JOINT MOMENTS

FOR RV $E[X]$, $VAR(X)$ IMPORTANT
FOR TWO RVS, ALSO NEED $COV(X,Y)$.
FOR R.P.S WE WILL PAY PARTICULAR ATTENTION TO FIRST TWO MOMENTS

DEFINE MEAN SEQUENCE

$$\mu_X[n] = E[X[n]] \qquad -\infty < n < \infty$$

VARIANCE SEQUENCE

$$\sigma_X^2[n] = VAR(X[n]) \qquad -\infty < n < \infty$$

COVARIANCE SEQUENCE

$$\begin{aligned} c_X[n_1,n_2] &= COV(X[n_1], X[n_2]) \\ &= E[(X[n_1]-E[X[n_1]])(X[n_2]-E[X[n_2]])] \\ &= E[(X[n_1]-\mu_X[n_1])(X[n_2]-\mu_X[n_2])] \\ &\qquad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty \end{aligned}$$

USUAL DEFINITIONS FOR RVS!

WE NOW DO NOT USE $E_{\underbrace{X_1,X_2}_{OMIT}}[\ ]$, SHOULD BE CLEAR, FOR EXAMPLE,

$$E[X[n_1]X[n_2]] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \, p_{X[n_1],X[n_2]}(x_1,x_2)\,dx_1\,dx_2$$

SOME OBVIOUS PROPERTIES:

$$C_X[n_2, n_1] = C_X[n_1, n_2] \qquad \text{WHY?}$$
$$C_X[n, n] = \sigma_X^2[n] \qquad \text{WHY?}$$

SEE SIMILAR DEFINITIONS FOR CONT.-TIME R.P.

<u>EXAMPLE</u>: WGN $X[n] \sim N(0, \sigma^2)$ AND IID

$$\Rightarrow \quad \mu_X[n] = E[X[n]] = 0 \qquad -\infty < n < \infty$$
$$\sigma_X^2[n] = VAR[X[n]] = \sigma^2 \qquad -\infty < n < \infty$$

$$C_X[n_1, n_2] = COV(X[n_1], X[n_2])$$
$$= E[X[n_1]X[n_2]] \qquad \text{MEAN} = 0$$
$$= \begin{cases} E[X[n_1]]E[X[n_2]] & n_1 \neq n_2 \text{ (IND)} \\ E[X^2[n_1]] & n_1 = n_2 \end{cases}$$

$$= \begin{cases} 0 & n_1 \neq n_2 \\ \sigma^2 & n_1 = n_2 \end{cases}$$
$$= \sigma^2 \delta[n_2 - n_1] \qquad \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

DISCRETE IMPULSE

<u>EXAMPLE</u>: MA R.P. $X[n] = \frac{1}{2}(U[n] + U[n-1])$

U ↑ WGN $N(0, \sigma_U^2)$

$$\mu_X[n] = E[X[n]]$$
$$= \frac{1}{2}(E(U[n]) + E[U[n-1]]) = 0$$
$$-\infty < n < \infty$$

$$C_X[n_1, n_2] = E[X[n_1]X[n_2]] \qquad \text{MEAN} = 0$$

$$= \frac{1}{4} E\left[(v[n_1] + v[n_1-1])(v[n_2] + v[n_2-1]) \right]$$
$$= \frac{1}{4} E\left[v[n_1] v[n_2] \right] + \frac{1}{4} E\left[v[n_1] v[n_2-1] \right]$$
$$+ \frac{1}{4} E\left[v[n_1-1] v[n_2] \right] + \frac{1}{4} E\left[v[n_1-1] v[n_2-1] \right]$$

BUT $E[v[k] v[l]] = \sigma_v^2 \delta[l-k]$

$$c_X[n_1, n_2] = \frac{1}{4}\left(\sigma_v^2 \delta[n_2-n_1] + \sigma_v^2 \delta[n_2-n_1-1] + \sigma_v^2 \delta[n_2-n_1+1] + \sigma_v^2 \delta[n_2-n_1] \right)$$

$$= \frac{\sigma_v^2}{2} \delta[n_2-n_1] + \frac{\sigma_v^2}{4} \delta[n_2-n_1-1] + \frac{\sigma_v^2}{4} \delta[n_2-n_1+1]$$

NOTE: DEPENDS ONLY ON $n_2 - n_1 = \Delta n$

$$= \frac{\sigma_v^2}{2} \delta[\Delta n] + \frac{\sigma_v^2}{4} \delta[\overbrace{\Delta n - 1}^{=0 \text{ UNLESS}}] + \frac{\sigma_v^2}{4} \delta[\underbrace{\Delta n + 1}_{=0 \text{ UNLESS}}]$$

