ELE 509

INTRODUCTION TO RANDOM PROCESSES

INSTRUCTOR : PROF. STEVEN KAY

TEXTBOOK : 1) INTUITIVE PROBABILITY AND RANDOM PROCESSES USING MATLAB, S. KAY (REQUIRED)

2) SCHAUM'S OUTLINE ON PROBABILITY, RANDOM VARIABLES, AND RANDOM PROCESSES, H. HSU (RECOMMENDED)

OTHER REFERENCES :

1) PROBABILITY AND RANDOM PROCESSES, W. DAVENPORT (OUT OF PRINT), McGRAW HILL, 1970
2) PROBABILITY AND RANDOM PROCESSES, A. LEON-GARCIA, 1994
3) ... SEE PAGE 9 OF TEXT

TOPICS : CHAPTERS 10-12, CHAPTER 14 (SOME SECTIONS), CHAPTERS 15-18, SELECTED PORTIONS OF CHAPTERS 20 AND 21

- CONTINUOUS RANDOM VARIABLES (CRV)
- EXPECTED VALUES
- MULTIPLE CRV

- N-DIMENSIONAL CRV
- LAW OF LARGE NUMBERS
- CENTRAL LIMIT THEOREM
- BASIC RANDOM PROCESSES (RP)
- STATIONARY RP
- LINEAR SYSTEMS AND RP
- GAUSSIAN RP
- POISSON RP

* REAL-WORLD APPLICATIONS
* COMPUTER SIMULATIONS (VIA MATLAB)

* THROUGHOUT COURSE

PREREQUISITES: 1) MTH 451 OR EQUIVALENT COURSE IN BASIC PROBABILITY (CHAPTERS 1-7 IN TEXT)
2) CALCULUS
3) LINEAR SYSTEMS AND TRANSFORMS
4) LINEAR AND MATRIX ALGEBRA

GRADING:

"HOURLY" EXAM - 30%
HOURLY EXAM - 30%
FINAL EXAM - 40%
WEEKLY HOMEWORK ASSIGNED.
HOMEWORK DUE ONE WEEK AFTER ASSIGNED!

GRADED AS EITHER PASS (✓) OR FAIL (✓-). MUST ATTEMPT ALL PROBLEMS TO PASS. (TOO MANY ✓-'s ⇒ LOWER GRADE!)

HOMEWORK = ANALYTICAL EXERCISES
+ MATLAB EXPERIMENTS

NOTES: COPIES OF TRANSPARENCIES AT
WWW.ELE.URI/COURSES
READ BEFORE CLASS!

OFFICE HOURS: TUES, THURS, FRI 9-11 AM
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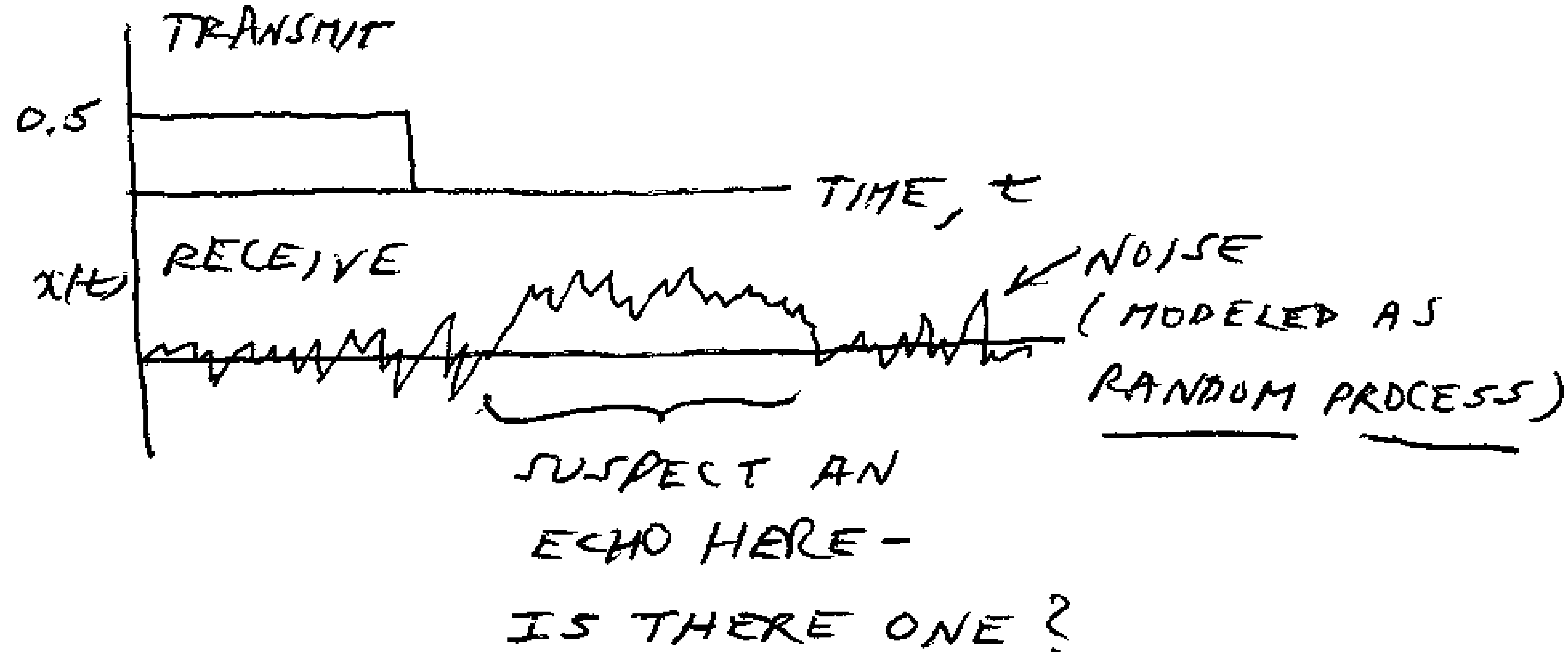
INTRODUCTION (14.10*)

COMMON PROBLEM: SIGNAL DETECTION

1) RADAR
2) SONAR
3) COMMUNICATIONS
4) CHEMICALS, BIOLOGICS, ETC

* SECTION NUMBER IN TEXT.

TRANSMIT PULSE AND LOOK FOR ECHO FROM "TARGET"



ASSUME WE SAMPLE $x(t) \Rightarrow x_i, \ i = 1, 2, \ldots, N$

HOW CAN WE MAKE A DECISION IF

$$x_i = 0.5 + w_i \qquad i = 1, 2, \ldots, N$$

OR

$$x_i = w_i$$

$\uparrow$ NOISE SAMPLE ?

SOLUTION: COMPUTE SAMPLE MEAN

$$\frac{1}{N}\sum_{i=1}^{N} x_i \longrightarrow \begin{array}{ll} 0 & \text{FOR NOISE ONLY} \\ 0.5 & \text{FOR SIGNAL + NOISE} \end{array}$$

WHY ?

