a) $$p_{S_3}(x) = \int_{-1}^{x+\frac{1}{2}} p_{S_2}(u)\cdot 1\, du \qquad -\tfrac{3}{2} \le x \le -\tfrac{1}{2}$$

$$= \tfrac{1}{2}x^2 + 3/2\, x + 9/8$$

b) $$p_{S_3}(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} p_{S_2}(u)\cdot 1\, du \qquad -\tfrac{1}{2} < x \le \tfrac{1}{2}$$

$$= -x^2 + 3/4$$

c) $$p_{S_3}(x) = \int_{x-\frac{1}{2}}^{1} p_{S_2}(u)\cdot 1\, du \qquad \tfrac{1}{2} < x \le 3/2$$

$$= \tfrac{1}{2}x^2 - 3/2\, x + 9/8$$

d) $= 0$ OTHERWISE

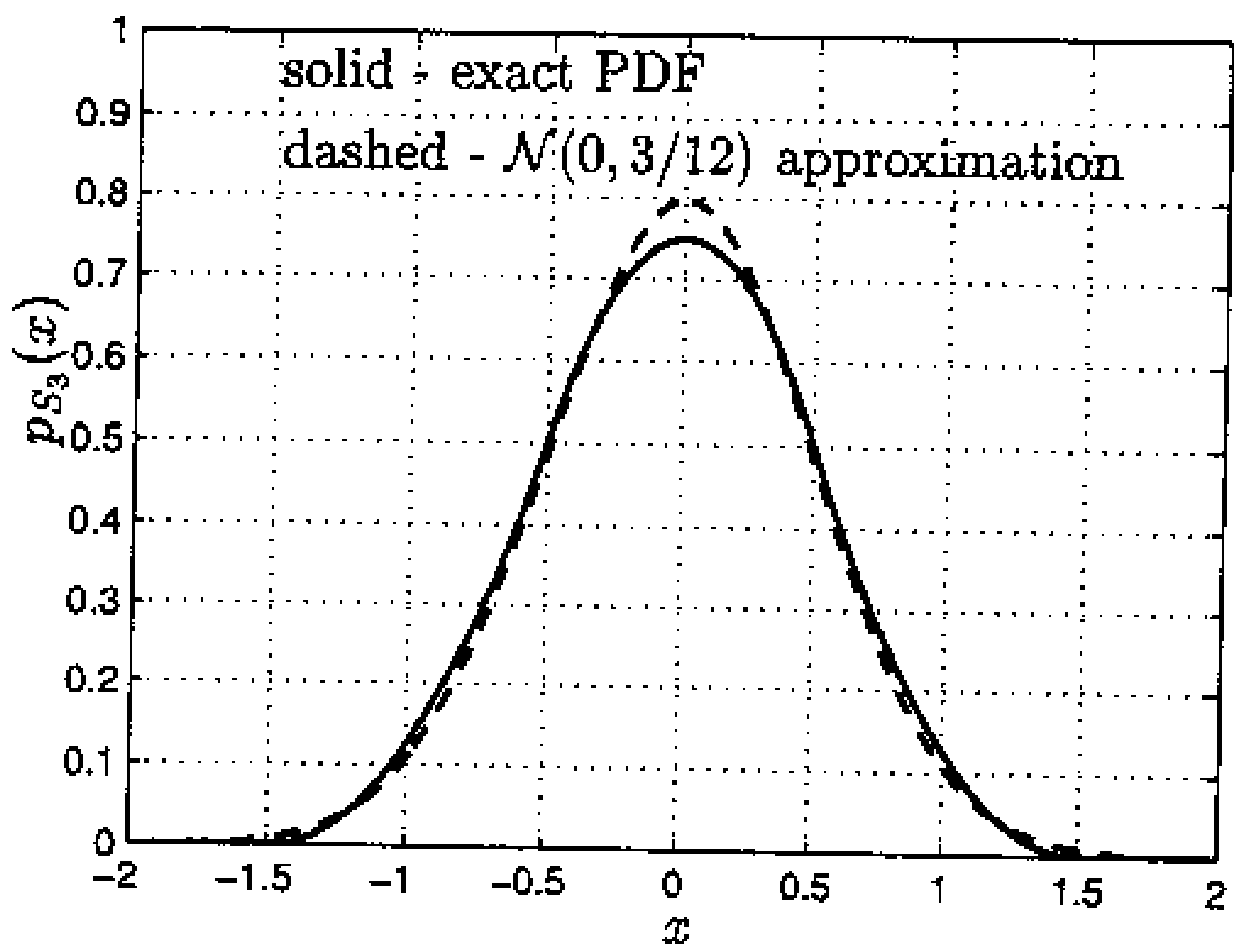


Figure 15.6: PDF for sum of 3 IID $\mathcal{U}(-1/2, 1/2)$ random variables and Gaussian approximation.

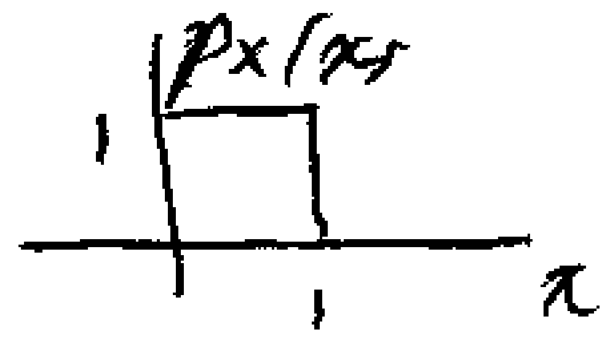
NOTE: $E[S_3] = E\left(\sum_{i=1}^{3} X_i\right) = 0 \qquad X_i \sim U(-\tfrac{1}{2}, \tfrac{1}{2})$

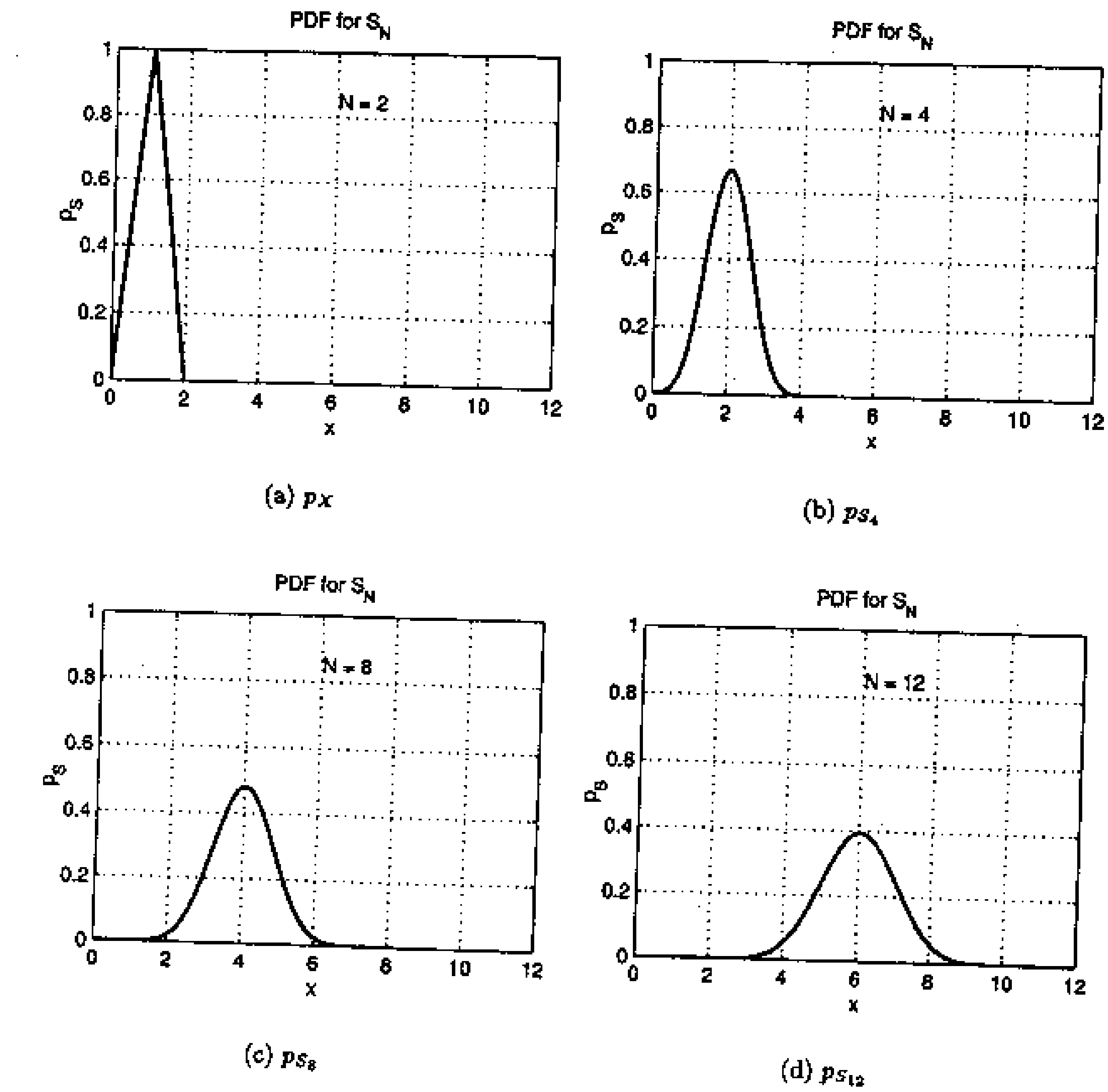
$VAR[S_3] = VAR\left(\sum_{i=1}^{3} X_i\right) = 3\, VAR(X_i) = \frac{3}{12}$

