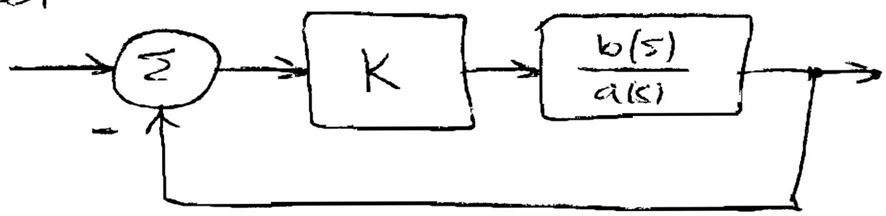
Root Locus Rules

Consider



Note: e.g. if $C(s) = \frac{K(s+1)}{s+2}$, $G(s) = \frac{5}{s+10}$

then $\frac{b(s)}{a(s)} = \frac{5(s+1)}{(s+2)(s+10)}$

In general, $\frac{b(s)}{a(s)}$ has m zeros and n poles

We are interested in the poles of the CLTF as K varies from 0 to ∞. The plot of the pole locations is called a root locus.

Rule 0 The root locus is symmetric about the real axis

Rule 1 The n branches of the locus start at the open-loop poles (roots of $a(s)$) and move toward the m open-loop zeros (roots of $b(s)$) with the remaining $n-m$ branches going to infinity.

Rule 2 The branches on the real axis lie to the left of an odd number of open loop poles and zeros

Rule 3 $n-m$ branches go to infinity at asymptotes centered at

p_i = roots of $a(s)$

z_i = roots of $b(s)$

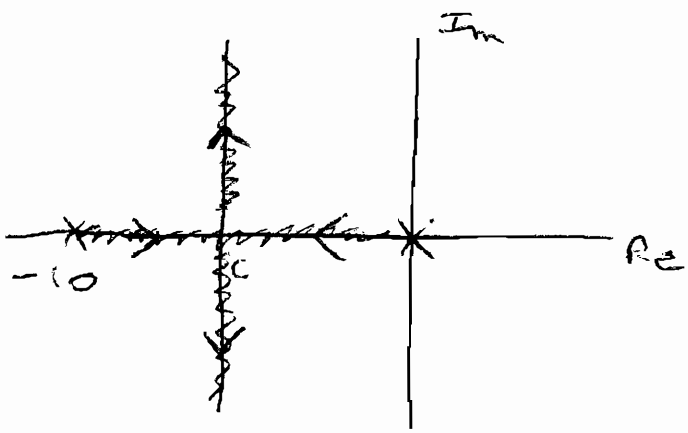
$$c = \frac{\sum p_i - \sum z_i}{n-m}$$

at angles $$\phi_\ell = \frac{180^\circ + (\ell-1)360^\circ}{n-m}, \quad \ell = 1, 2, \ldots, n-m$$