DECIDE SIGNAL PRESENT IF

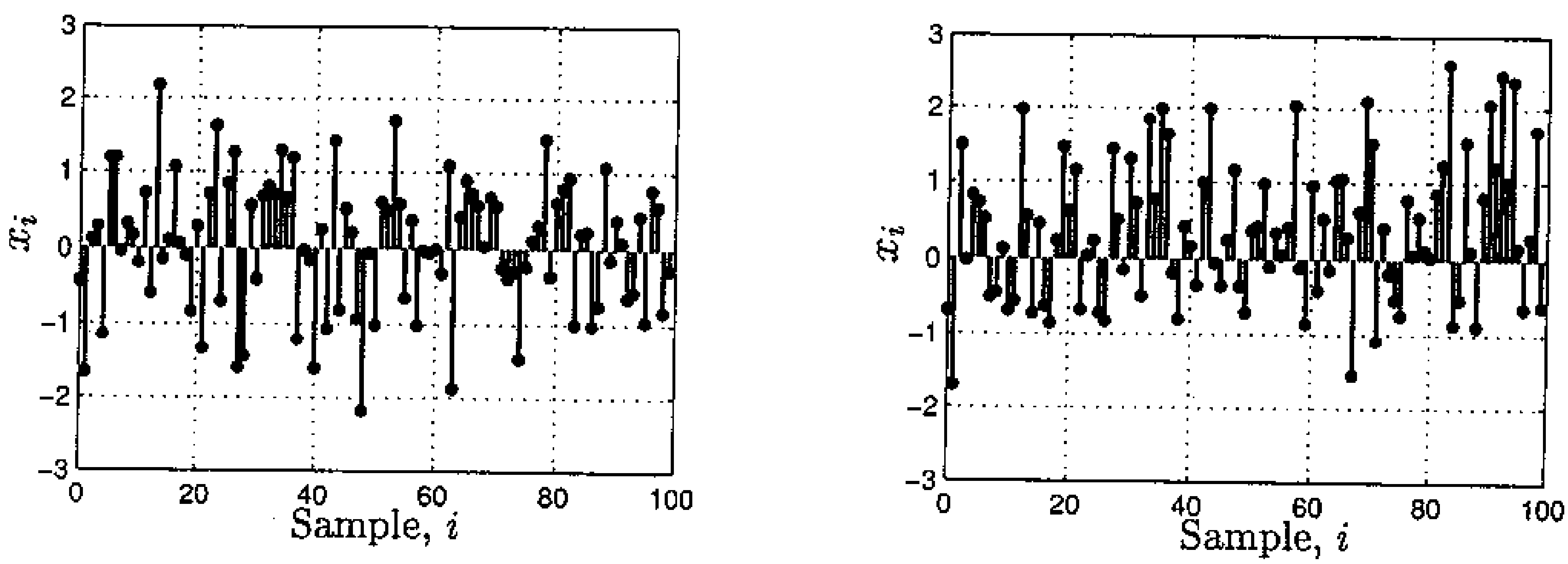
$$\frac{1}{N}\sum_{i=1}^{N} x_i > 0.25 = \text{THRESHOLD}$$

IS THIS A GOOD DETECTOR ?
(TAKE ELE 665 FOR ANSWER!)

CAN USE A COMPUTER SIMULATION TO "ASSESS" PERFORMANCE BY :

1) GENERATE USING MATLAB $N = 100$ SAMPLES OF NOISE
2) COMPUTE SAMPLE MEAN (> 0.25 ?)
3) REPEAT (1) BUT NOW ADD 0.5 TO EACH SAMPLE
4) COMPUTE SAMPLE MEAN (> 0.25 ?)

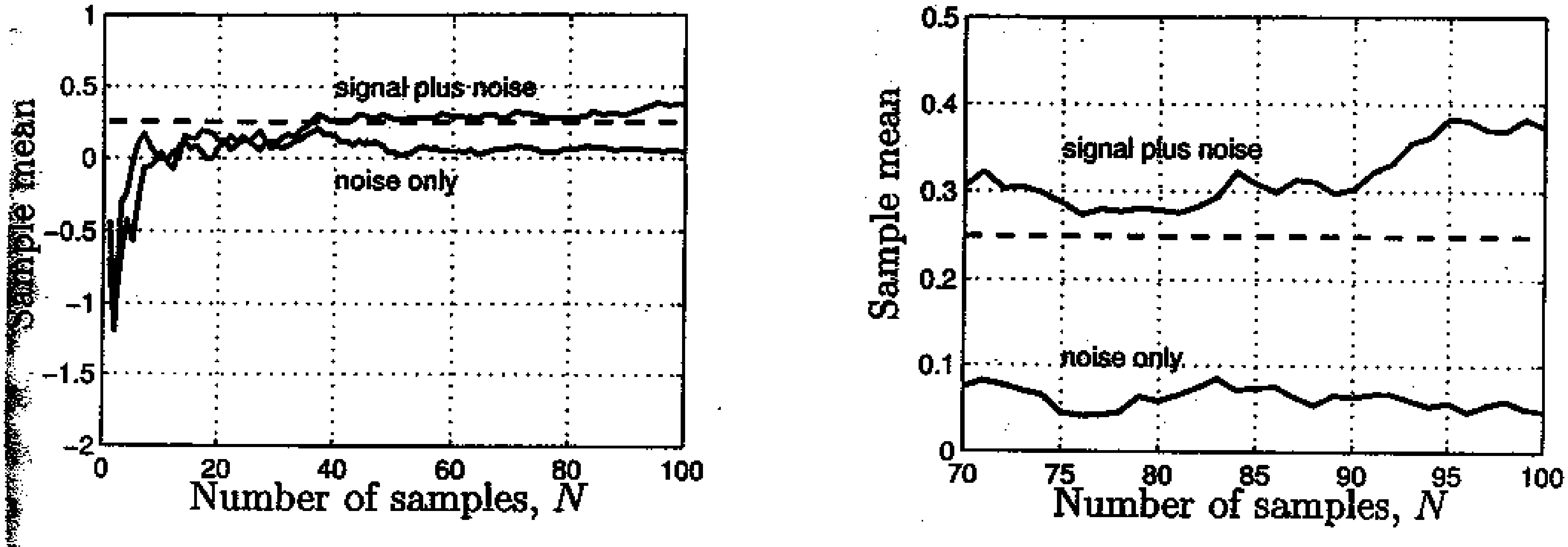
MIGHT WANT TO REPEAT EXPERIMENT SEVERAL TIMES - WHY ?



(a) Noise only

(b) Signal plus noise

Figure 14.5: Received data samples. Signal is $s_i = A = 0.5$ and noise consists of IID standard Gaussian random variables.



(a) Total view

(b) Expanded view for $70 \leq N \leq 100$

Figure 14.6: Value of sample mean versus the number of data samples averaged.

QUESTIONS:

1) WHAT WILL SAMPLE MEAN CONVERGE TO (IF IT CONVERGES)?
2) HOW LARGE DOES N HAVE TO BE?
3) HOW DO NOISE STATISTICS AFFECT RESULTS?
4) HOW DOES AMPLITUDE OF PULSE AFFECT RESULTS?

⇨ ALL THESE QUESTIONS CAN BE ANSWERED BY STUDYING RANDOM PROCESS THEORY (AND SOME DETECTION THEORY!)

CHAPTER 10 - CONT. R.V.s

RECALL THAT A RANDOM VARIABLE IS A <u>MAPPING</u> FROM <u>SAMPLE SPACE</u> TO A <u>NUMBER</u>

<u>EXAMPLE</u>: TOSS A DIE

$s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6$

$S = \{$  $\}$

↑ SAMPLE SPACE (ALL POSSIBLE EXPERIMENTAL OUTCOMES)

$X(s_i) = i \qquad i = 1, 2, 3, 4, 5, 6$

↑

RANDOM VARIABLE (NOT REALLY RANDOM!)
(R.V.)

NOTE: SIZE OF $S = 6$
NUMBER OF VALUES OF $X = 6$

THIS IS A DISCRETE R.V. ⇒ <u>COUNTABLE</u> NUMBER OF VALUES. COULD HAVE

$X(s_i) = i \qquad i = 1, 2, 3, \ldots$

STILL A DISCRETE R.V.

FOR THE DIE EXAMPLE $P[X(s_i) = i] = \frac{1}{6}$

FOR THIS EXAMPLE COULD HAVE

$$P[X(s_i) = i] = \frac{1}{2^i} \qquad i = 1, 2, 3, \dots$$

DOES $\sum_{i=1}^{\infty} P[X(s_i) = i] = 1$?