GAUSSIAN APPROXIMATION VERY GOOD NEAR $x = 0$. HOW ABOUT FOR $x > 2$?

<u>EXAMPLE</u>: $S_N = \sum_{i=1}^{N} X_i \qquad X_i \sim U(0,1)$ IID

USING clt_demo.m (APPENDIX 15A) FOR REPEATED CONVOLUTION OF





(a) p_X (b) p_{S_4} (c) p_{S_8} (d) $p_{S_{12}}$

Figure 15.7: PDF of sum of N IID $\mathcal{U}(0,1)$ random variables. The plots were obtained using clt_demo.m listed in Appendix 15A.

DOES p_{S_N} CONVERGE TO ANYTHING AS $N \to \infty$?

NOTE:

$$E[S_N] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i] = N\,E[X] = N/2$$

ALSO $\mathrm{VAR}(S_N) = N\,\mathrm{VAR}(X) = N/12$

MUST SOMEHOW <u>NORMALIZE</u> TO TALK ABOUT CONVERGENCE.

SIMILAR TO FOLLOWING:

$$\bar{n}_N = \frac{1}{N}\sum_{n=1}^{N} n = \frac{\frac{N}{2}(N+1)}{N} \to \infty \quad \text{AS } N \to \infty$$

OR AVERAGE OF FIRST N POSITIVE INTEGERS $\to \infty$ AS N GETS LARGE. BUT WE WOULD LIKE TO SAY THAT THIS AVERAGE $\simeq N/2$.

$$\text{EXAMINE} \quad \frac{\bar{n}_N}{N} = \frac{\frac{N}{2}(N+1)}{N^2} \to \frac{1}{2} \quad \text{AS } N \to \infty$$

$$\text{OR} \quad \frac{\bar{n}_N}{N} \simeq \frac{1}{2} \Rightarrow \bar{n}_N \simeq N/2$$

TO NORMALIZE S_N WE FORM <u>STANDARDIZED</u> SUM (RECALL STANDARD NORMAL HAD $X \sim N(0,1)$ OR $E[X]=0$, $VAR(X)=1$). SAME ~~HERE~~!

$$Z_N = \frac{S_N - E[S_N]}{\sqrt{VAR(S_N)}}$$

HAS MEAN $= 0$, VARIANCE $= 1$ WHY?

$$Z_N = \frac{S_N - N E_X[X]}{\sqrt{N\,VAR(X)}} \qquad \text{WELL DEFINED EVEN AS } N \to \infty$$

FINALLY CENTRAL LIMIT THEOREM (CLT) SAYS THAT AS $N \to \infty$, $Z_N \to N(0,1)$

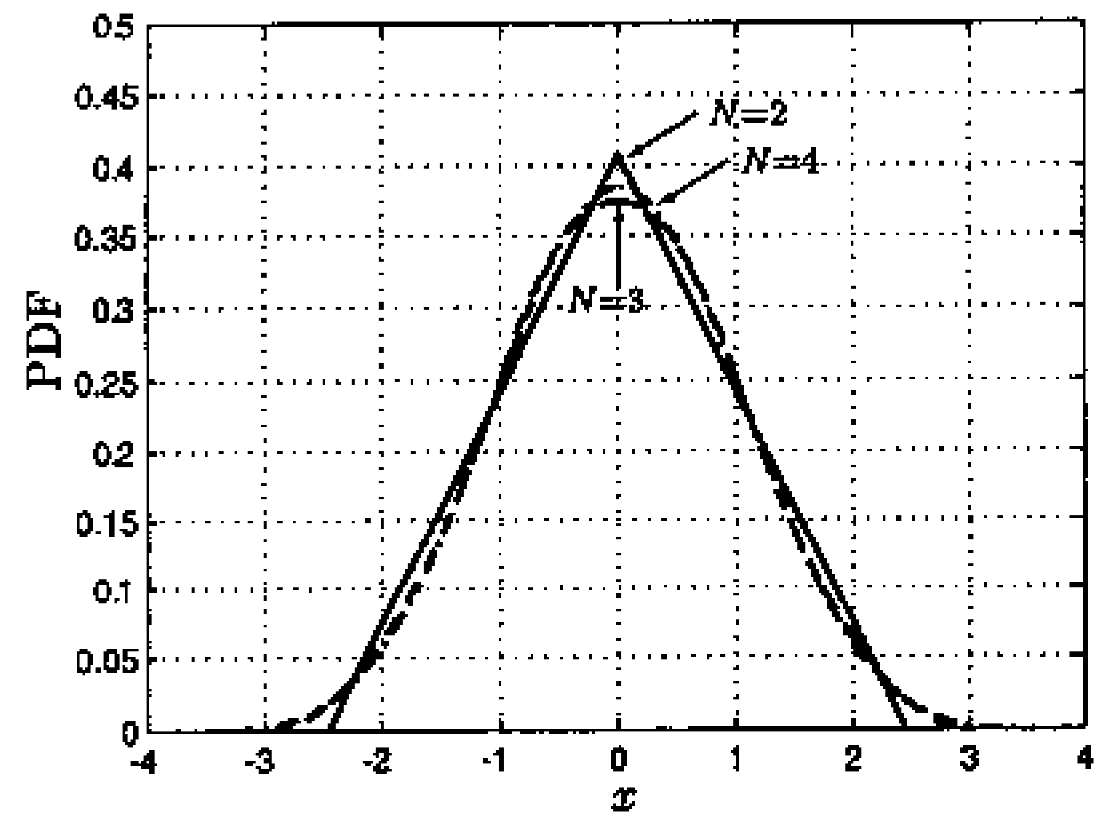


Figure 15.8: PDF of standardized sum of N IID $\mathcal{U}(0,1)$ random variables.

SAME EXAMPLE AS BEFORE – MUCH BETTER "BEHAVED" MATHEMATICALLY

CLT: IF $X_1, X_2, \ldots, X_N$ ARE CONT. IID RVS EACH WITH MEAN $E_X[X]$ AND VARIANCE $VAR(X)$, THEN AS $N \to \infty$

$$\frac{\sum_{i=1}^{N} X_i - N E_X[X]}{\sqrt{N VAR(X)}} \to N(0,1)$$

EXAMPLE: $X_i \sim N(0,1)$

EXAMINE PDF OF $Y = \sum_{i=1}^{N} X_i^2$ AS $N \to \infty$

TRUE PDF IS $Y \sim \chi_N^2$. TO APPLY CLT NOTE:

IID

1) $X_1, X_2, \ldots, X_N$ ARE IND $\Rightarrow X_1^2, X_2^2, \ldots, X_N^2$ ARE INDEPENDENT (CAN YOU JUSTIFY THIS? SEE PG. 200 IN BOOK)

2) $X_1, X_2, \ldots, X_N$ HAVE SAME PDF $\Rightarrow X_1^2, X_2^2, \ldots, X_N^2$ HAVE SAME PDF

AS PER CLT

$$\frac{\sum_{i=1}^{N} x_i^2 - N\,E_X(x^2)}{\sqrt{N\,VAR(x^2)}} \rightarrow N(0,1)$$