Example 1: $\dfrac{b(s)}{a(s)} = \dfrac{5}{s(s+10)}$

$m = 0$, no z_i

$n = 2$, $p_1 = 0$, $p_2 = -10$



$c = \dfrac{-10}{2} = -5$

$\phi_1 = 90^\circ$

$\phi_2 = 270^\circ$

Example 2: $\dfrac{b(s)}{a(s)} = \dfrac{5}{s(s+10)\left(\frac{s}{100}+1\right)}$

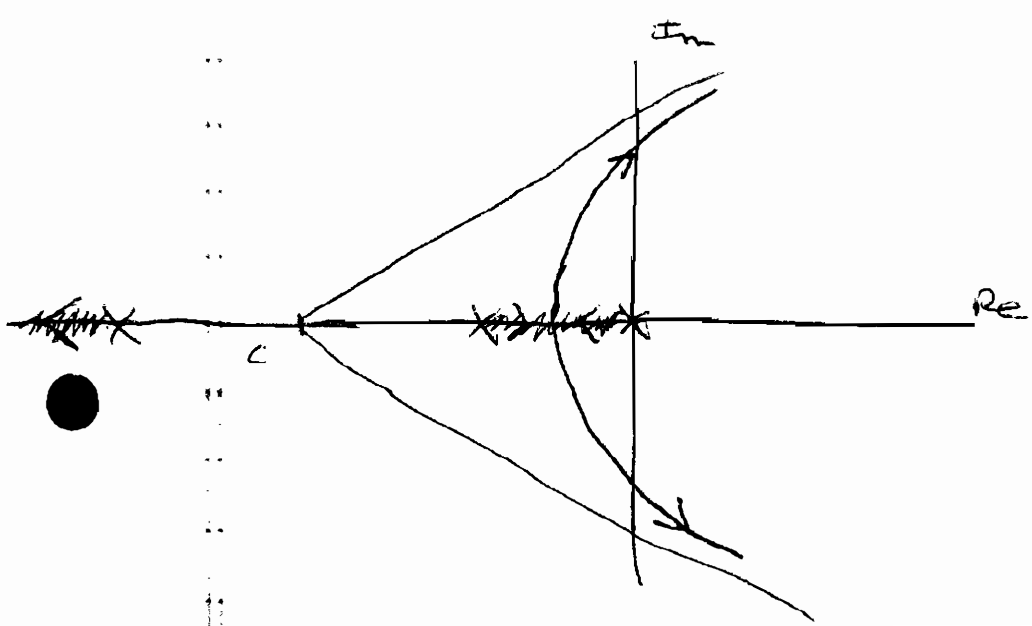
$m = 0$

$n = 3$

$p_1 = 0$

$p_2 = -10$

$p_3 = -100$



$c = \dfrac{-110}{3} \approx -37$

$\phi_1 = 60^\circ$

$\phi_2 = 180^\circ$

$\phi_3 = 300^\circ$

Example 3

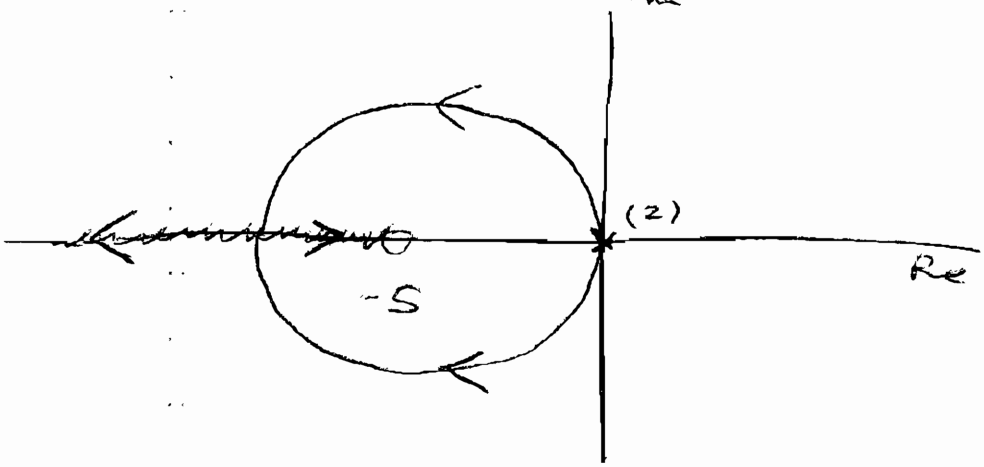
$$\frac{b(s)}{a(s)} = \frac{s+5}{s^2}$$

$m = 1$ $\quad z_1 = -5$

$p = 2$ $\quad p_1 = 0$, $p_2 = 0$

$c = 5$

$\phi_1 = 180°$



Example 4

$$\frac{b(s)}{a(s)} = \frac{s+5}{s^2(s+10)}$$

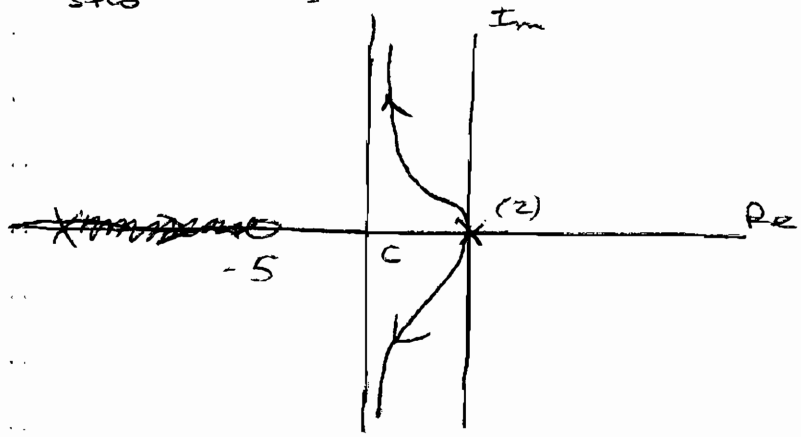
$C(s) = k\frac{s+5}{s+10}, \quad G(s) = \frac{1}{s^2}$

$m = 1, \; n = 3$

$c = -\frac{5}{2} = -2.5$