IN GENERAL DENOTE R.V. BY CAPITAL LETTER X AND VALUE BY LOWER CASE x, $X(s_i) = x_i$ (SIMILAR TO FUNCTION DEFINITION $y = F(x)$).

$$\underset{x_i}{\underset{\uparrow}{y}} = \underset{X}{\underset{\uparrow}{F}}(\underset{s_i}{\underset{\uparrow}{x}})$$

NOW CONSIDER A DARTBOARD EXPT.

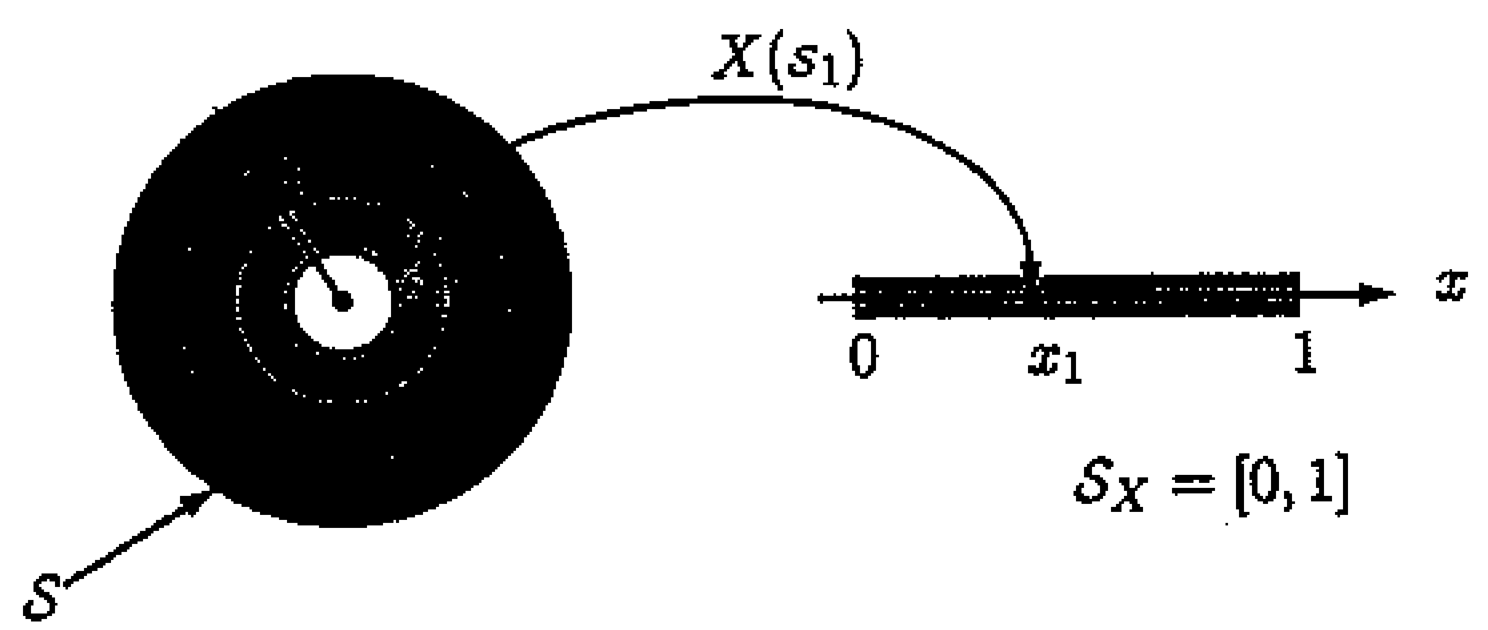


Figure 10.1: Mapping of the outcome of a thrown dart to the real line (example of continuous random variable).

X IS DISTANCE FROM BULLSEYE (CENTER).

HOW MANY OUTCOMES ARE THERE?

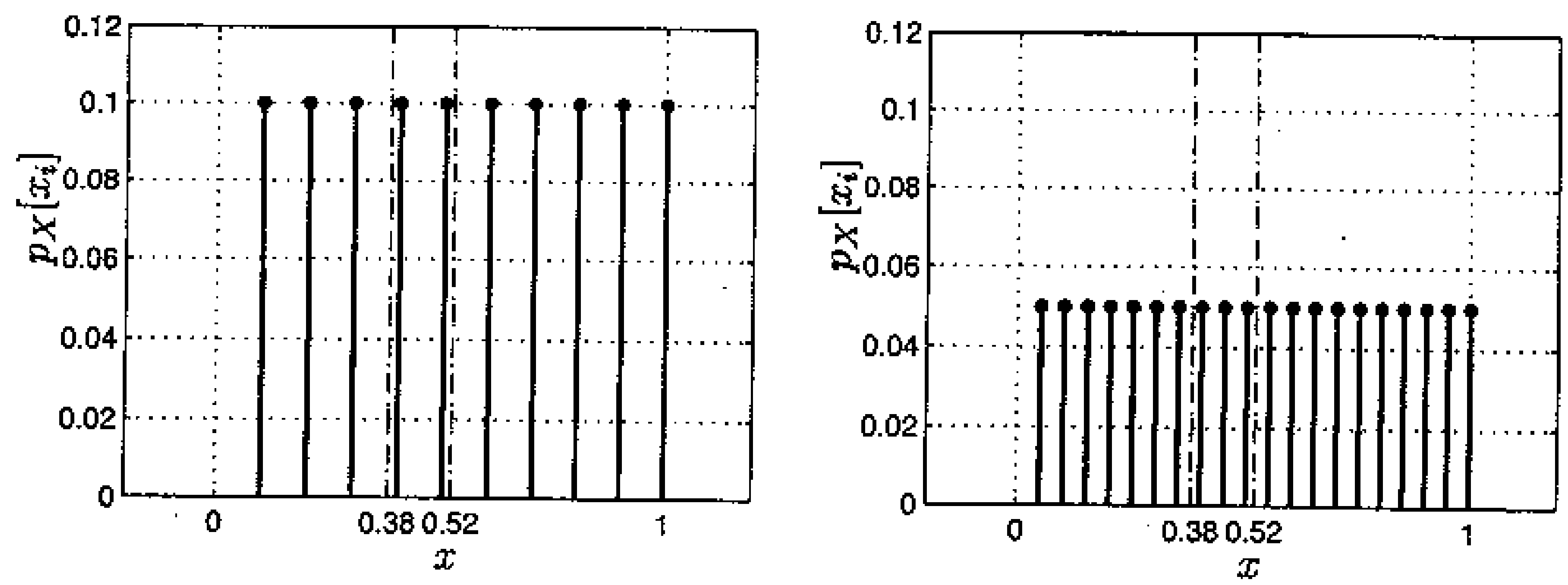
CAN WE LET $P[X(s_i) = x_i] = p_i$ FOR $i = 1, 2, 3, \dots$?

ASSUME ALL OUTCOMES ARE EQUALLY LIKELY $\Rightarrow P[0 \le X \le \frac{1}{2}] = P[\frac{1}{4} \le X \le \frac{3}{4}]$
$= P[\frac{1}{3} \le X \le \frac{1}{3}+\frac{1}{2}]$ ETC.

ASSIGN UNIFORM PROBABILITIES OR

$$P[a \le X \le b] = b - a \qquad 0 \le a \le b \le 1$$
$$= \text{LENGTH OF INTERVAL}$$

IS THIS REASONABLE? IF OUTCOMES WERE DISCRETE OR $x_i = \Delta x, 2\Delta x, \ldots, M\Delta x$ AND EQUALLY LIKELY



(a) $M = 10$, $\Delta x = 0.1$ (b) $M = 20$, $\Delta x = 0.05$

Figure 10.2: Approximating the probability of an interval for a continuous random variable by using a PMF.

