a) \( ps_3(x) = \int_{-\frac{1}{2}}^{x} ps_1(u) \, du \quad -\frac{3}{2} \leq x \leq -\frac{1}{2} \)

= \frac{1}{2} x^2 + 3/2 x + 9/8

b) \( ps_3(x) = \int_{-\frac{1}{2}}^{x} ps_1(u) \, du \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \)

= -x^2 + 3/4

c) \( ps_3(x) = \int_{-\frac{1}{2}}^{x} ps_1(u) \, du \quad \frac{1}{2} \leq x \leq \frac{3}{2} \)

= \frac{1}{2} x^3 - \frac{3}{2} x + 9/8

d) = 0 \quad \text{OTHERWISE}

![Figure 15.6: PDF for sum of 3 IID U(-1/2, 1/2) random variables and Gaussian approximation.](image)

**NOTE**: \( E(S_3) = E\left(\sum_{i=1}^{3} x_i\right) = 0 \quad x_i \sim U\left(-\frac{1}{2}, \frac{1}{2}\right) \)

\( \text{VAR}(S_3) = \text{VAR}\left(\sum_{i=1}^{3} x_i\right) = 3 \text{VAR}(x_i) = \frac{3}{12} \)
Gaussian approximation very good near $x = 0$. How about for $x > 2$?

**Example:**

$$S_N = \sum_{i=1}^{N} X_i, \quad X_i \sim \mathcal{N}(0,1)$$

Using `clt-demo.m` (Appendix 15A) for repeated convolution of $\frac{1}{\sqrt{N}}$

Does $p_{S_N}$ converge to anything as $N \to \infty$?

**Note:**

$$E[S_N] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i] = N E[X] = N \sigma^2$$

Also, $\text{VAR}(S_N) = \frac{1}{N} \text{VAR}(X) = \frac{N}{12}$

**MUST SOMEHOW Normalize TO TALK ABOUT CONVERGENCE.**

Figure 15.7: PDF of sum of $N$ IID $\mathcal{N}(0,1)$ random variables. The plots were obtained using `clt-demo.m` listed in Appendix 15A.
Similar to following:

\[ \bar{y}_n = \frac{1}{N} \sum_{n=1}^{N} y_n = \frac{N}{2} (N+1) \to \infty \text{ as } N \to \infty \]

or average of first \( N \) positive integers \( \to \infty \) as \( N \) gets large. But we would like to say that this average \( \approx N/2 \).

Examine \( \bar{y}_n = \frac{N(N+1)}{2N} \to \frac{1}{2} \text{ as } N \to \infty \)

or \( \frac{\bar{y}_n}{N} \approx \frac{1}{2} \Rightarrow \bar{y}_n \approx N/2 \)

To normalize \( S_N \) we form \textbf{standardized sum} (recall standard normal had \( X \sim N(0,1) \) or \( E[X] = 0, \text{ VAR}(X) = 1 \)).

\( S_N \) having same here:

\[ Z_N = \frac{S_N - E(S_N)}{\sqrt{\text{VAR}(S_N)}} \]

has mean = 0, variance = 1. Why?

\[ Z_N = \frac{S_N - N \cdot E[X]}{\sqrt{N \cdot \text{VAR}(X)}} \text{ well defined even as } N \to \infty \]

Finally, \textbf{central limit theorem (CLT)} says that as \( N \to \infty \), \( Z_N \to N(0,1) \).
SAME EXAMPLE
AS BEFORE -
MUCH BETTER
"BEHAVED"
MATHEMATICALLY

Figure 15.8: PDF of standardized sum of $N$ IID $U(0,1)$ random variables.

**CLT:** If $X_1, X_2, \ldots, X_N$ are cont. IID rvs each with mean $\mu_X(x)$ and variance $\text{VAR}(x)$, then as $N \to \infty$

$$\frac{\sum_{i=1}^{N} X_i - N\mu_X(x)}{\sqrt{N\text{VAR}(x)}} \to N(0,1)$$

**Example:** $X_1 \sim N(0,1)$

Examine PDF of $Y = \sum_{i=1}^{N} X_i^2$

as $N \to \infty$

true PDF is $Y \sim \chi_n^2$. To apply CLT

**Note:**

1) $X_1, X_2, \ldots, X_N$ are ind $\Rightarrow X_1^2, X_2^2, \ldots, X_n^2$

are independent (can you justify this? see pg. 200 in book)

2) $X_1, X_2, \ldots, X_N$ have same PDF

$\Rightarrow X_1^2, X_2^2, \ldots, X_n^2$ have same PDF
As per CLT

\[
\frac{\sum_{i=1}^{N} x_i^2 - N E(x^2)}{\sqrt{N \text{VAR}(x^2)}} \to N(0,1)
\]