BUT IF $x \sim N(0,1)$,

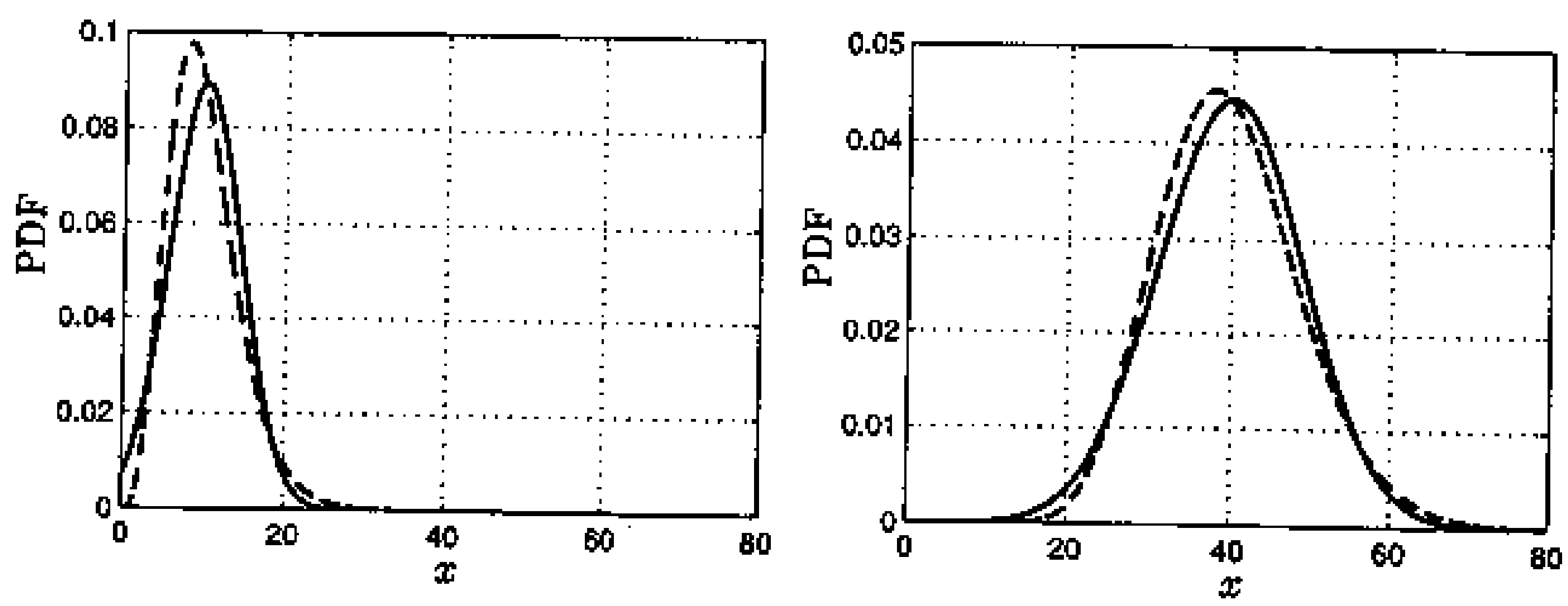
$$E_X(x^2) = VAR(x) + \underbrace{E_X^2(x)}_{=0} = 1$$

$$VAR(x^2) = E_X(x^4) - \underbrace{E_X^2(x^2)}_{1} = 2$$

CAN BE SHOWN TO $= 3$
(TRY INTEGRATION)

$$Z_N = \frac{\sum_{i=1}^{N} x_i^2 - N}{\sqrt{2N}} \rightarrow N(0,1)$$

$$Y_N = \sum_{i=1}^{N} x_i^2 = \sqrt{2N}\,\underbrace{Z_N}_{\rightarrow N(0,1)} + N \approx N(N, 2N)$$



(a) $N = 10$

(b) $N = 40$

Figure 15.9: χ^2_N PDF (dashed curve) and Gaussian PDF approximation of $\mathcal{N}(N, 2N)$ (solid curve).

"PROOF" OF CLT : LET $Z_N = \frac{S_N - N E_X[X]}{\sqrt{N VAR(X)}}$

TO SHOW $Z_N \to N(0,1) = Z$ WE CAN EQUIVALENTLY SHOW THAT CHARACTERISTIC FUNCTION $\phi_{Z_N}(\omega) \to \phi_Z(\omega) = e^{-\frac{1}{2}\omega^2}$ ← FROM TABLES!
(RECALL CHARACTERISTIC FUNCTION IS FOURIER TRANSFORM OF PDF)

$$\phi_{Z_N}(\omega) = E_{Z_N}[e^{j\omega Z_N}]$$

$$= E_{\underline{X}}\left[e^{j\omega \frac{\sum X_i - N E_X[X]}{\sqrt{N VAR(X)}}}\right]$$

RECALL $Y = g(\underline{X})$
$E_Y[Y] = E_{\underline{X}}[g(\underline{X})]$

$$= E_{\underline{X}}\left[\prod_{i=1}^{N} e^{j\omega \frac{X_i - E_X[X]}{\sqrt{N VAR(X)}}}\right]$$

$$= \prod_{i=1}^{N} E_{X_i}\left[e^{j\omega \frac{X_i - E_X[X]}{\sqrt{N VAR(X)}}}\right]$$

X, Y IND ⇒
$E_{XY}[g(X)h(Y)] = E_X[g(X)]E_Y[h(Y)]$

$$= \left[E_X\left[e^{j\omega \frac{X - E_X[X]}{\sqrt{N VAR(X)}}}\right]\right]^N$$

X_i'S SAME PDF
⇒ SAME $E_X[g(X)]$

NOW $e^{z} = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

$$\Rightarrow E_X\left[e^{j\omega \frac{X - E_X[X]}{\sqrt{N VAR(X)}}}\right] = E_X\left[\sum_{k=0}^{\infty} \frac{(j\omega)^k}{k!}\left(\frac{X - E_X[X]}{\sqrt{N VAR(X)}}\right)^k\right]$$

$$= \sum_{k=0}^{\infty} \frac{(j\omega)^k}{k!} E_X\left[\left(\frac{X - E_X[X]}{\sqrt{N VAR(X)}}\right)^k\right]$$

$$= 1 + j\omega E_X\left[\frac{X - E_X[X]}{\sqrt{N VAR(X)}}\right] + \frac{1}{2}(j\omega)^2 E_X\left[\left(\frac{X - E_X[X]}{\sqrt{N VAR(X)}}\right)^2\right]$$

+ HIGHER ORDER TERMS IN $1/\sqrt{N}$

$$\text{BUT } E_X\left[\frac{X - E_X[X]}{\sqrt{N\,VAR(X)}}\right] = 0$$

$$E_X\left[\left(\frac{X - E_X[X]}{\sqrt{N\,VAR(X)}}\right)^2\right] = \frac{E_X\left[(X - E_X[X])^2\right]}{N\,VAR(X)}$$

$$= 1/N$$

$$\phi_{Z_N}(\omega) = \left(1 + \tfrac{1}{2}(j\omega)^2/N\right)^N \quad \text{DROP HIGHER ORDER TERMS}$$

$$= \left(1 - \tfrac{1}{2}\omega^2/N\right)^N$$

$$\to e^{-\frac{1}{2}\omega^2} \quad \text{AS } N \to \infty \quad \text{(SEE PROB 5.15)}$$

