ELE 509

INTRODUCTION TO RANDOM PROCESSES

INSTRUCTOR: PROF. STEVEN KAY

TEXTBOOK: 1) INTUATIVE PROBABILITY AND RANDOM PROCESSES USING MATLAB, S. KAY (REQUIRED)
2) SCHAUM'S OUTLINE ON PROBABILITY, RANDOM VARIABLES, AND RANDOM PROCESSES, H. HSU (RECOMMENDED)

OTHER REFERENCES:
1) PROBABILITY AND RANDOM PROCESSES, W. DAVENPORT (OUT OF PRINT), McGRAW HILL, 1970
2) PROBABILITY AND RANDOM PROCESSES, A. LEON-GARCIA, 1994
3) ... SEE PAGE 9 OF TEXT

TOPICS: CHAPTERS 10-12, CHAPTER 14 (SOME SECTIONS), CHAPTERS 15-18, SELECTED PORTIONS OF CHAPTERS 20 AND 21

- CONTINUOUS RANDOM VARIABLES (CRV)
- EXPECTED VALUES
- MULTIPLE CRV
* N-DIMENSIONAL CRV
* LAW OF LARGE NUMBERS
* CENTRAL LIMIT THEOREM
* BASIC RANDOM PROCESSES (RP)
* STATIONARY RP
* LINEAR SYSTEMS AND RP
* GAUSSIAN RP
* POISSON RP
* REAL-WORLD APPLICATIONS
* COMPUTER SIMULATIONS (VIA MATLAB)

* THROUGHOUT COURSE

**PREREQUISITES:**
1) MTH 451 OR EQUIVALENT COURSE IN BASIC PROBABILITY (CHAPTERS 1-7 IN TEXT)
2) CALCULUS
3) LINEAR SYSTEMS AND TRANSFORMS
4) LINEAR AND MATRIX ALGEBRA

**GRADING:**

"HOURLY" EXAM - 30%
HOURLY EXAM - 30%
FINAL EXAM - 40%
WEAKLH HOMEWORK ASSIGNED.
HOMEWORK DUE ONE WEEK AFTER ASSIGNED!
GRADED AS EITHER PASS (V) OR FAIL (V-). MUST ATTEMPT ALL PROBLEMS TO PASS. (TOO MANY V-'S \implies LOWER GRADE!)

HOMEWORK = ANALYTICAL EXERCISES + MATLAB EXPERIMENTS

NOTES: COPIES OF TRANSPARENCIES AT WWW.ELE.URICOURSES
READ BEFORE CLASS!

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INTRODUCTION (14,16*)

COMMON PROBLEM: SIGNAL DETECTION
1) RADAR
2) SONAR
3) COMMUNICATIONS
4) CHEMICALS, BIOLOGICS, ETC

SECTION NUMBER IN TEXT.
TRANSMIT PULSE AND LOOK FOR
ECHO FROM "TARGET"

TIME, t

NOISE
(MODELED AS
RANDOM PROCESS)

SUSPECT AN
ECHO HERE-
IS THERE ONE?

ASSUME WE SAMPLE \( x(i) \) \( \Rightarrow x_i \), \( i = 1, 2, \ldots, N \)

HOw CAN WE MAKE A DECISION IF

\[ x_i = 0.5 + w_i \quad i = 1, 2, \ldots, N \]

OR \( x_i = w_i \) ↑ NOISE SAMPLE

SOLUTION: COMPUTE SAMPLE MEAN

\[ \frac{1}{N} \sum_{i=1}^{N} x_i \rightarrow 0 \quad \text{FOR NOISE ONLY} \]

\[ 0.5 \quad \text{FOR SIGNAL} \]

\[ + \text{NOISE} \]

WHY?

DECIDE SIGNAL PRESENT IF

\[ \frac{1}{N} \sum_{i=1}^{N} x_i > 0.25 = \text{THRESHOLD} \]
IS THIS A GOOD DETECTOR? \
(TAKE ELE 665 FOR ANSWER!)

CAN USE A COMPUTER SIMULATION TO \n"ASSESS" PERFORMANCE BY:

1) GENERATE USING MATLAB $N = 100$ \nSAMPLES OF NOISE
2) COMPUTE SAMPLE MEAN ($> 0.25$?)
3) REPEAT (1) BUT NOW ADD 0.5 \nTO EACH SAMPLE
4) COMPUTE SAMPLE MEAN ($> 0.25$?)