$p_X[x_i]$ = PROBABILITY MASS FUNCTION (PMF)
NOTE

$= 1/M$

RECALL $\displaystyle P[a \le X \le b] = \sum_{\{i : a \le x_i \le b\}} p_X[x_i]$

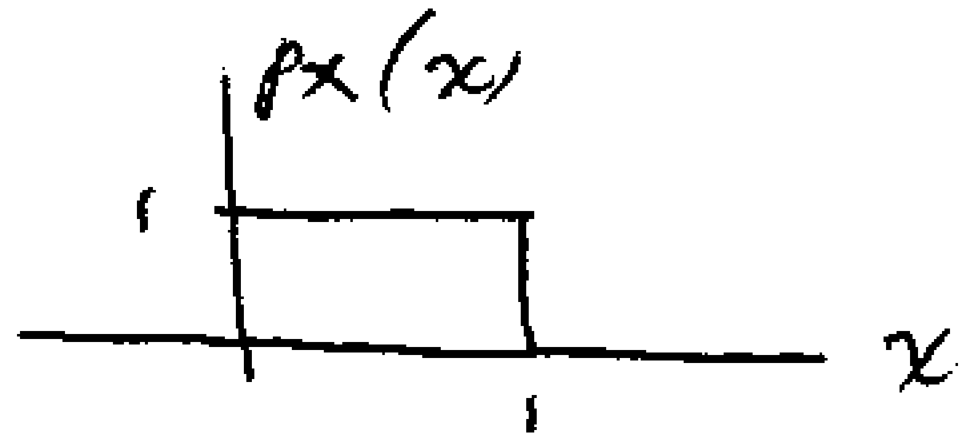
BUT $1/M = 1 \cdot \Delta x$ AND AS $\Delta x \to 0$ WE RECOVER ALL OUTCOMES.

$$\Rightarrow P[a \le X \le b] = \sum_{\{i: a \le x_i \le b\}} 1 \cdot \Delta x$$

$$= \sum_{\{i: a \le x_i \le b\}} p_X(x_i) \Delta x \qquad p_X(x) = 1 \text{ FOR } 0 < x < 1$$

$$\to \int_a^b p_X(x)\, dx$$

AS $\Delta x \to 0$. $p_X(x)$ CALLED THE <u>PROBABILITY</u> <u>DENSITY</u> <u>FUNCTION</u> (PDF).



TO FIND PROBABILITY OF DART LANDING IN INTERVAL $[a, b]$ JUST INTEGRATE PDF OR

$$P[a \le X \le b] = \int_a^b p_X(x)\, dx$$

$$= \int_a^b 1\, dx = b - a$$

OUR ORIGNAL ASSIGNMENT FOR <u>EQUALLY</u> <u>LIKELY</u> OUTCOMES.

NOTE : $p_X[x_i]$ DENOTES PMF

$p_X(x)$ DENOTES PDF

THINK OF PDF AS PROBABILITY PER UNIT LENGTH SINCE ASSUMING THE PREVIOUS DISCRETE APPROXIMATION

$$P[x_0 - \Delta x/2 \le X \le x_0 + \Delta x/2] =$$

$$\sum_{\{i:\ x_0-\Delta x/2 \le x_i \le x_0+\Delta x/2\}} p_X(x_i)\,\Delta x = \sum_{\{i:\ x_i = x_0\}} p_X(x_i)\,\Delta x$$

Δx SMALL AND $x_0 = k\Delta x$

$$= p_X(x_0)\,\Delta x$$

$$\Rightarrow \quad p_X(x_0) = \frac{P[x_0 - \Delta x/2 \le X \le x_0 + \Delta x/2]}{\Delta x}$$

PDF FOR WHICH $p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{OTHERWISE} \end{cases}$

CALLED A UNIFORM PDF.

USE rand IN MATLAB TO GENERATE AN OUTCOME.

ANOTHER EXAMPLE : IN GENERAL WE FIND PROBABILITIES BY $P[a \le X \le b] = \int_a^b p_X(x)\,dx$

$p_X(x) = 2(1-x)$

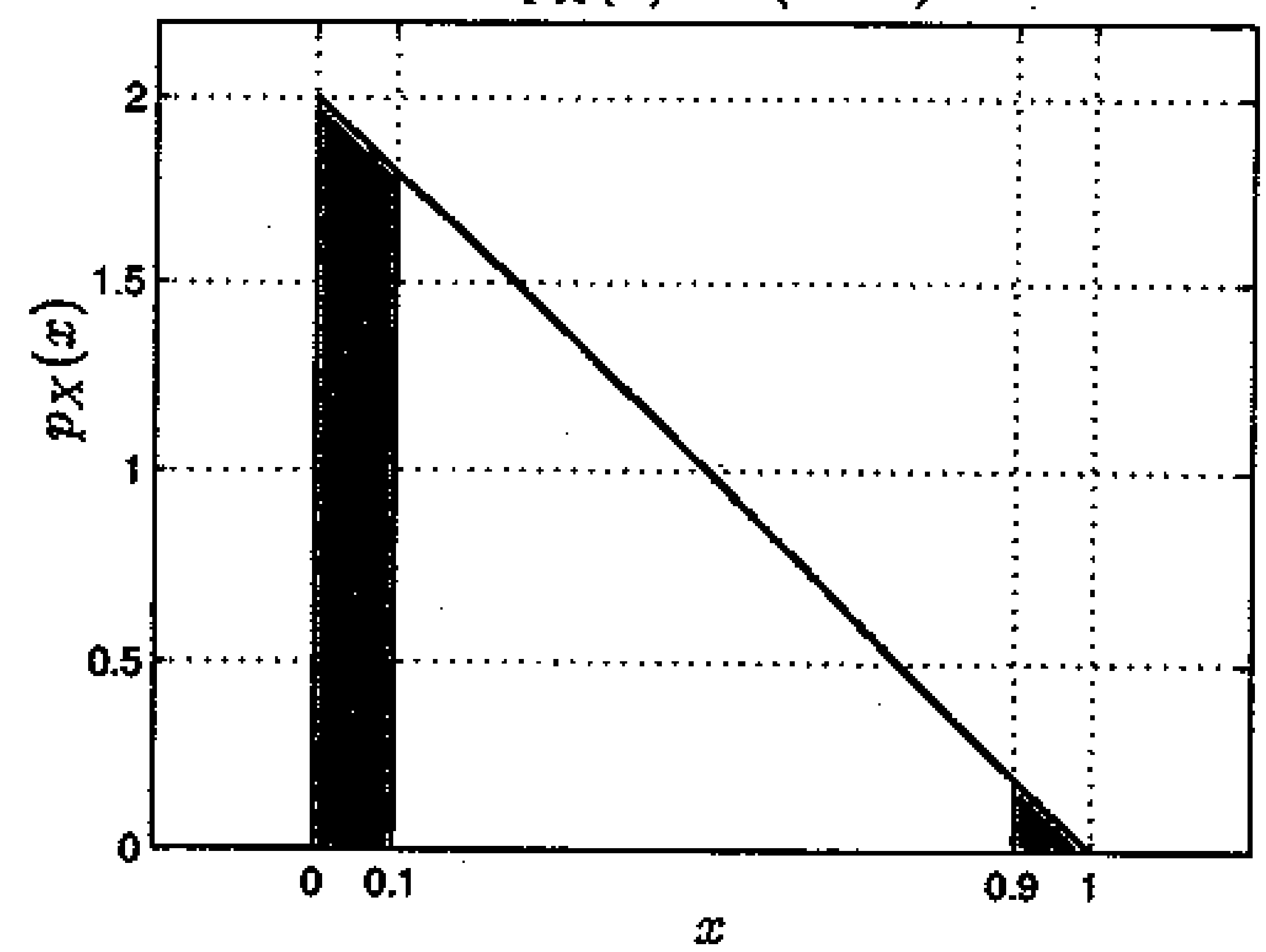


Figure 10.4: Nonuniform PDF.