$\phi_1 = 90°$

$\phi_2 = 270°$



Example 5

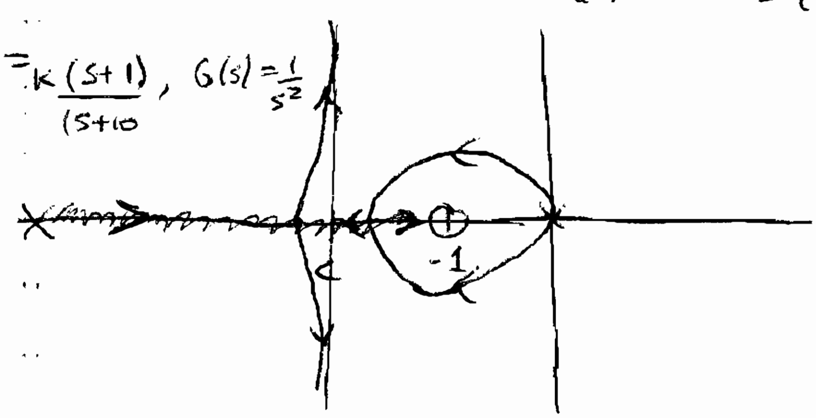
$$\frac{b(s)}{a(s)} = \frac{s+1}{s^2(s+10)}$$

$C(s) = k\frac{(s+1)}{(s+10)}, \quad G(s) = \frac{1}{s^2}$

$m = 1, \; n = 3$

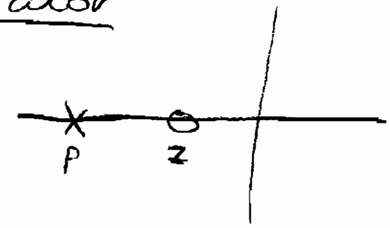
$c = -\frac{9}{2} = -4.5$

$\phi_{1,2} = \frac{90°}{270°}$



Phase-Lead Compensator

$$C(s) = K\,\frac{(s-z)}{(s-p)}$$



Effect: pulls the root locus "to the left" resulting in a stable, "fast" system.

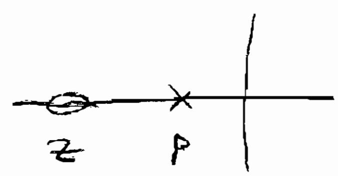
[Settling Time]

Design Procedure: Get a root locus plot of desired shape. Use Matlab to meet numerical specifications.

Consider effect of p & z on center of asymptotes

Phase Lag Compensator

$$C(s) = K\,\frac{(s-z)}{(s-p)}$$



Effect: pulls root locus "to the right". Thus, need to be careful about stability.