MIGHT WANT TO REPEAT EXPERIMENT \nSEVERAL TIMES - WHY?

Figure 14.5: Received data samples. Signal is $s_i = A = 0.5$ and noise consists of \nIID standard Gaussian random variables.
Figure 14.6: Value of sample mean versus the number of data samples averaged.

Questions:

1) What will sample mean converge to (if it converges)?

2) How large does N have to be?

3) How do noise statistics affect results?

4) How does amplitude of pulse affect results?

2) All these questions can be answered by studying random process theory (and some detection theory!)
CHAPTER 10 - CONT. R.V.'S

RECALL THAT A RANDOM VARIABLE IS A MAPPING FROM SAMPLE SPACE TO A NUMBER

**EXAMPLE:** Toss A Die

\[ S = \{ s_1, s_2, s_3, s_4, s_5, s_6 \} \]

↑SAMPLE SPACE (ALL POSSIBLE EXPERIMENTAL OUTCOMES)

\[ X(s_i) = i \quad i = 1, 2, 3, 4, 5, 6 \]

↑RANDOM VARIABLE (NOT REALLY RANDOM!) (R.V.)

**NOTE:** SIZE OF \( S \) = 6

NUMBER OF VALUES OF \( X \) = 6

THIS IS A DISCRETE R.V. ⇒ COUNTABLE NUMBER OF VALUES. COULD HAVE

\[ X(s_i) = i \quad i = 1, 2, 3, \ldots \]

STILL A DISCRETE R.V.

FOR THE DIE EXAMPLE \( P(X(s_i) = i) = \frac{1}{6} \)
For this example could have
\[ P \left( X(S_i) = i \right) = \frac{1}{2^i}, \quad i = 1, 2, 3, \ldots \]

Does \( \sum_{i=1}^{\infty} P \left( X(S_i) = i \right) = 1 \) ?

In general denote R.V. by capital letter \( X \) and value by lower case \( x \), \( X(S_i) = x_i \) (similar to function definition \( y = F(x) \)).

\[ \uparrow \uparrow \uparrow \]
\[ x_i \quad X \quad S_i \]

Now consider a dartboard expt.

\[ x \quad S_X = [0,1] \]

**Figure 10.1:** Mapping of the outcome of a thrown dart to the real line (example of continuous random variable).

\( X \) is distance from bullseye (center).

How many outcomes are there?

Can we let \( P \left[ X(S_i) = x_i \right] = p_i \)

for \( i = 1, 2, 3, \ldots \) ?
Assume all outcomes are equally likely \( \Rightarrow P\left(0 \leq x \leq \frac{1}{2}\right) = P\left[\frac{1}{2} < x \leq \frac{3}{4}\right] = P\left[\frac{1}{2} < x \leq \frac{3}{4} + \frac{1}{2}\right] \text{ etc.} \)

Assign uniform probabilities or
\[ P\{a \leq x \leq b\} = b - a \quad 0 \leq a \leq b \leq 1 \]

Is this reasonable? If outcomes were discrete or \( x_i = a x_0, 2 a x_0, \ldots, M a x_0 \)
and equally likely

Figure 10.2: Approximating the probability of an interval for a continuous random variable by using a PMF.

\[ P_X\{x_i\} = \text{PROBABILITY MASS FUNCTION (PMF)} \]

\[ \sum_{i=1}^{\infty} P_X\{x_i\} = 1 \]

Recall \( P\{a \leq x \leq b\} = \sum_{i=1}^{\infty} P_X\{x_i\} \quad \{x_i: a \leq x_i \leq b\} \)
BUT \( \frac{1}{m} = 1 \cdot \Delta x \) AND AS \( \Delta x \to 0 \) WE RECOVER ALL OUTCOMES.

\[
\Rightarrow P[a \leq x \leq b] = \sum_{i: a \leq x_i \leq b} 1 \cdot \Delta x
\]

\[
= \sum_{i: a \leq x_i \leq b} p(x) \Delta x
\]

\[
\to \int_{a}^{b} p(x) \, dx
\]

AS \( \Delta x \to 0 \). \( p(x) \) CALLED THE PROBABILITY DENSITY FUNCTION (PDF).