$$= \phi_Z(\omega) \qquad Z \sim N(0,1)$$

CHAPTER 16 - BASIC RANDOM PROCESSES

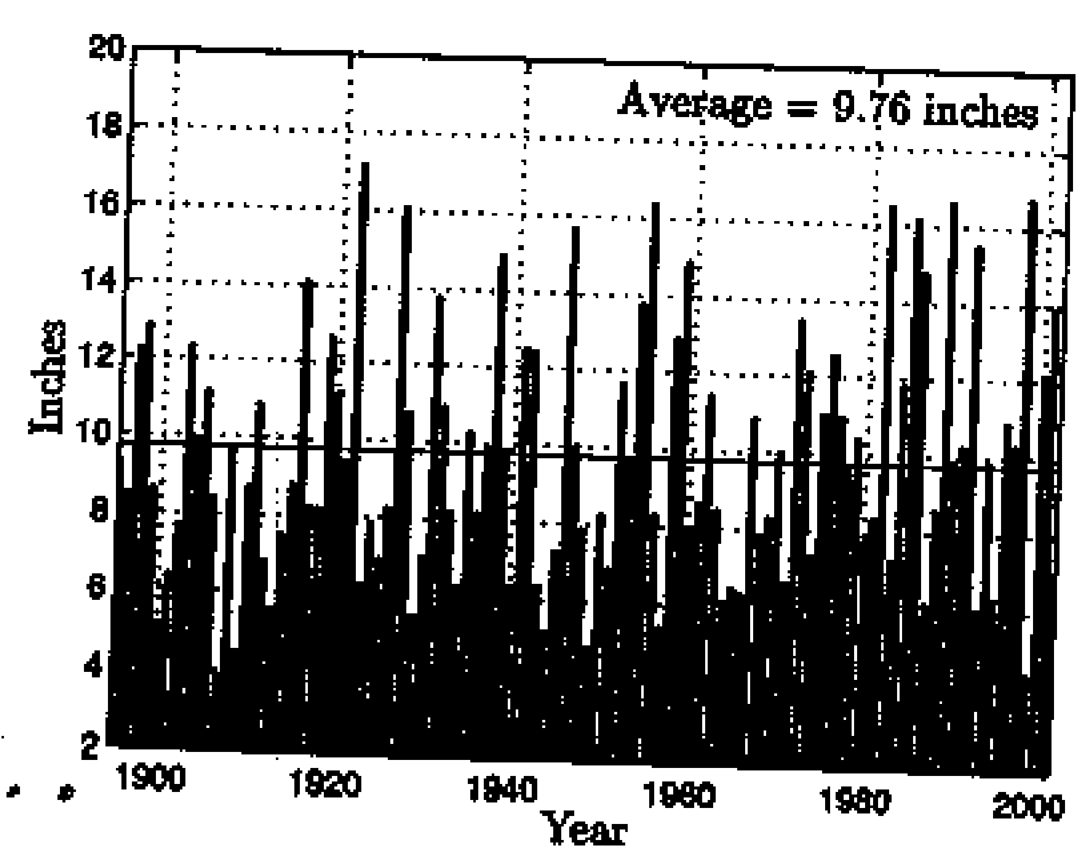
STUDIED RANDOM VARIABLE X

RANDOM VECTOR $(X_1, X_2, \ldots, X_N)$

NOW RANDOM PROCESS $(\ldots, X_{-1}, X_0, X_1, \ldots)$

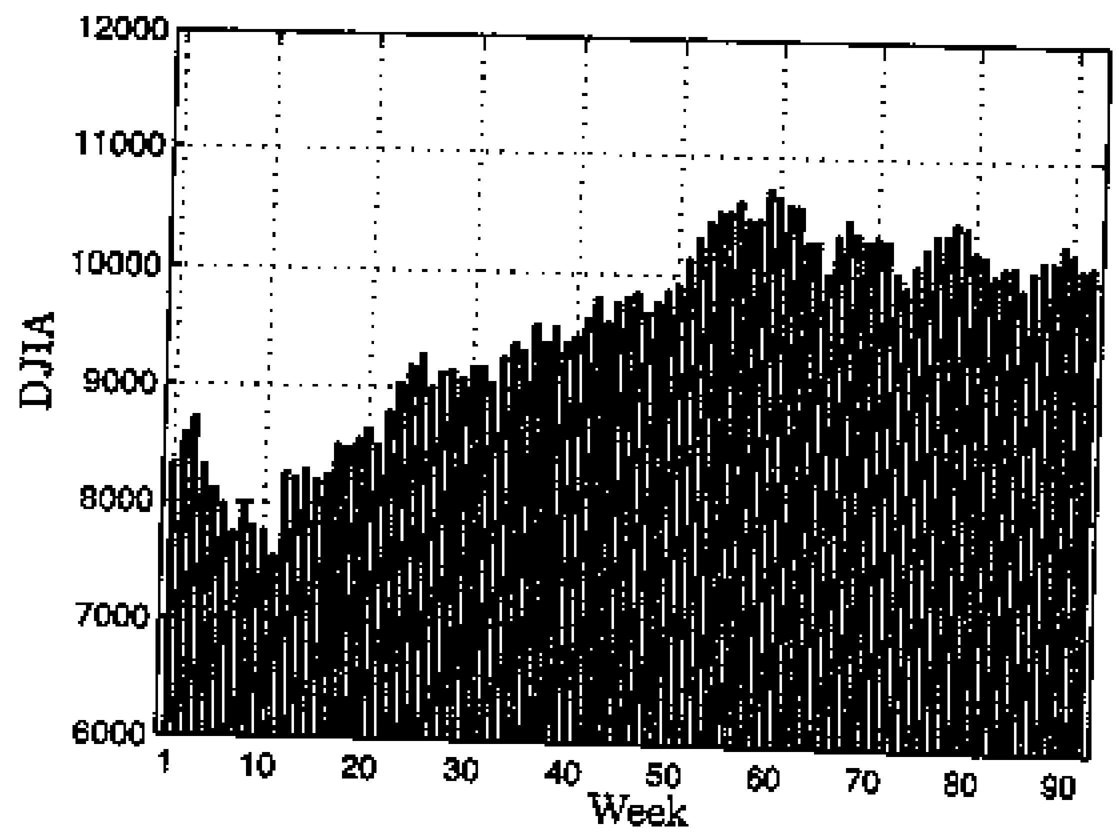
EXAMPLES:

WHAT IS OF INTEREST HERE?



$\leftarrow \ldots \qquad \ldots \rightarrow$

Figure 16.1: Annual summer rainfall in Rhode Island from 1895 to 2002.



WHAT IS OF INTEREST HERE?

Figure 16.2: Dow-Jones industrial average at the end of each week from January 8, 2003 to September 29, 2004 [DowJones.com 2004].

WHAT IS A RANDOM PROCESS (RP)?

EXAMPLE: START TOSSING COIN AT SOME TIME $n=0$ AND CONTINUE INDEFINITELY $(n=0,1,2,\ldots)$

$\Rightarrow$ INFINITE SEQUENCE OF COIN TOSSES

OUTCOMES ARE

$$S = \{(H,H,T,\ldots),(H,T,H,\ldots),(T,T,H,\ldots),\ldots\}$$

IF WE DEFINE A R.V. AS

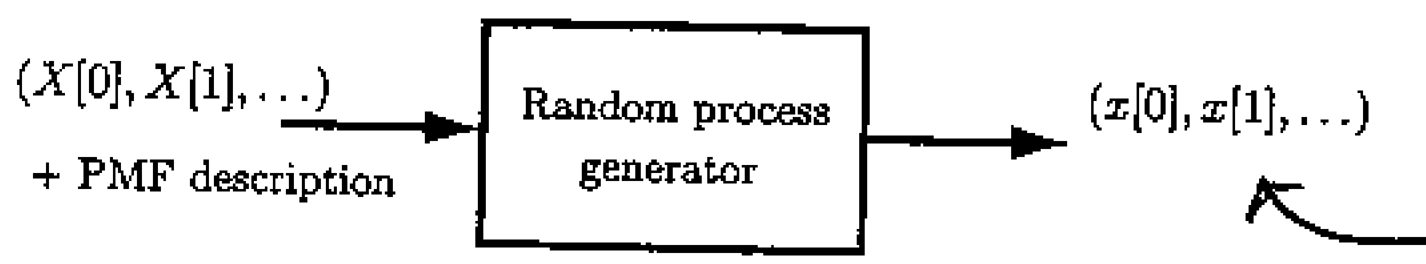
$$\left.\begin{array}{ll} X = 0 & \text{IF TAIL} \\ \ 1 & \text{IF HEAD} \end{array}\right\} \text{BERNOULLI RV}$$

THEN WE CALL THIS A BERNOULLI RP.

THE OUTCOMES OF THE <u>RANDOM PROCESS</u> ARE

$S_x = \{(1,1,0,\ldots), (1,0,1,\ldots), (0,0,1,\ldots), \ldots\}$

NOW DENOTE THE RANDOM VARIABLES AS $X[0], X[1], \ldots$ AND THEIR OUTCOMES AS $x[0], x[1], \ldots$ OR IN GENERAL $X[n], x[n]$



CALLED AN <u>OUTCOME</u> OR <u>SAMPLE SEQUENCE</u> OR <u>REALIZATION</u> ↑ OUR CHOICE

Figure 16.3: A conceptual random process generator. The input is an infinite sequence of random variables with their probabilistic description and the output is an infinite sequence of numbers.

NOTE: EACH REALIZATION IS AN <u>INFINITE</u> SEQUENCE OF NUMBERS

RECALL FOR A SINGLE R.V., A MAPPING FROM S TO S_X, WE DENOTE IT MORE EXPLICITLY AS THE SET FUNCTION $X(s)$.