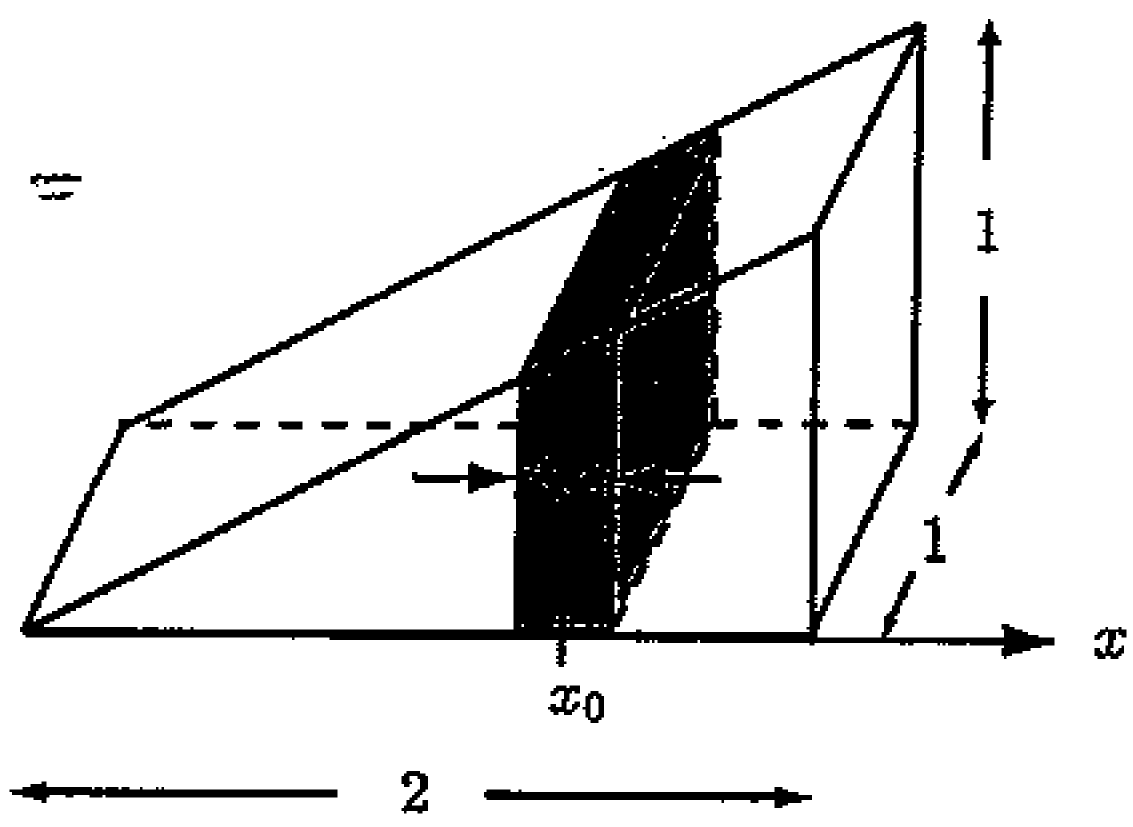
$$\begin{aligned} P[0 \le X \le 0.1] &= \int_0^{0.1} 2(1-x)dx = 2(x - x^2/2)\Big|_0^{0.1} = 0.19 \\ P[0.9 \le X \le 1] &= \int_{0.9}^{1} 2(1-x)dx = 2(x - x^2/2)\Big|_{0.9}^{1} = 0.01. \end{aligned}$$

NOTE : $p_X(x) \ge 0$ $\quad \int_{-\infty}^{\infty} p_X(x)\,dx = 1$

PDF IS ANALOGOUS TO A MASS DENSITY OR MASS PER UNIT LENGTH

CONSIDER WEDGE OF CHEESE BELOW. IF M = MASS, AND TOTAL MASS = 1,

$M([a,b]) = \int_a^b \underbrace{\tfrac{1}{2} x}_{\text{MASS DENSITY}} \, dx$



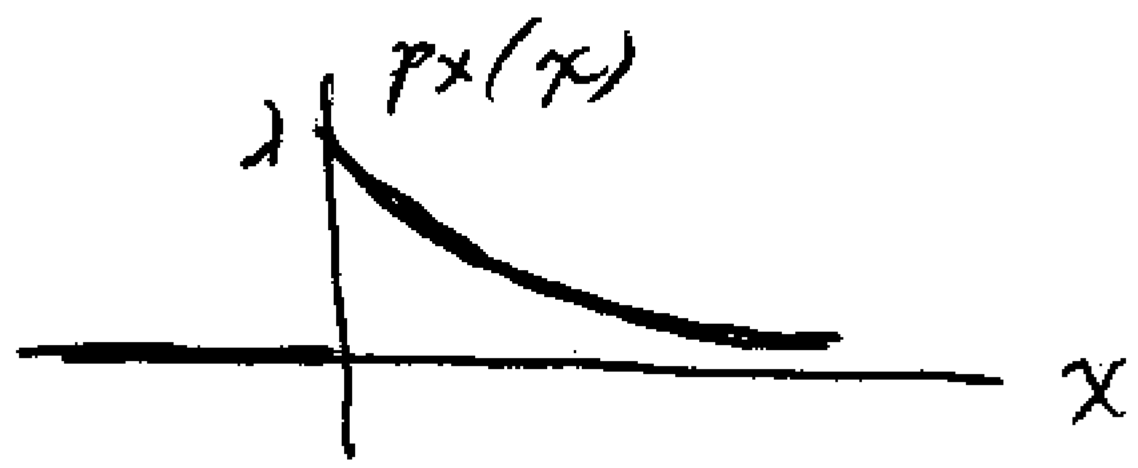
(SEE BOOK)

PDF PROPERTIES

1) $p_X(x) \geq 0$ WHY?

2) $\int_{-\infty}^{\infty} p_X(x)\, dx = 1$ WHY?

EXAMPLE : EXPONENTIAL PDF



$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\lambda > 0$

ALSO, CAN $\lambda = 2$? $p_X(0) = \lambda = 2$?

FOR $\lambda > 0 \Rightarrow p_X(x) \geq 0$

$$\int_{-\infty}^{\infty} p_X(x)\, dx = \int_0^{\infty} \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x} \Big|_0^{\infty} = 1$$

WHAT ABOUT DISCONTINUITY AT $x = 0$,

$p_X(0^-) = 0$, $p_X(0) = p_X(0^+) = \lambda$?

IS THIS A PROBLEM? WHAT IS $P[-\epsilon \le X \le \epsilon]$ AS $\epsilon \to 0$?