But if \( x \sim N(0,1) \),

\[
E(x^2) = \text{VAR}(x) + \frac{E(x)}{\sigma^2} = 1
\]

\[
\text{VAR}(x^2) = E(x^4) - E(x^2)^2 = 2
\]

can be shown to be 3

(TRY INTEGRATION)

\[
Z_N = \frac{\sum_{i=1}^{N} x_i^2 - N}{\sqrt{2N}} \to N(0,1)
\]

\[
Y_N = \frac{\sum_{i=1}^{N} x_i^2}{\sqrt{N}} = \sqrt{2N} Z_N + N \sim N(N, 2N)
\]

\[
\to N(0,1)
\]

(a) \( N = 10 \)  
(b) \( N = 40 \)

Figure 15.9: \( \chi^2_N \) PDF (dashed curve) and Gaussian PDF approximation of \( N(N, 2N) \) (solid curve).
"Proof" of CLT: Let \( Z_N = \frac{S_N - N \mu}{\sqrt{N \text{VAR}(X)}} \)

To show \( Z_N \Rightarrow N(0, 1) = Z \), we can equivalently show that characteristic function \( \phi_{Z_N}(w) \Rightarrow \phi_Z(w) = e^{-\frac{1}{2}w^2} \) from tables! (Recall characteristic function is Fourier transform of PDF.

\[ \phi_{Z_N}(w) = E_{Z_N} \left[ e^{jwZ_N} \right] \]

\[ = E_X \left[ e^{jw \frac{X - N \mu}{\sqrt{N \text{VAR}(X)}}} \right] \]

\[ = E_X \left[ \prod_{i=1}^{N} e^{jw \frac{X_i - E_X(X)}{\sqrt{N \text{VAR}(X)}}} \right] \]

\[ = \prod_{i=1}^{N} E_X \left[ e^{jw \frac{X_i - E_X(X)}{\sqrt{N \text{VAR}(X)}}} \right] \]

\[ = \prod_{i=1}^{N} E_X \left[ e^{jw \frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}}} \right] \]

\[ = \left[ E_X \left[ e^{jw \frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}}} \right] \right]^N \]

Now \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \)

\[ = E_X \left[ e^{jw \frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}}} \right] = E_X \left[ \sum_{k=0}^{\infty} \frac{(jw)^k}{k!} \left( \frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}} \right)^k \right] \]

\[ = \sum_{k=0}^{\infty} \frac{(jw)^k}{k!} E_X \left[ (\frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}})^k \right] \]

\[ = 1 + jw E_X \left[ \frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}} \right] + \frac{1}{2} (jw)^2 E_X \left[ (\frac{X - E_X(X)}{\sqrt{N \text{VAR}(X)}})^2 \right] \]

+ higher order terms in \( \frac{1}{\sqrt{N}} \)
\[
\text{BUT \( E_X \left[ \frac{X-E_X(X)}{\sqrt{N \text{VAR}(X)}} \right] = 0 \)}
\]
\[
E_X \left[ \left( \frac{X-E_X(X)}{\sqrt{N \text{VAR}(X)}} \right)^2 \right] = \frac{E_X \left[ (X-E_X(X))^2 \right]}{N \text{VAR}(X)}
\]
\[= \sqrt{N} \]
\[
\phi_{Z_n}(\omega) = (1 + \frac{1}{2}\left(\frac{1}{N} \omega^{-1} \right))^N \]
\[= (1 - \frac{1}{2} \omega^2)^N \]
\[\rightarrow e^{-\frac{1}{2} \omega^2} \quad \text{as} \quad N \rightarrow \infty \quad (\text{see Prob 5.15})
\]
\[\phi_z(\omega) \quad Z \sim N(0, 1) \]

---

**CHAPTER 16 - BASIC RANDOM PROCESSES**

**STUDIED RANDOM VARIABLE** \( X \)

**RANDOM VECTOR** \((X_1, X_2, \ldots, X_N)\)

**NOW RANDOM PROCESS** \((\ldots, X_{-1}, X_0, X_1, \ldots)\)

**EXAMPLES:**

**WHAT IS OF INTEREST HERE?**

![Figure 16.1: Annual summer rainfall in Rhode Island from 1895 to 2002.](image)
What is a random process (RP)?