NOW S = SET OF INFINITE EXPERIMENTAL OUTCOMES (COIN TOSSES)

S_X = SET OF INFINITE SEQUENCES OF 1'S AND 0'S (REALIZATIONS)

SET OF ALL REALIZATION CALLED THE ENSEMBLE OF REALIZATIONS.

NOW INSTEAD OF $X(S)$, USE $X[n,S]$ = MAPPING FROM S TO S_X. $0 \le n < \infty$

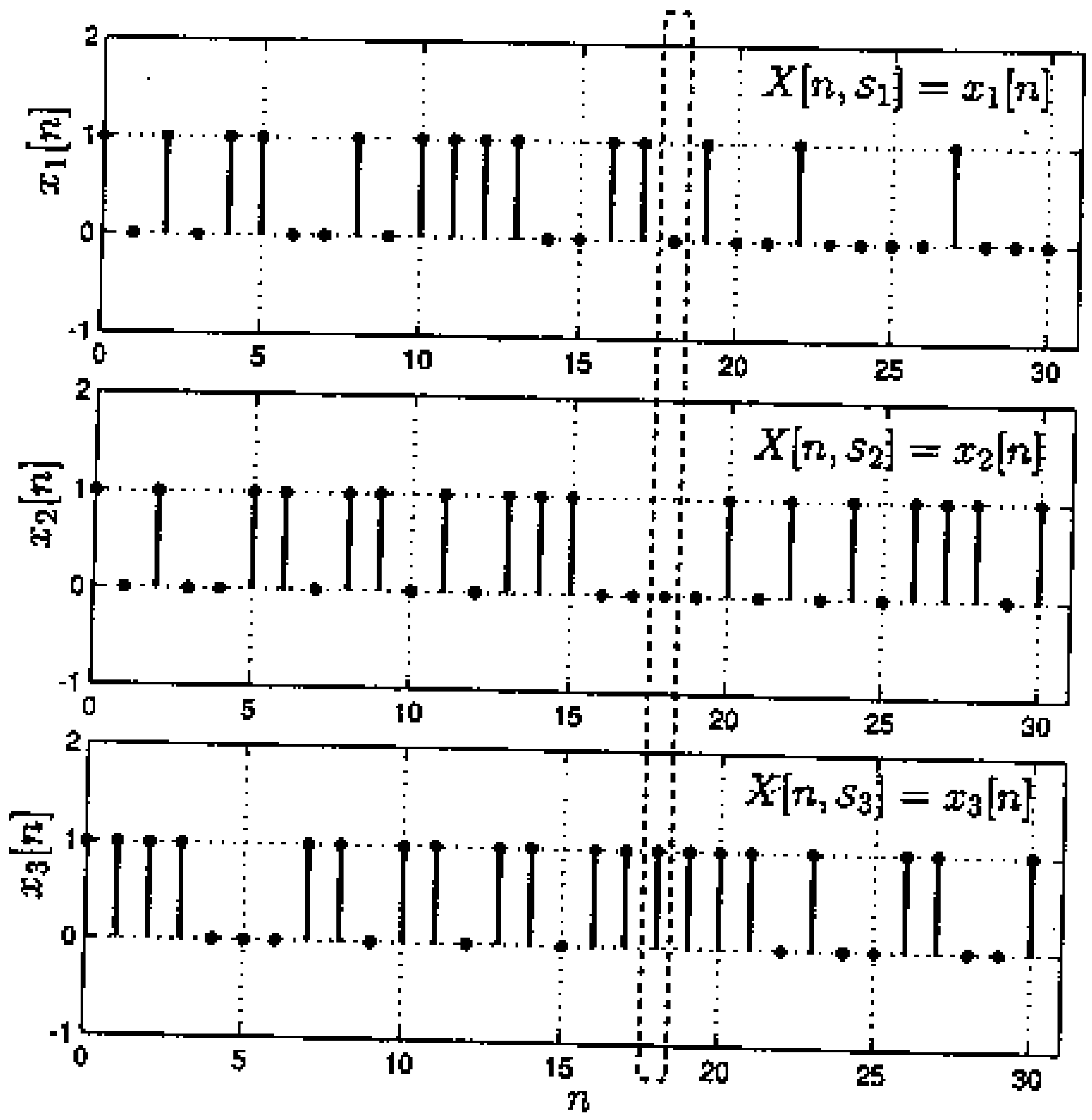


Figure 16.4: Typical outcomes of Bernoulli random process with $p = 0.5$. The realization starts at $n = 0$ and continues indefinitely. The dashed box indicates the realizations of the *random variable* $X[18, s]$.

← FIRST REALIZATION

← SECOND REALIZATION

⋮

NOTE THAT $X[18,S]$ IS JUST A R.V. ⇒ HAS A PMF, MEAN, VARIANCE, ETC

WILL DENOTE THE R.P. BY $X[n]$ (DROP THE S)

BRACKET ⇒ DISCRETE-TIME OR FOR $n = 0, 1, \ldots$ AS OPPOSED TO $X(t)$ $t \ge 0$

ALSO, $X[n]$ WILL DENOTE THE ENTIRE R.P. $\{X[0], X[1], \ldots\}$

(SOMETIMES AUTHORS USE $\{X[n]\}\big|_{n=0,1,\ldots}$

AND $X[n_0]$ WILL DENOTE R.P. AT <u>FIXED TIME</u> $n = n_0$

EXAMPLE: BERNOULLI R.P. (IID TOSSES)
WHAT IS PROB. OF FIRST 5 TOSSES COMING UP HEADS?

$$P[x[0]=1, x[1]=1, x[2]=1, x[3]=1, x[4]=1, x[5]=0 \text{ OR } 1, x[6]=0 \text{ OR } 1, \ldots]$$

SINCE WE DON'T CARE WHAT $x[n]$ FOR $n \geq 5$ IS, WE CAN RESTRICT ATTENTION TO

$$P[x[0]=1, x[1]=1, x[2]=1, x[3]=1, x[4]=1]$$

$$= \prod_{n=0}^{4} P[x[n]=1] = p^5$$

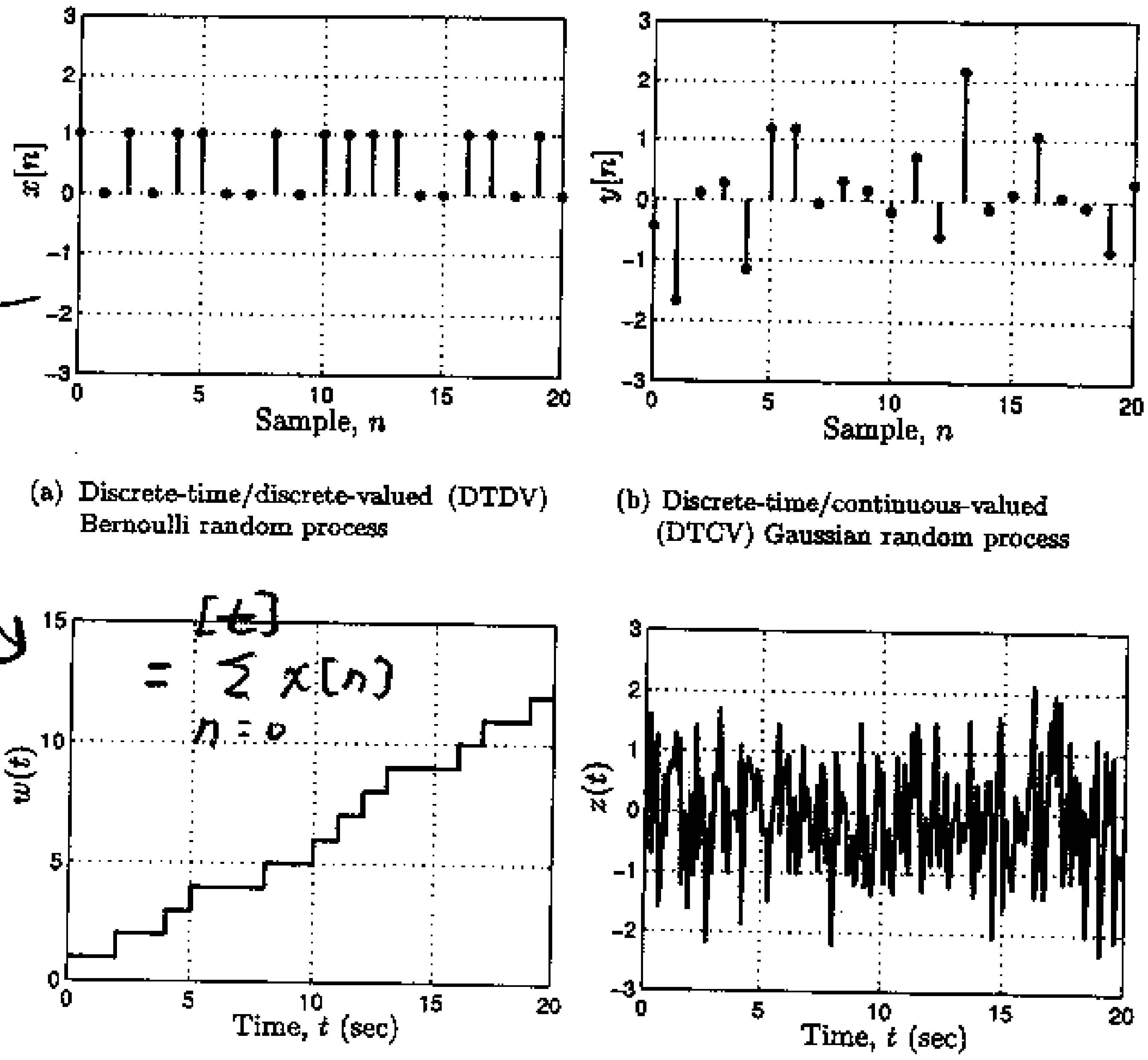
IN ESSENCE WE REPLACED R.P. BY RANDOM VECTOR
$\Rightarrow$ EASY PROB. COMPUTATION