$$\int_{-\epsilon}^{\epsilon} p_X(x)\,dx = \int_0^{\epsilon} p_X(x)\,dx$$

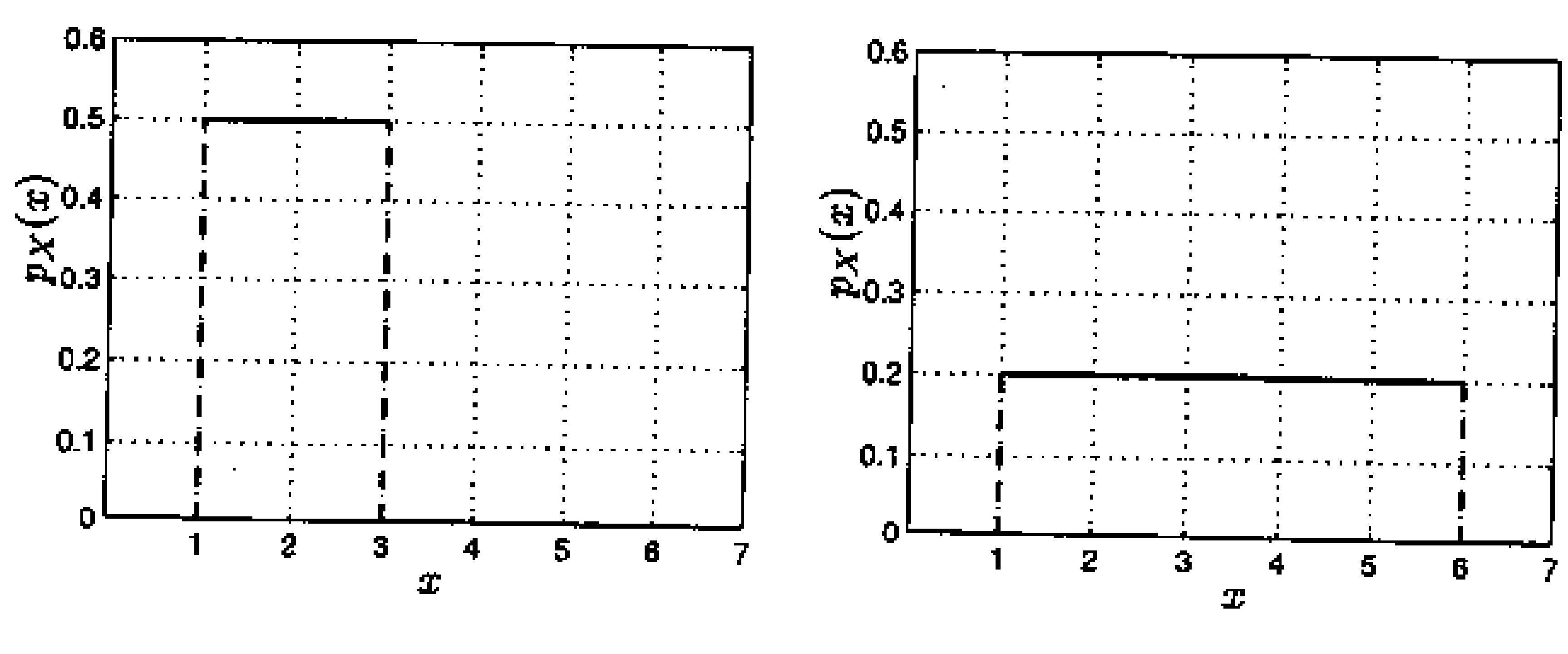
$$= \epsilon\, p_X(0) \to 0$$

REGARDLESS OF VALUE OF $p_X(0)$.
FOR CONTINUOUS RANDOM VARIABLE
P[SMALL INTERVAL] $\to 0$ AND
$P[X = x] = 0$ (WIDTH OF INTERVAL $= 0$).

HENCE, $P[a \le X \le b] = P[a < X \le b]$
$= P[a \le X < b] = P[a < X < b]$

IMPORTANT PDFS

1) UNIFORM
$$p_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{OTHERWISE} \end{cases}$$



(a) $a = 1, b = 3$

(b) $a = 1, b = 6$

Figure 10.7: Examples of uniform PDF.

$X \sim U(a, b)$
"IS DISTRIBUTED ACCORDING TO"

INTEGRATE TO 1?

IN MATLAB $U(0,1)$ = rand

2) EXPONENTIAL $p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

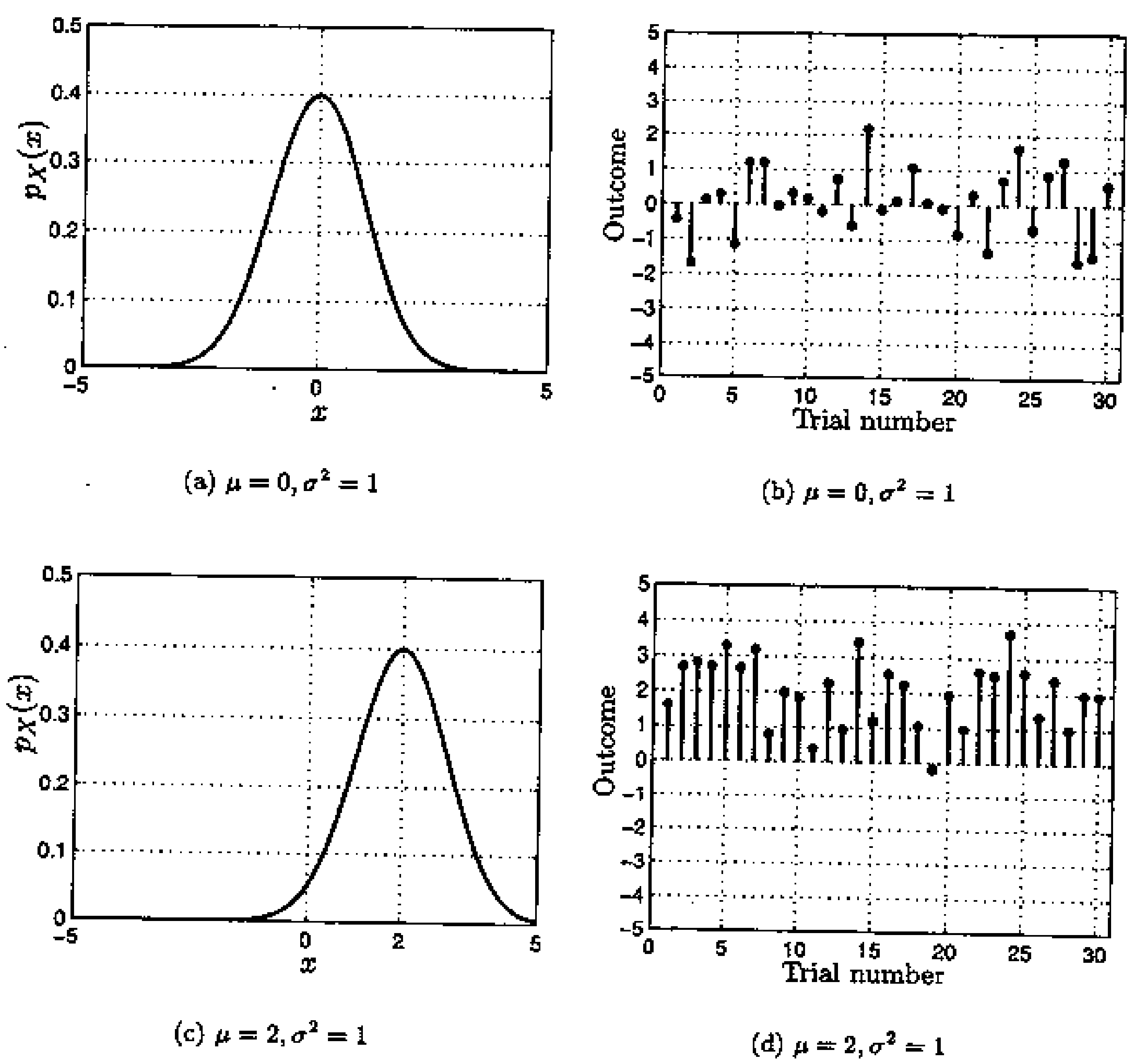
$X \sim EXP(\lambda)$

3) GAUSSIAN OR NORMAL OR "BELL CURVE"

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad -\infty < x < \infty$$

$-\infty < \mu < \infty$ SPECIFIES CENTER

$\sigma^2 > 0$ SPECIFIES SPREAD



(a) $\mu = 0, \sigma^2 = 1$

(b) $\mu = 0, \sigma^2 = 1$

(c) $\mu = 2, \sigma^2 = 1$

(d) $\mu = 2, \sigma^2 = 1$

Figure 10.8: Examples of Gaussian PDF with different μ's.