**Example:** Start tossing coin at some time $n=0$ and continue indefinitely ($n=0, 1, 2, \ldots$)

$\implies$ infinite sequence of coin tosses

Outcomes are

$s = \{ (H, H, T, \ldots), (H, T, H, \ldots), (T, T, H, \ldots), \ldots \}$

If we define a R.V. as

$x = 0$ if tail

1 if head

Then we call this a **Bernoulli** R.V.
The outcomes of the random process are
\[ S_x = \{(1, 0, \ldots), (1, 0, \ldots), (0, 0, 1, \ldots), \ldots\} \]

Now denote the random variables as \( X[0], X[1], \ldots \) and their outcomes as \( x[0], x[1], \ldots \) or in general \( X[n], X[n] \)

\[ (X[0], X[1], \ldots) \xrightarrow{\text{PMF description}} \text{Random process generator} \xrightarrow{\text{PMF description}} (x[0], x[1], \ldots) \]

Figure 16.3: A conceptual random process generator. The input is an infinite sequence of random variables with their probabilistic description and the output is an infinite sequence of numbers.

Note: Each realization is an infinite sequence of numbers.

Recall for a single R.V., a mapping from \( S \) to \( S_x \), we denote it more explicitly as the set function \( X(S) \).

Now \( S = \) set of infinite experimental outcomes (coin tosses)
\[ S_x = \) set of infinite sequences of 1's and 0's (realizations) \]
Set of all realization called the ensemble of realizations.

Now instead of $X(s)$, use $X(n, s) = Z_n$, mapping from $s$ to $S_n$.

![Graphs showing realizations](image)

**First realization**

**Second realization**

Note that $X(18, s)$ is just a r.v. $\Rightarrow$ has a p.m.f., mean, variance, etc.

Will denote the r.p. by $X(n)$ (drop the $s$).

Also, $X(0)$ will denote the entire r.p.

\{ $x(0), x(1), ...$ \}

Sometimes authors use \{ $x(n)$ \}$_{n=0}^{\infty}$ and $X(0)$ will denote r.p. at fixed time $n = 0$.

---

Figure 16.4: Typical outcomes of Bernoulli random process with $p = 0.5$. The realization starts at $n = 0$ and continues indefinitely. The dashed box indicates the realizations of the random variable $X(18, s)$. 
**Example:** Bernoulli R.P. (IID tosses)

What is prob. of first 5 tosses coming up heads?

\[ P\left( x_{10} = 1, x_{11} = 1, x_{12} = 1, x_{13} = 1, x_{14} = 1 \right) \]

\[ x_{15} = 0 \text{ or } 1, x_{16} = 0 \text{ or } 1, \ldots \]

Since we don’t care what \( x_{n+1} \) for \( n \geq 5 \)

is, we can restrict attention to

\[ P\left( x_{10} = 1, x_{11} = 1, x_{12} = 1, x_{13} = 1, x_{14} = 1 \right) \]

\[ = \prod_{n=0}^{4} P(x_{n+1} = 1) = p^5 \]

In essence we replaced R.P. by random vector

⇒ easy prob. computation

**What is prob. of ever observing 5 consecutive heads?**

**How to find this?**

Types of random processes

\[ x[n] \quad n = 0, 1, \ldots \quad \text{semi-infinite} \]

\[ x[n] \quad n = \ldots, -1, 0, 1 \ldots \quad \text{infinite} \]

\[ x[n] \quad \text{discrete-time} \quad t \geq 0 \]

\[ x(t) \quad \text{continuous-time} \quad -\infty < t < \infty \]
Can also categorize according to discrete or cont. outcomes (same as discrete or cont. RVs)

\[ [t] = \text{largest integer } \leq t \]

![Graphs showing different types of random processes](image)

(a) Discrete-time/discrete-valued (DTDV) Bernoulli random process
(b) Discrete-time/continuous-valued (DTCV) Gaussian random process
(c) Continuous-time/discrete-valued (CTDV) binomial random process
(d) Continuous-time/continuous-valued (CTCV) Gaussian random process

Figure 16.5: Typical realizations of different types of random processes.

Will generally focus on (b) since discrete-time used extensively in practice and cont.-valued outcomes correspond to cont. RVs, which we have already studied.