WHAT IS PROB. OF EVER OBSERVING 5 CONSECUTIVE HEADS = 1? HOW TO FIND THIS?

TYPES OF RANDOM PROCESSES

$x[n]$ $\quad n = 0, 1, \ldots$ $\quad$ SEMI-INFINITE
$x[n]$ $\quad n = \ldots, -1, 0, 1, \ldots$ $\quad$ INFINITE
$x[n]$ $\quad$ DISCRETE-TIME
$x(t)$ $\quad$ CONTINUOUS-TIME $\quad -\infty < t < \infty$ $\quad t \geq 0$

CAN ALSO CATEGORIZE ACCORDING TO DISCRETE OR CONT. OUTCOMES (SAME AS DISCRETE OR CONT. RVS)



RUNNING SUM

$$= \sum_{n=0}^{[t]} x[n]$$

$[t]$ = LARGEST INTEGER $\leq t$

← COULD BE SAMPLES OF

(a) Discrete-time/discrete-valued (DTDV) Bernoulli random process

(b) Discrete-time/continuous-valued (DTCV) Gaussian random process

(c) Continuous-time/discrete-valued (CTDV) binomial random process

(d) Continuous-time/continuous-valued (CTCV) Gaussian random process

Figure 16.5: Typical realizations of different types of random processes.

WILL GENERALLY FOCUS ON (b) SINCE DISCRETE-TIME USED EXTENSIVELY IN PRACTICE AND CONT.-VALUED OUTCOMES CORRESPOND TO CONT. RVS WHICH WE HAVE ALREADY STUDIED

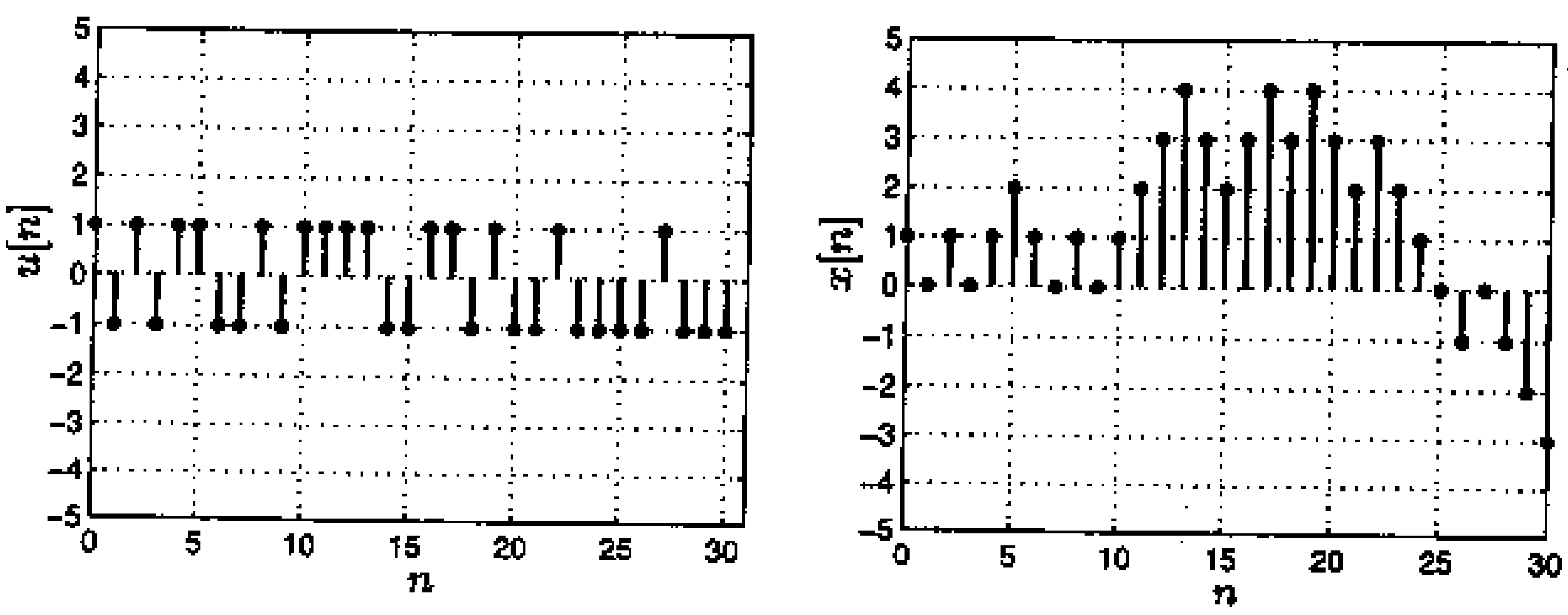
<u>EXAMPLE</u>: RANDOM WALK (USED AS MODEL FOR MANY PHYSICAL PROCESSES - "A RANDOM WALK DOWN WALL STREET")

DEFINED AS $X[n] = \sum_{i=0}^{n} U[i] \quad n = 0, 1, \ldots$

$U[n]$ IS BERNOULLI WITH OUTCOMES ± 1

AND $p_U[k] = \begin{cases} \frac{1}{2} & k = -1 \\ \frac{1}{2} & k = 1 \end{cases}$

AND $U[n]$'S ARE IID (BERNOULLI RP ± 1 OUTCOMES INSTEAD OF $0, 1$)



(a) Realization of Bernoulli random process $U[n]$

(b) Realization of random walk $X[n]$

Figure 16.6: Typical realization of a random walk.

POSITION OF DRUNK AFTER n STEPS, PRICE OF STOCK THAT MOVES UP OR DOWN BY \$1. (PRICE CHANGES UNPREDICTABLE?)

AN IMPORTANT QUESTION IS BEHAVIOR FOR LARGE n.

BY CLT (WHY?) $X[n] \sim$ GAUSSIAN

$$E[X[n]] = E\left[\sum_{i=0}^{n} U[i]\right] = (n+1)E[U[0]] = 0$$

$$VAR[X[n]] = VAR\left[\sum_{i=0}^{n} U[i]\right] = (n+1)\underbrace{VAR[U[0]]}_{=1}$$

$$VAR(U[0]) = \underbrace{E[U^2[0]]}_{=1} - \underbrace{E^2[U[0]]}_{=0} = 1$$

$\Rightarrow X[n] \sim \mathcal{N}(0, n+1)$ MAKE SENSE?

STATIONARITY

DO CHARACTERISTICS OF RP CHANGE WITH TIME? BERNOULLI RP – NO
RANDOM WALK – YES

TO QUANTIFY THIS NEED TO DESCRIBE PROBABILITIES OF R.P. AND EXAMINE THEM OVER TIME.