To find probability of dart landing in interval \([a, b]\) just integrate PDF or

\[
P(a \leq x \leq b) = \int_{a}^{b} p(x) \, dx
\]

\[
= \int_{a}^{b} 1 \, dx = b - a
\]

Our original assignment for equally likely outcomes.
NOTE: \( p_X [x_i]^k \) denotes PMF

\( p_X(x)^k \) denotes PDF

Think of PDF as probability per unit length since assuming the previous discrete approximation

\[
p \left[ \frac{x_0 - \Delta x}{2} \leq x \leq x_0 + \Delta x/2 \right] =
\]

\[
\sum p_X(x_i) \Delta x = \sum p_X(x_i) \Delta x
\]

\[
\{ i : x_0 - \Delta x/2 \leq x_i \leq x_0 + \Delta x/2 \}
\]

\[
\{ i : x_i = x_0 \}
\]

\[\uparrow \Delta x \text{ small and} \]
\[x_0 = k\Delta x\]

\[
= p_X(x_0) \Delta x
\]

\[\Rightarrow p_X(x_0) = \frac{p \left[ \frac{x_0 - \Delta x}{2} \leq x \leq x_0 + \Delta x/2 \right]}{\Delta x}
\]

PDF for which \( p_X(x) = 1 \) \( 0 < x < 1 \) and \( 0 \) otherwise called a uniform PDF.

Use \texttt{rand} in MATLAB to generate an outcome.
Another example: In general we find probabilities by 
\[ P(a \leq x \leq b) = \int_a^b p_X(x) \, dx \]

\[ p_X(x) = 2(1-x) \]

Figure 10.4: Nonuniform PDF.

\[ P[0 \leq X \leq 0.1] = \int_0^{0.1} 2(1-x) \, dx = 2(x - x^2/2) \bigg|_0^{0.1} = 0.19 \]

\[ P[0.9 \leq X \leq 1] = \int_{0.9}^1 2(1-x) \, dx = 2(x - x^2/2) \bigg|_{0.9}^1 = 0.01. \]

Note: \( p_X(x) \geq 0 \quad \int_{-\infty}^{\infty} p_X(x) \, dx = 1 \)

PDF is analogous to a mass density or mass per unit length.

Consider wedge of cheese below. If \( M = \text{mass}, \) and total mass \( = 1, \)
**PDF Properties**

1) \( p_X(x) \geq 0 \) why?

2) \( \int_{-\infty}^{\infty} p_X(x) \, dx = 1 \) why?

**Example: Exponential PDF**

\[
 p_X(x) = \begin{cases} 
 \lambda e^{-\lambda x} & x \geq 0 \\
 0 & x < 0 
\end{cases}
\]

Also can \( \lambda = 2 \)? \( p_X(0) = \lambda = 2 \)?

For \( \lambda > 0 \) \( \Rightarrow p_X(x) \geq 0 \)

\[
 \int_{-\infty}^{\infty} p_X(x) \, dx = \int_{0}^{\infty} \lambda e^{-\lambda x} \, dx
\]

\[
 = -e^{-\lambda x} \bigg|_{0}^{\infty} = 1
\]

What about discontinuity at \( x = 0 \),

\( p_X(0^-) = 0 \), \( p_X(0) = p_X(0+) = \lambda \)?
Is this a problem? What is \( P[-\epsilon \leq x \leq \epsilon] \) as \( \epsilon \to 0 \)?

\[
\int_{-\epsilon}^{\epsilon} p_X(x) \, dx = \int_{0}^{\epsilon} p_X(x) \, dx = \epsilon p_X(0) \to 0
\]

Regardless of value of \( p_X(0) \), for continuous random variable \( p[small \, interval] \to 0 \) and \( p[x = x] = 0 \) (width of interval \( = 0 \)).

Hence, \( p[a \leq x \leq b] = p[a < x \leq b] = p[a \leq x < b] = p[a < x < b] \)

**IMPORTANT PDFS**

1) **Uniform** \( p_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \)

![Graph of Uniform PDF](image)

Figure 10.7: Examples of uniform PDF.