Example: random walk (used as model for many physical processes - "A random walk down Wall Street")
DEFINED AS \( x(n) = \sum_{i=0}^{n} u(i) \) \( n = 0, 1, ... \)

\( u(n) \) IS BERNouLLI WITH OUTCOMES \( \pm 1 \)

\[ \begin{align*}
    & p_u(k) = \frac{1}{2} \quad k = -1 \\
    & \frac{1}{2} \quad k = 1
\end{align*} \]

AND \( u(n)'s \) ARE I.I.D (i.e., BERNouLLI RAP \( \pm 1 \) OUTCOMES INSTEAD OF 0, 1)

\[ \text{POSITION OF DRUNK AFTER} \]
\[ \n \text{N STEPS, PRICE OF STOCK THAT} \]
\[ \n \text{MOVES UP OR} \]
\[ \n \text{DOWN BY \$1.} \]
\[ \n \text{(PRICE CHANGES UNPREDICTABLE?)} \]

\[ \text{AN IMPORTANT QUESTION} \]
\[ \n \text{IS BEHAVIOR FOR LARGE N.} \]

\[ \text{BY CLT (WHY?)} \]
\[ x(n) \sim \text{GAUSSIAN} \]
\[ E[x(n)] = E[\sum_{i=0}^{n} u(i)] = (n+1) E[u(0)] = 0 \]
\[ \text{VAR}(x(n)) = \text{VAR}[\sum_{i=0}^{n} u(i)] = (n+1) \text{VAR}(u(0)) \]
\[ \text{VAR}(u(0)) = E[u^2(0)] - E[u(0)]^2 = 1 - 0 = 1 \]
\[ \Rightarrow \]
\[ x(n) \sim N(0, n+1) \]

MAKE SENSE?
STATIONARITY

Do characteristics of RP change with time? Bernoulli RP - no
Random walk - yes

to quantify this need to describe probabilities of RP, and examine
then over time.

Example: Bernoulli RP

This is example of IID RP.
To compute probs. must constrain
ourselves to a finite set of times.

\[ P_{x[n_1], x[n_2], \ldots, x[n_N]} = \prod_{i=1}^{N} P_{x[n_i]} \]

Joint PMF

called a finite dimensional distribution.

Note here that prob of first 5
samples \( n_1 = 0, n_2 = 1, \ldots, n_5 = 4 \) being
all 1's is \( p^5 \) and prob of
second 5 samples \( n_6 = 5, \ldots, n_{10} = 9 \)
being all 1's is also \( p^5 \), etc.
THIS R.P. IS STATIONARY. NOTE THAT

\[ P(x_1, \ldots, x_{\rho}) = P(x_1, \ldots, x_{\rho}) \]

OR \[ P(x_{n}, \ldots, x_{n+\rho}) = P(x_{n+n_0}, \ldots, x_{n+n_0+\rho}) \]

\[ n = 0 \]
\[ n_0 = 5 \]

IN GENERAL, A R.P. IS DEFINED TO BE STATIONARY IF

\[ P(x_{n+n_0}, x_{n+n_0}, \ldots, x_{n+n_0}) = P(x_{n_1}, x_{n_2}, \ldots, x_{n_\rho}) \]

FOR ALL \( n_1, n_2, \ldots, n_\rho \) \( (\forall n) \) AND ALL \( n_0 \).

EVERY FINITE DIMENSIONAL DISTRIBUTION (GIVEN \( n_1, n_2, \ldots, n_\rho \)) DOES NOT CHANGE IF SAMPLE TIMES ARE ALL SHIFTED BY \( n_0 \).

EXAMPLE: IID R.P. IS STATIONARY

\[ P(x_{n+n_0}, \ldots, x_{n+n_0}) = \prod_{i=1}^{N} P(x_{n+n_0}) \]

\[ = \prod_{i=1}^{N} P(x_{n_i}) \]

\[ = P(x_{n_1}, \ldots, x_{n_\rho}) \]
Note that if R.P. is stationary, so are all joint moments since

\[ E[x_{(n+1)}, \ldots, x_{(2n+1)}] = E[x_{(n-1)}, \ldots, x_{1}] \]

⇒ If moments are not stationary, then R.P. cannot be stationary.

\[ \text{if you just look at these realizations, can you tell if R.P. is stationary?} \]

(a) Mean increasing with n
(b) Variance decreasing with n

Figure 18.7: Random processes that are not stationary.

Example:

\[ \text{sum R.P. } \]

\[ x[n] = \sum_{i=0}^{2} u[i] \]

\[ u[i] \] is i.i.d. arbitrary pmf or pdf

\[ \text{stationary? } \]

\[ E(x[n]) = (n+1)E[u[0]] \]

\[ \text{var}(x[n]) = (n+1) \text{var}(u[0]) \]

Sometimes can convert a nonstationary R.P. to stationary one (by processing it)
Previous example let \( y(n) = x(n) - x(n-1) \)
where \( x(-1) = 0 \). Then,

\[
y(n) = \sum_{i=0}^{n} v(i) - \sum_{i=0}^{n-1} v(i) = v(n)
\]  \( \text{IID} \implies \text{STATIONARY} \)