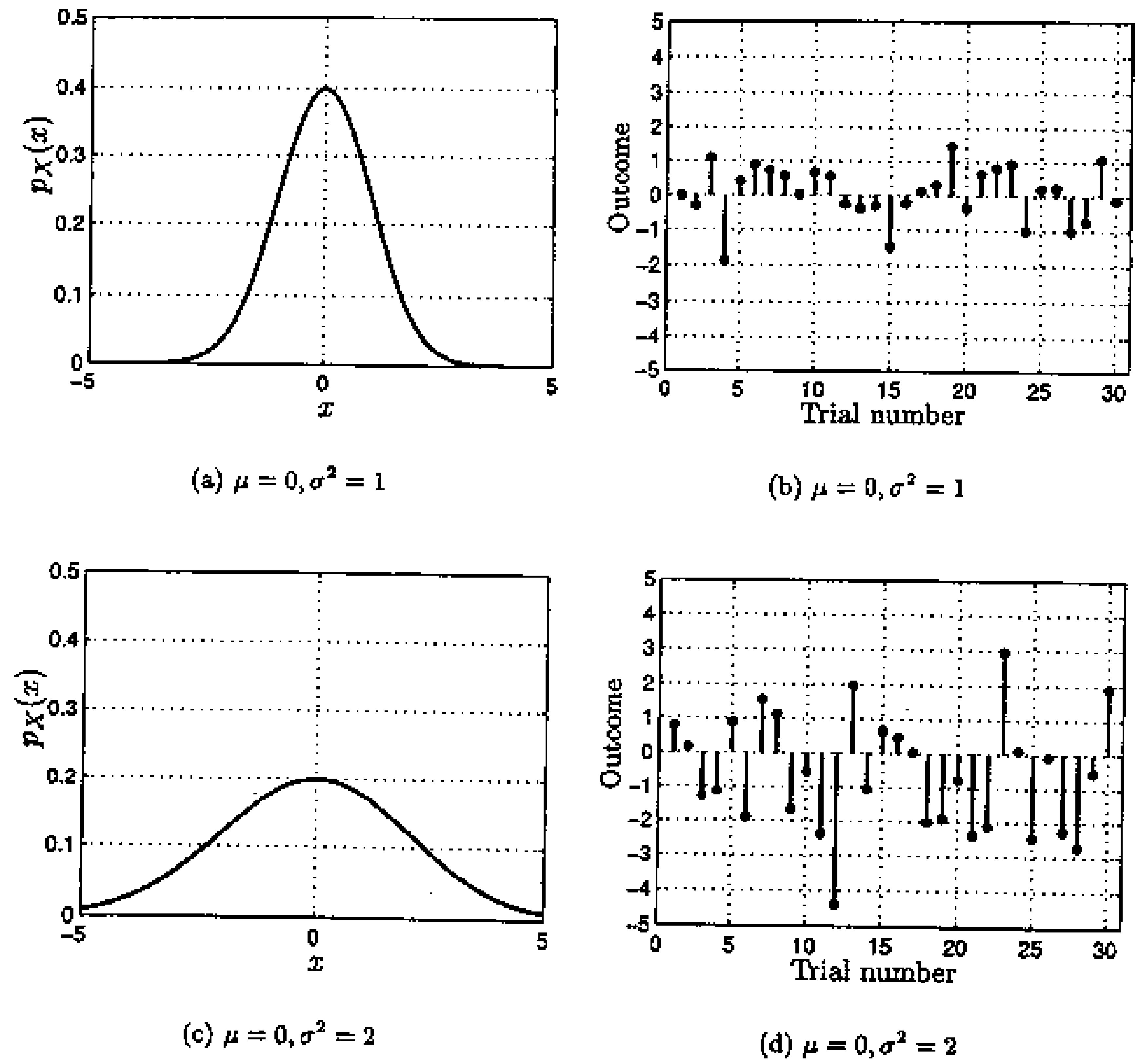


Figure 10.9: Examples of Gaussian PDF with different σ^2's.

CAN'T INTEGRATE $\int e^{-\frac{1}{2}x^2} dx$ ANALYTICALLY - CAN HOWEVER EVALUATE NUMERICALLY (SEE H.W.)

IF $\mu = 0, \sigma^2 = 1$ $\quad p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

CALLED STANDARD NORMAL PDF

IN MATLAB randn GENERATES STANDARD NORMAL R.V.

NOTATION : $X \sim N(\mu, \sigma^2)$

$N(0, 1)$ = STANDARD NORMAL PDF

HOW DO WE KNOW $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$

<u>PROOF</u> : $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = I$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= \iint \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy$$

(ANALOGOUS TO $\sum_{i=1}^{M} x_i \sum_{j=1}^{N} y_j = \sum_{i=1}^{M}\sum_{j=1}^{N} (x_i y_j)$

PROVE THIS IF UNFAMILIAR, LET
M = N = 3 FOR EXAMPLE)

TRANSFORM TO POLAR COORDINATES

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial(\underline{x}\,\underline{y})}{\partial(\underline{r}\,\underline{\theta})} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

WHAT IS THIS?

$$\left| \det \frac{\partial(\underline{x}\,\underline{y})}{\partial(\underline{r}\,\underline{\theta})} \right| = |r\cos^2\theta + r\sin^2\theta| = r > 0$$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r\, d\theta\, dr$$

$$= \int_0^{\infty} r e^{-\frac{1}{2}r^2} \underbrace{\int_0^{2\pi} \frac{1}{2\pi} d\theta}_{=1}\, dr = \underbrace{-e^{-\frac{1}{2}r^2}\Big|_0^{\infty}}_{=1}$$

$\Rightarrow \quad I = 1 \qquad$ (WHY NOT $I = -1$?)

4) LAPLACIAN

$$p_X(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}\,|x|} \qquad -\infty < x < \infty$$

