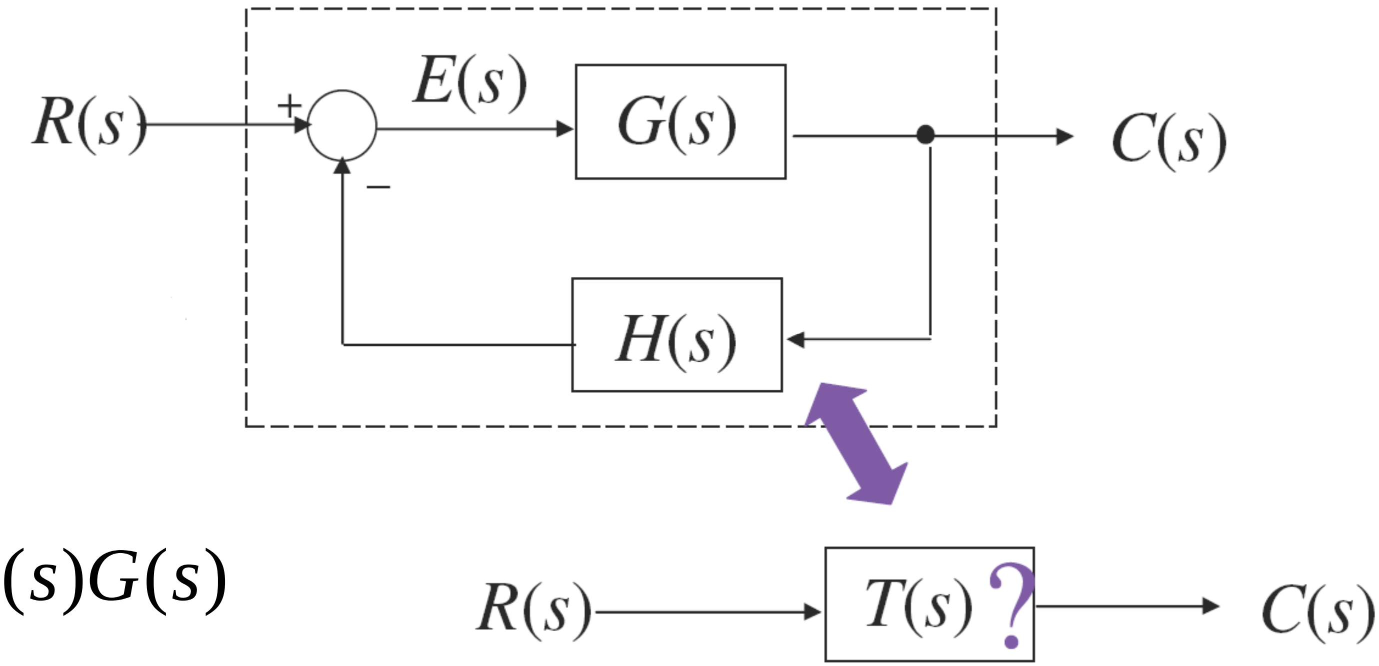
Block Diagram

$$C(s) = G_1(s)E_1(s) + G_2(s)E_2(s) - G_3(s)E_3(s)$$

parallel connection



Block Diagram: Eliminating Variables



$$C(s) = E(s)G(s)$$

$$E(s) = \frac{C(s)}{G(s)} = R(s) - H(s)C(s)$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$