Note more generally that for \( n_4 > n_3 \geq n_2 > n_1 \),

\[
x(n_2) - x(n_1) = \sum_{i=n_1+1}^{n_2} v(i) \quad \text{CALLED INCREMENTS OF RS.}
\]

\[
x(n_4) - x(n_3) = \sum_{i=n_3+1}^{n_4} v(i)
\]
are independent of each other and if \( n_4 - n_3 = n_2 - n_1 \) (same number of \( v(i) \) terms),
they have same pmf/pdf.

\[
x(n_2) - x(n_1), \quad x(n_4) - x(n_3) \quad \text{ARE CALLED STATIONARY INDEPENDENT INCREMENTS.}
\]

\[
x(n_4) - x(n_3) = x(n_2+2) - x(n_1+2)
\]
shift by 2

Allows easier calculation of probs - see ex 16.5

More Examples

1) White Gaussian Noise (WGN) - used extensively in radar/sonar/communications
DTCV R.P., \( x(n) \) is IID R.P. with \( x(n) \sim N(0, \delta^2) - \infty < n < \infty \). See Fig 16.56 for realization.

**Note:** \( E\{x(\infty)\} = 0 \) "Noise"

**Average Power** = \( E\{x^2(\infty)\} = \text{VAR}(x(\infty)) = \delta^2 \)

**PDF is**

\[
\begin{align*}
    f(x(1), \ldots, x(n)) = \frac{1}{\sqrt{\pi \delta^2}} \frac{1}{(2\pi \delta^2)^{n/2}} e^{-\frac{1}{2\delta^2} \sum_{i=1}^{n} x_i^2} \\
    \text{Also} \quad e^{-\frac{1}{2} \begin{pmatrix} x \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x \end{pmatrix}} \quad \text{for} \quad \Sigma = \delta^2 I \quad \text{or} \quad N(0, \delta^2 I)
\end{align*}
\]

Called **White Gaussian Noise since its power is equally distributed in frequency** (~ white light) - Chapter 17, Ex. 17.9

**2) Moving Average R.P.**

\[
\begin{align*}
    x(n) &= \frac{1}{2} (u(n) + u(n-1)) \quad -\infty < n < \infty \\
    x(0) &= \frac{1}{2} (u[0] + u[-1]) \\
    x(1) &= \frac{1}{2} (u[1] + u[0])
\end{align*}
\]
\[ x(n) = \frac{1}{2}(u(n)+u(n-1)) \]

Averaging "moves" in time. \( u(n) \) is WGN with variance \( \sigma^2 \).

\[ u(n) \]

\[ x(n) \]

Figure 16.9: Typical realization of moving average random process. The realization of the \( U[n] \) random process is shown in Figure 16.5b.

\[ x(n) \] is "smoother" (averager acts as a linear filter)

To find joint PDF of \( x(n) \) note that transformation from \( u(n) \) to \( x(n) \) is linear. For example,

\[
\begin{bmatrix}
  x(0) \\
  x(1)
\end{bmatrix} =
\begin{bmatrix}
  \frac{1}{2} & 0 \\
  0 & 1/2
\end{bmatrix}
\begin{bmatrix}
  u(n-1) \\
  u(n)
\end{bmatrix}
\]

\[ x = G \cdot u \]

Recall for WGN \( u \sim N(0, \sigma^2 I) \)

\[ x = G u \sim N(0, G C_u G^T) \]

Since \( E[x] = E[G u] = G E[u] = 0 \)
\[ G = \begin{bmatrix} 0 \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} \]

Also, \( G G^T = G^T G = \sigma G G^T \)

and
\[ GG^T = \begin{bmatrix} \frac{\sigma}{2} & \frac{\sigma}{2} \\ \frac{\sigma}{2} & \frac{\sigma}{2} \end{bmatrix} \]

\[ \begin{bmatrix} x_1[n] \\ x_1[n] \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{bmatrix} \right) \]

Also can show that this RPF is stationary.

**Example:** Randomly Phased Sinusoid

\[ x[n] = \cos(2\pi(0.1)n + \Theta) \quad -\infty < n < \infty \]

where \( \Theta \sim U(0, 2\pi) \),

\[ n = \begin{bmatrix} 0 & 3 \end{bmatrix}^T \]

In MATLAB

\[ x = \cos(2\pi(0.1)n) + \cos(2\pi(0.1)n + 2\pi \cdot \text{rand}(1,1)) \]

![Graphs showing typical realizations for randomly phased sinusoid](image)

(a) \( \theta = 5.9598 \)  
(b) \( \theta = 1.4523 \)  
(c) \( \theta = 3.8129 \)

Figure 16.10: Typical realizations for randomly phased sinusoid.