Integrate to 1?
In MATLAB, \( \text{unif}(0,1) = \text{rand} \).

\[ p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \]

\( X \sim \text{Exp}(\lambda) \)

\[ p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \]

\(-\infty < \mu < \infty\) specifies center

\(\sigma^2 > 0\) specifies spread

Figure 10.8: Examples of Gaussian PDF with different \( \mu \)'s.
Figure 10.9: Examples of Gaussian PDF with different $\sigma^2$'s.

\[ \int e^{-\frac{1}{2}x^2} \, dx \]

Cannot integrate analytically - can however evaluate numerically (see H.W.)

If $\mu = 0, \sigma^2 = 1$, $p(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

called **standard normal PDF**

In MATLAB `randn` generates standard normal RV.

**Notation:** $x \sim N(\mu, \sigma^2)$

$N(0, 1) = $ standard normal PDF
How do we know \( \int_{-\infty}^{\infty} p(x) / \sqrt{2\pi} dx = 1 \)

**Proof:** \( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx = 1 \)

\[ I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy \]

\[ = \iint \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \, dx \, dy \]

(Analogous to \( \sum_{i=1}^{M} x_i \cdot \sum_{j=1}^{N} y_j = \sum_{i=1}^{M} \sum_{j=1}^{N} (x_i \cdot y_j) \) when \( M = N = 3 \) for example)

Prove this if unfamiliar, let \( M = N = 3 \) for example.

**Transform to polar coordinates**

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

\[ \begin{vmatrix} x \ y \end{vmatrix} = \begin{vmatrix} \cos \theta \ & -r \sin \theta \\ \sin \theta \ & \ r \cos \theta \end{vmatrix} \]

\[ = r \cos^2 \theta + r \sin^2 \theta = r > 0 \]

\[ I^2 = \iint_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} r^2} r \, dr \, d\theta \]

\[ = \int_{0}^{\infty} r e^{-\frac{1}{2} r^2} \left. \int_{0}^{2\pi} \, d\theta \, dr \right|_{r=0}^{r=\infty} \]

\[ = e^{-\frac{1}{2} r^2} \bigg|_{r=0}^{r=\infty} = 1 \]
$\Rightarrow x = 1$ (Why not $x = -1$?)

4) Laplacian

$p(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad -\infty < x < \infty$

See Book for Examples

Example: Find $P[|X| > 1]$ for $\sigma^2 = 1$

$P[|X| > 1] = P[x > 1] + P[x < -1]$

$= 2P[x > 1]$  \hspace{1cm} \text{Why?}

$= 2 \int_1^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \, dx$

$= -e^{-\frac{x^2}{2\sigma^2}} \bigg|_1^\infty = e^{-\frac{1}{2\sigma^2}} \approx 0.24$

(a) $\sigma^2 = 1$

(b) $\sigma^2 = 1$

$12/30 \approx 0.4$
5) **CAUCHY**

\[ p_X(x) = \frac{1}{\pi(1+x^2)} \quad -\infty < x < \infty \]

**NOTE:**

\[ p_X(-x) = p_X(x) \quad \text{(EVEN)} \]

**WHAT OTHER PDFS ARE EVEN?**

\[ \text{Figure 10.11: Cauchy PDF.} \]

6) **GAMMA**

Can model wide range of PDFs for nonnegative rvs.

\[ p_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \]

\( \lambda > 0, \alpha > 0 \) (to integrate to 1)

\( \Gamma(\alpha) \) called gamma function

\[ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt \]

**NOTATION:** \( X \sim \Gamma(\alpha, \lambda) \)
Properties of $\Gamma(z)$:

1) $\Gamma(z+1) = z \Gamma(z)$ \textit{see Prob. 10.16}
2) $\Gamma(N) = (N-1)!$ \textit{from property 1}
3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

\textbf{Proof:} $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$

\textit{Let} $t = u^2/2 \Rightarrow dt = u du$

$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{\sqrt{u^2}} e^{-\frac{1}{2}u^2} u du$

$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \sqrt{2} e^{-\frac{1}{2}u^2} du = \sqrt{\pi}$

\textit{Special cases of $\Gamma(\alpha, \lambda)$}

1) $\alpha = 1 \Rightarrow$

$p_x(x) = \frac{1}{\Gamma(1)} x^{1-1} e^{-\lambda x} \quad x > 0$