EXAMPLE: BERNOULLI RP

THIS IS EXAMPLE OF IID R.P.
TO COMPUTE PROBS. MUST CONSTRAIN OURSELVES TO A FINITE SET OF TIMES.

$$p_{X[n_1],X[n_2],\ldots,X[n_N]}[x_1,x_2,\ldots,x_N] = \prod_{i=1}^{N} p_{X[n_i]}[x_i]$$

JOINT PMF

CALLED A FINITE DIMENSIONAL DISTRIBUTION.

NOTE HERE THAT PROB OF FIRST 5 SAMPLES $n_1=0, n_2=1, \ldots, n_5=4$ BEING ALL 1'S IS p^5 AND PROB. OF SECOND 5 SAMPLES $n_6=5, \ldots, n_{10}=9$ BEING ALL 1'S IS ALSO p^5, ETC.

THIS R.P. IS <u>STATIONARY</u>. NOTE THAT

$$p_{X[0],\ldots,X[4]} = p_{X[5],\ldots,X[9]}$$

OR $p_{X[n],\ldots,X[n+4]} = p_{X[n+n_0],\ldots,X[n+4+n_0]}$

$n=0$
$n_0=5$

IN GENERAL, A R.P. IS DEFINED TO BE STATIONARY IF

$$p_{X[n_1+n_0],X[n_2+n_0],\ldots,X[n_N+n_0]} = p_{X[n_1],X[n_2],\ldots,X[n_N]}$$

FOR ALL $n_1, n_2, \ldots, n_N$ (ALL N) AND ALL n_0

EVERY FINITE DIMENSIONAL DISTRIBUTION (GIVEN $n_1, n_2, \ldots, n_N$) DOES NOT CHANGE IF SAMPLE TIMES ARE ALL SHIFTED BY n_0.

<u>EXAMPLE</u>: IID R.P. IS STATIONARY

$$p_{X[n_1+n_0],\ldots,X[n_N+n_0]} = \prod_{i=1}^{N} p_{X[n_i+n_0]} \qquad \text{IND.}$$

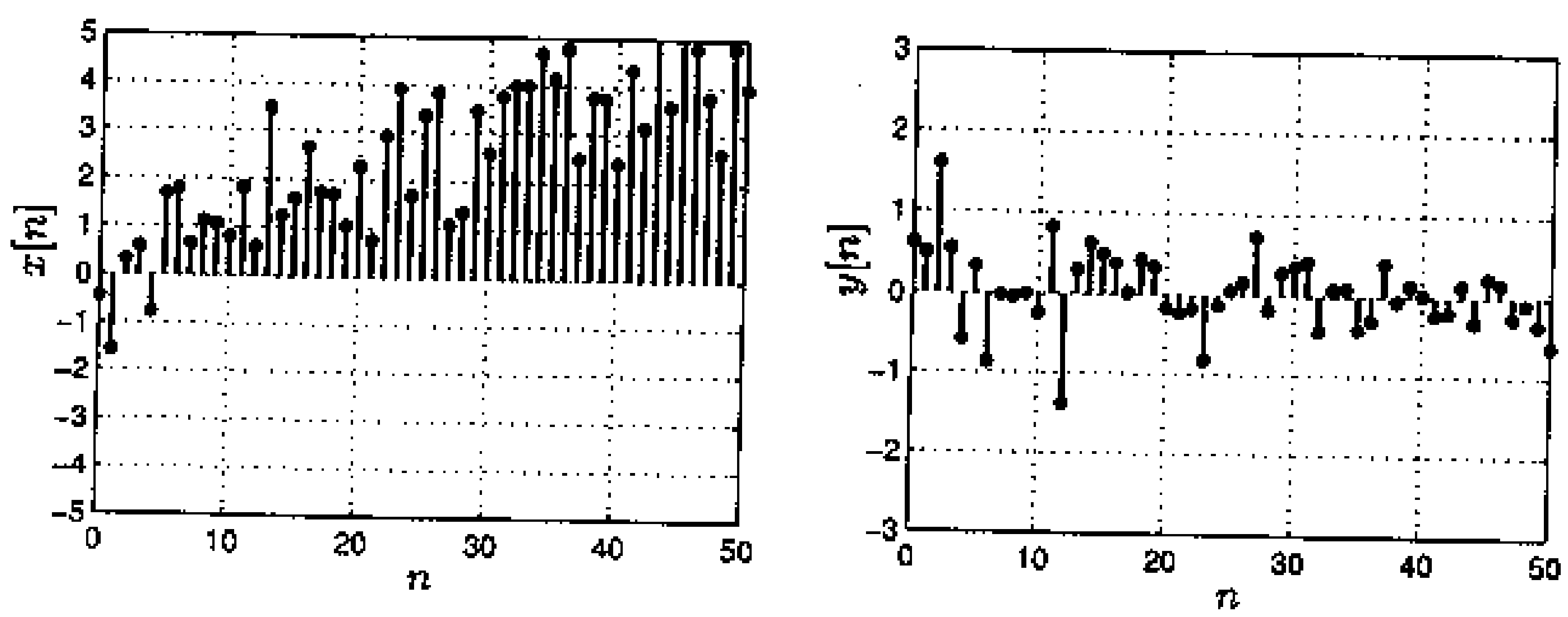
$$= \prod_{i=1}^{N} p_{X[n_i]} \qquad \text{IDENTICALLY DIST.}$$

$$= p_{X[n_1],\ldots,X[n_N]} \qquad \text{IND.}$$

NOTE THAT IF R.P. IS STATIONARY SO ARE ALL JOINT MOMENTS SINCE

$$E_{X[n_1+n_0],\ldots,X[n_N+n_0]}[\cdot] = E_{X[n_1],\ldots,X[n_N]}[\cdot]$$

$\Rightarrow$ IF MOMENTS ARE NOT STATIONARY THEN R.P. CANNOT BE STATIONARY.



(a) Mean increasing with n (b) Variance decreasing with n

Figure 16.7: Random processes that are not stationary.

IF YOU JUST LOOK AT THESE REALIZATIONS, CAN YOU TELL IF R.P. IS STATIONARY?

EXAMPLE: SUM R.P.

$$X[n] = \sum_{i=0}^{n} U[i]$$

$U[i]$'S IID ARBITRARY PMF OR PDF

STATIONARY?

$$E(X[n]) = (n+1)E_U[U[0]]$$
$$VAR(X[n]) = (n+1)\,VAR(U[0])$$

SOMETIMES CAN CONVERT A NONSTATIONARY R.P. TO STATIONARY ONE (BY PROCESSING IT)

PREVIOUS EXAMPLE LET $y[n] = x[n] - x[n-1]$ WHERE $x[-1] = 0$. THEN,

$$y[n] = \sum_{i=0}^{n} u[i] - \sum_{i=0}^{n-1} u[i] = u[n] \quad \text{IID} \Rightarrow \text{STATIONARY}$$

NOTE MORE GENERALLY THAT FOR

$n_4 > n_3 \geq n_2 > n_1$

$$\left.\begin{aligned} x[n_2] - x[n_1] &= \sum_{i=n_1+1}^{n_2} u[i] \\ x[n_4] - x[n_3] &= \sum_{i=n_3+1}^{n_4} u[i] \end{aligned}\right\} \text{CALLED } \underline{\text{INCREMENTS}} \text{ OF R.P.}$$

ARE IND. OF EACH OTHER AND IF $n_4 - n_3 = n_2 - n_1$ (SAME NUMBER OF $u[i]$ TERMS), THEY HAVE SAME PMF/PDF.

$x[n_2] - x[n_1]$, $x[n_4] - x[n_3]$ ARE CALLED <u>STATIONARY INDEPENDENT INCREMENTS</u>.

$x[4] - x[3] = x[2+2] - x[1+2]$

SHIFT BY 2

ALLOWS EASIER CALCULATION OF PROBS - SEE EX 16.5

<u>MORE EXAMPLES</u>

1) WHITE GAUSSIAN NOISE (WGN) - USED EXTENSIVELY IN RADAR/SONAR/COMMUNICATIONS

DTCV R.P., $X[n]$ IS IID R.P.
WITH $X[n] \sim N(0, \sigma^2)$ $-\infty < n < \infty$
SEE FIG 16.5b FOR REALIZATION.