[Steady-state error]

This is why phase lag is needed → * May be necessary to reduce steady-state error.

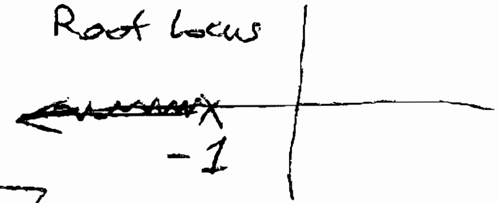
If we let $p=0$ then $C(s)$ contains a pure integrator, and this special phase-lag compensator is called a PI (proportional plus integral) compensator.

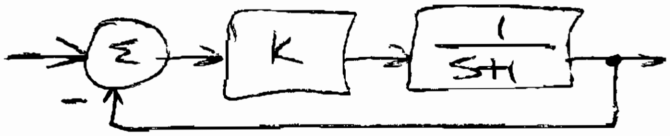
A PI compensator increases the system Type by 1.

Example: $C(s) = K \quad G(s) = \frac{1}{s+1}$

$$\frac{b(s)}{a(s)} = \frac{1}{s+1}$$

Root Locus





Suppose we want $T_s = 2$ sec

CLTF is $\frac{K}{s+1+K}$. Pole @ $s = -(1+K)$

$$T_s = 4.62/\alpha \Rightarrow \alpha = 2.31 \Rightarrow \text{pole} = -2.31$$

$$1+K = 2.31$$

$$\boxed{K = 1.31}$$

For this value of K what is the steady-state error to a step of height R?

$$Y_{ss} = (\text{DC gain}) \cdot (R) = \frac{1.31}{1+1.31} \cdot R = .57R$$

$$e_{ss} = R - Y_{ss} = (1-.57)R = .43R$$

the steady-state error is 43% of the reference input.

Suppose we want $e_{ss} = 0$ and $T_s = 2$?

To get $e_{ss} = 0$ we need a Type 1 system.

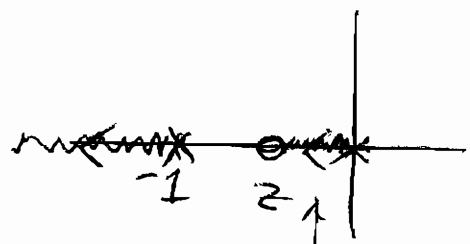
Let $C(s) = \underbrace{K \cdot \frac{(s-z)}{s}}_{\text{PI Compensator}}$ i.e. a phase lag compensator

(pole to right of zero)

$$\frac{b(s)}{a(s)} = \frac{s-z}{s(s+1)}. \quad \text{Two possibilities}$$

$z < -1$ $\qquad\qquad$ $z > -1$

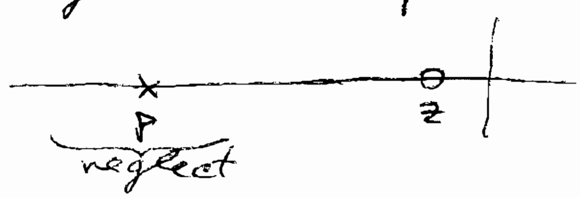
This will work well for $z \simeq -2$



This pole is too slow

In Matlab, check $\dfrac{b(s)}{a(s)} = \dfrac{s+2}{s(s+1)}$

A phase-lead compensator $K\dfrac{(s-z)}{(s-p)}$ with a large value of p



at low frequencies $\approx K'(s-z) = K's - Kz$

derivative $\qquad$ proportional

This is a PD (proportional plus derivative) compensator.

A practical PID (proportional/integral/derivative) compensator is a combination of phase lag (PI) and phase lead:

$$\text{PID}: \quad C(s) = \frac{K(s-z_1)(s-z_2)}{s(s-p)}$$