SEE BOOK FOR EXAMPLES

<u>EXAMPLE</u>: FIND $P[|X| > 1]$ FOR $\sigma^2 = 1$

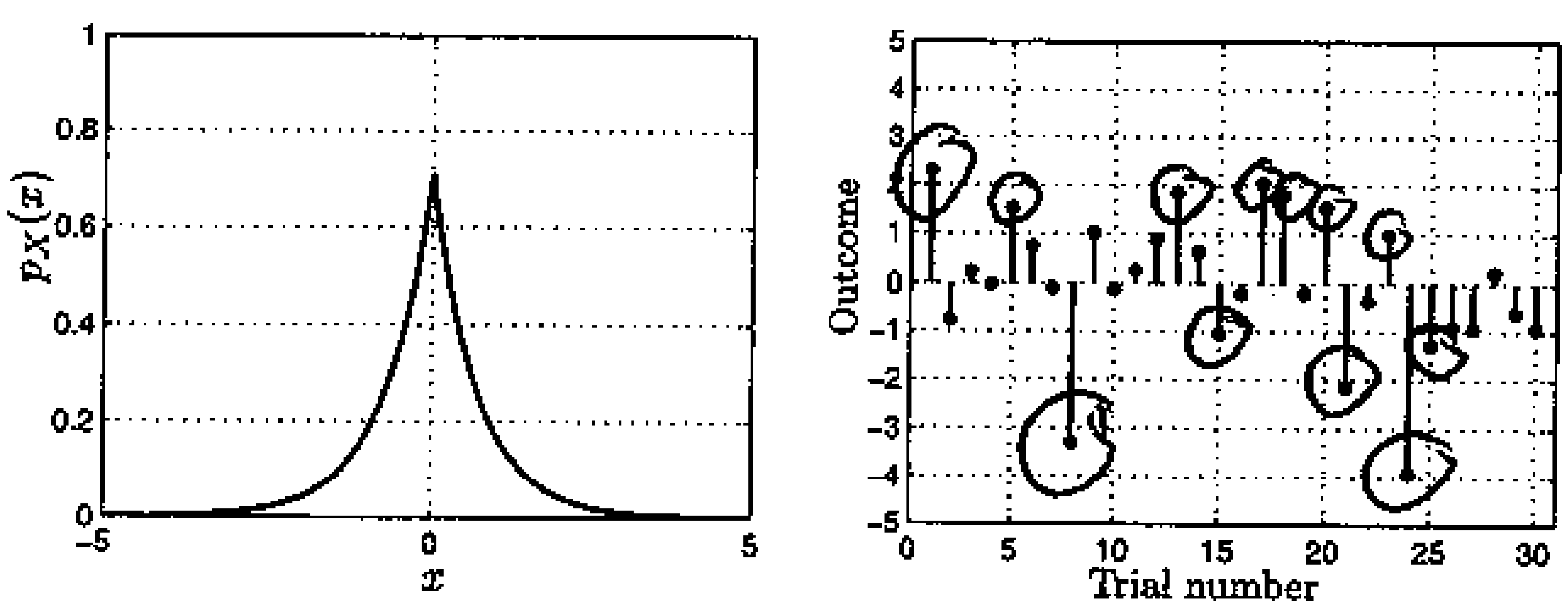
$$P[|X| > 1] = P[X > 1 \cup X < -1]$$

$$= P[X > 1] + P[X < -1] \qquad \text{WAY ?}$$

$$= 2P[X > 1] \qquad \text{WHY ?}$$

$$= 2\int_1^\infty \frac{1}{\sqrt{2}} e^{-\sqrt{2}x}\, dx$$

$$= -e^{-\sqrt{2}x}\Big|_1^\infty = e^{-\sqrt{2}} \approx 0.24$$



(a) $\sigma^2 = 1$ (b) $\sigma^2 = 1$

$12/30 \approx 0.4$

? ?

5) CAUCHY

$$p_X(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty$$

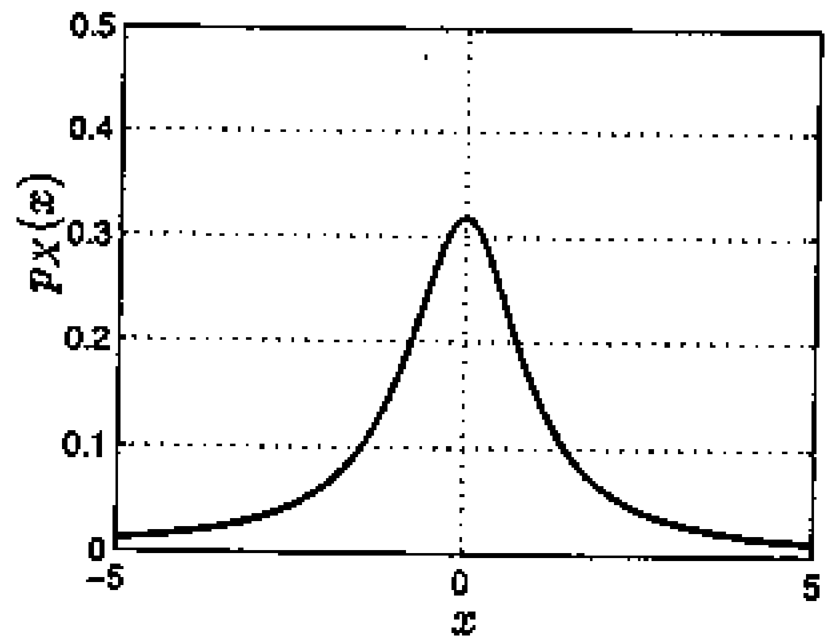


Figure 10.11: Cauchy PDF.

NOTE:
$p_X(-x) = p_X(x)$
(EVEN)

WHAT OTHER PDFS ARE EVEN?

6) GAMMA

CAN MODEL WIDE RANGE OF PDFS FOR NONNEGATIVE RVS.

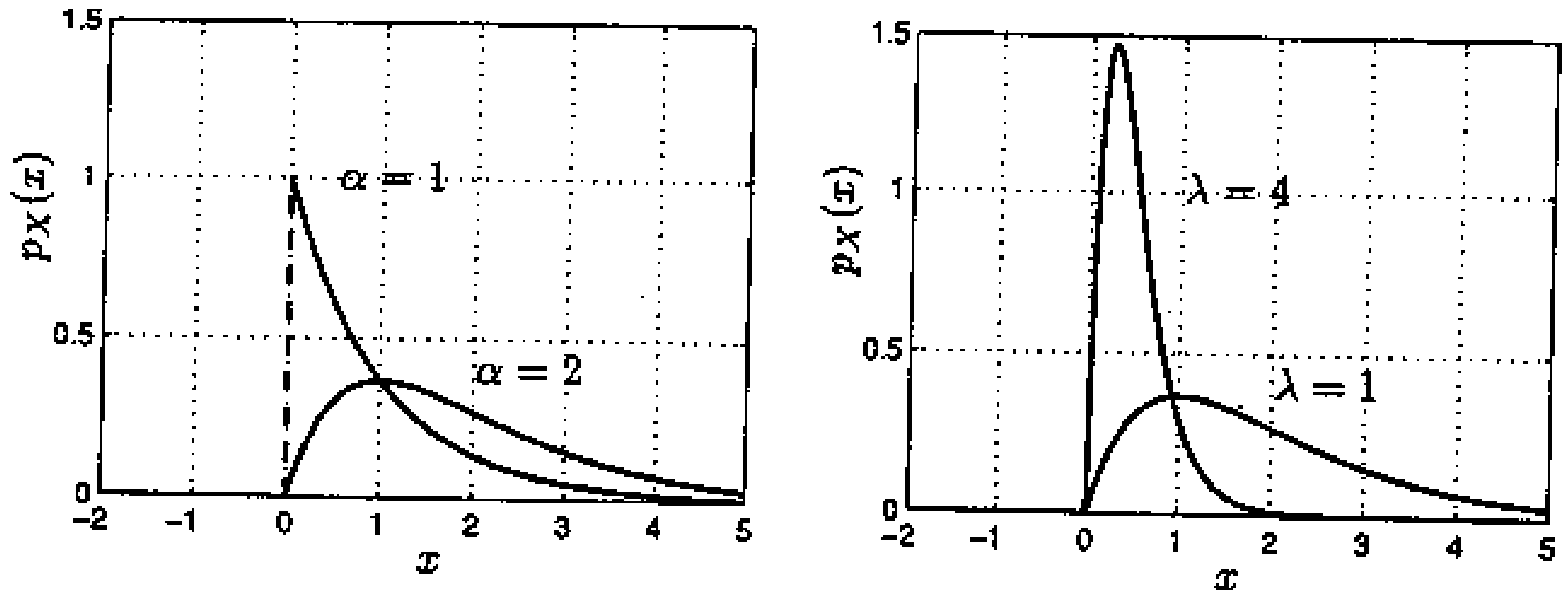
$$p_X(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\lambda > 0$, $\alpha > 0$ (TO INTEGRATE TO 1)

$\Gamma(z)$ CALLED GAMMA FUNCTION

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

NOTATION: $X \sim \Gamma(\alpha, \lambda)$



(a) $\lambda = 1$ (b) $\alpha = 2$

Figure 10.12: Examples of Gamma PDF.

PROPERTIES OF $\Gamma(z)$:

1) $\Gamma(z+1) = z\Gamma(z)$ SEE PROB. 10.16

2) $\Gamma(N) = (N-1)!$ FROM PROPERTY 1

3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

PROOF : $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$

LET $t = u^2/2 \Rightarrow dt = u\,du$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{u^2/2}} e^{-\frac{1}{2}u^2} u\,du$$

WHY ?

$$= \int_0^\infty \sqrt{2}\, e^{-\frac{1}{2}u^2} du = \frac{\sqrt{2}}{2} \underbrace{\int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du}_{= \sqrt{2\pi}}$$

WHY ?

$$= \sqrt{\pi}$$

SPECIAL CASES OF $\Gamma(\alpha, \lambda)$

1) $\alpha = 1 \Rightarrow$

$$p_X(x) = \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x} \qquad x > 0$$