NOTE: $E[X[n_0]] = 0$ "NOISE"
AVERAGE POWER $= E(X^2[n_0]) = VAR(X[n_0]) = \sigma^2$
PDF IS

$$p_{X[n_1],\dots,X[n_N]}(x_1,\dots,x_N) = \prod_{i=1}^{N} p_{X[n_i]}(x_i)$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x_i^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} x_i^2}$$

ALSO $$= \frac{1}{(2\pi)^{N/2} DET^{1/2}(\underline{C})} e^{-\frac{1}{2}\underline{x}^T \underline{C}^{-1}\underline{x}}$$

FOR $\underline{C} = \sigma^2 \underline{I}$ OR $N(\underline{0}, \sigma^2 \underline{I})$

CALLED <u>WHITE</u> GAUSSIAN NOISE SINCE ITS POWER IS EQUALLY DISTRIBUTED IN FREQUENCY (~ WHITE LIGHT) - CHAPTER 17, EX. 17.9

2) MOVING AVERAGE (MA) R.P.

$$X[n] = \tfrac{1}{2}(U[n] + U[n-1]) \qquad -\infty < n < \infty$$

$$X[0] = \tfrac{1}{2}(U[0] + U[-1])$$

$$X[1] = \tfrac{1}{2}(U[1] + U[0])$$

$$X(2) = \tfrac{1}{2}\left(U[2] + U[1]\right)$$

$\vdots$

AVERAGING "MOVES" IN TIME. $U[n]$ IS WGN WITH VARIANCE σ_U^2.

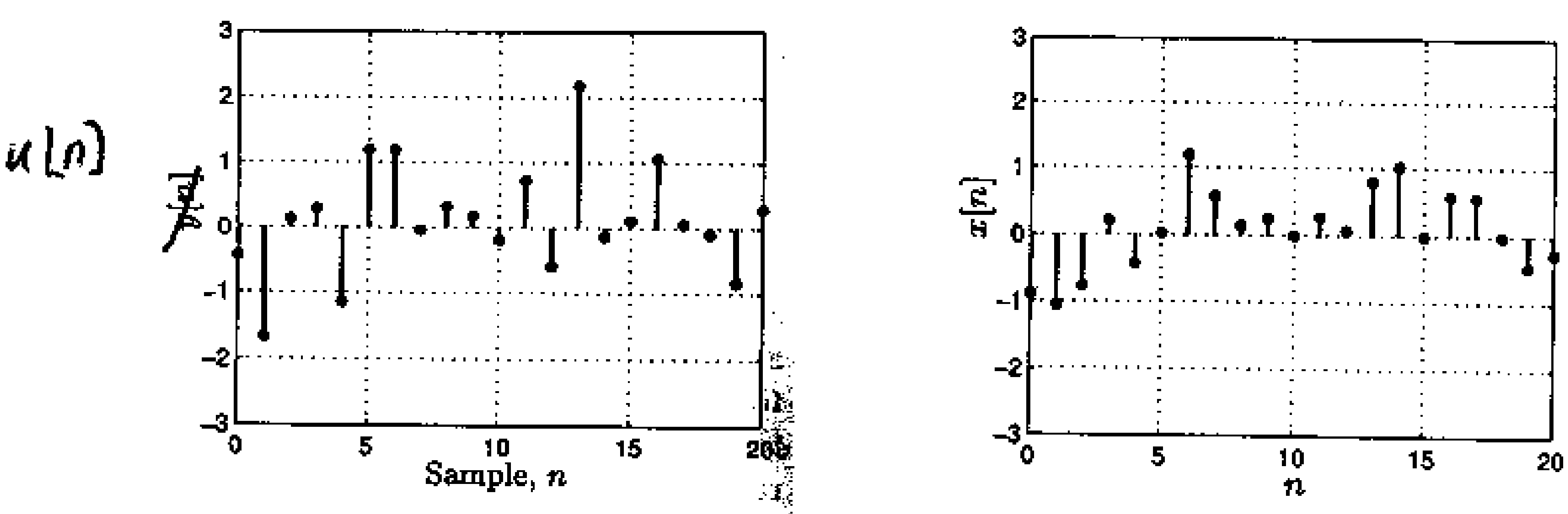


Figure 16.9: Typical realization of moving average random process. The realization of the $U[n]$ random process is shown in Figure 16.5b.

NOTE THAT $X[n]$ IS "SMOOTHER" (AVERAGER ACTS AS A LINEAR FILTER)

TO FIND JOINT PDF OF $X[n]$ NOTE THAT TRANSFORMATION FROM $U[n]$ TO $X[n]$ IS LINEAR. FOR EXAMPLE,

$$\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} U[-1] \\ U[0] \\ U[1] \end{bmatrix}$$

$$\underline{X} \quad = \quad \underline{G} \quad \underline{U}$$

RECALL FOR WGN $\underline{U} \sim N(\underline{0}, \sigma^2 \underline{I})$

$$\Rightarrow \quad \underline{X} = \underline{G}\,\underline{U} \sim N(\underline{0}, \underline{G}\,\underline{C}_U\,\underline{G}^T)$$

SINCE $E[\underline{X}] = E[\underline{G}\,\underline{U}] = \underline{G}\,\underbrace{E[\underline{U}]}_{\underline{0}} = \underline{0}$

ALSO $\underline{G}\,\underline{C}_v\underline{G}^T = \underline{G}\,\sigma^2\underline{I}\,\underline{G}^T = \sigma^2\underline{G}\,\underline{G}^T$

AND

$$\underline{G}\,\underline{G}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\begin{pmatrix} X[0] \\ X[1] \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2/2 & \sigma^2/4 \\ \sigma^2/4 & \sigma^2/2 \end{bmatrix} \right)$$

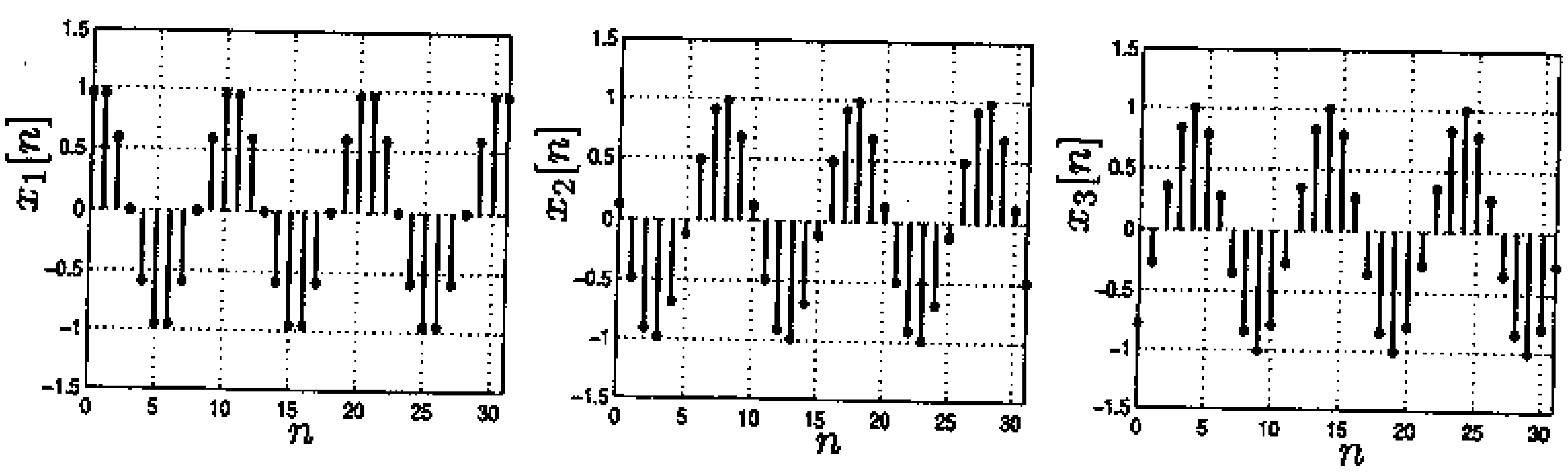
ALSO CAN SHOW THAT THIS R.P. IS STATIONARY.

__EXAMPLE__: RANDOMLY PHASED SINUSOID

$$X[n] = \cos(2\pi(0.1)n + \Theta) \quad -\infty < n < \infty$$

WHERE $\Theta \sim U(0, 2\pi)$

IN MATLAB
```
n = [0:31]'
x = cos(2*pi*0.1*n + 2*pi*rand(1,1))
```



(a) $\theta = 5.9698$ (b) $\theta = 1.4523$ (c) $\theta = 3.8129$

Figure 16.10: Typical realizations for randomly phased sinusoid.

NEARLY DETERMINISTIC – CAN PREDICT FUTURE SAMPLES PERFECTLY BASED ON PAST.