$\Delta n - 1 = 0 \Rightarrow \Delta n = 1$

$\Delta n + 1 = 0 \Rightarrow \Delta n = -1$

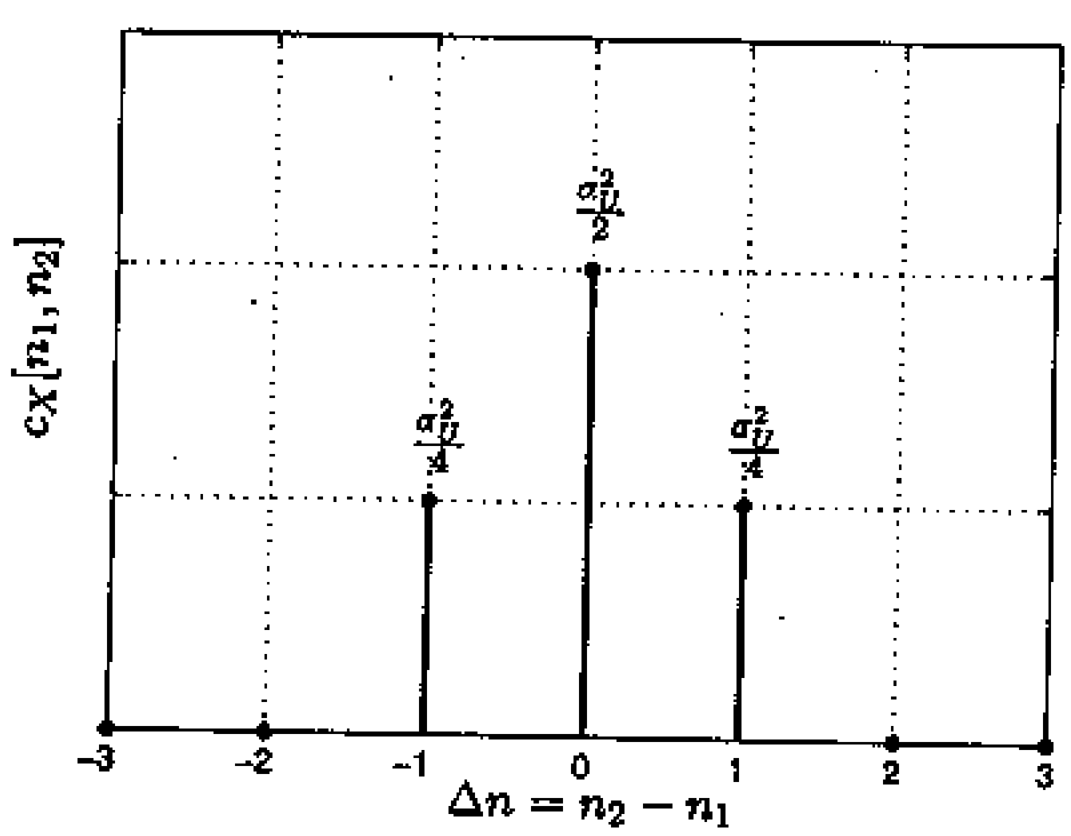


Figure 16.13: Covariance sequence for moving average random process.

= COVARIANCE BETWEEN TWO $X[n]$ RP SAMPLES SPACED Δn TIME SAMPLES APART

$\Rightarrow$ IF TWO R.P. SAMPLES SPACED MORE THAN 1 SAMPLE APART $\Rightarrow$ UNCORRELATED

SINCE $x[0] = \frac{1}{2}(v[0] + v[-1])$
$x[2] = \frac{1}{2}(v[2] + v[1])$ WHY?

<u>EXAMPLE</u>: RANDOMLY PHASE SINUSOID

$$\mu_x[n] = E[x[n]] = E[\cos(2\pi(0.1)n + \Theta)]$$

$$= \int_0^{2\pi} \underbrace{\cos(2\pi(0.1)n + \theta)}_{g(\theta)} \underbrace{\frac{1}{2\pi}}_{p(\theta)} d\theta = 0$$

$$c_x[n_1, n_2] = E[x[n_1]x[n_2]]$$
$$= \int_0^{2\pi} \underbrace{\cos(2\pi(0.1)n_1 + \theta)\cos(2\pi(0.1)n_2 + \theta)}_{g(\theta)} \cdot \frac{1}{2\pi} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}\cos(2\pi(0.1)(n_2 - n_1)) + \frac{1}{2}\cos(2\pi(0.1)(n_1 + n_2) + 2\theta)\right] \cdot \frac{d\theta}{2\pi}$$

↑ NOT DEPENDENT ON θ

$$= \frac{1}{2}\cos(2\pi(0.1)(n_2 - n_1)) + \underbrace{\frac{1}{2}\frac{1}{4\pi}\sin(2\pi(0.1)(n_1 + n_2) + 2\theta)\Big|_0^{2\pi}}_{=0}$$

$$= \frac{1}{2}\cos(2\pi(0.1)\underbrace{(n_2 - n_1)}_{\Delta n})$$

AGAIN COVARIANCE ONLY DEPENDS ON SPACING BETWEEN SAMPLES, Δn.

NOTE: $c_x[n_1, n_2] = f(\Delta n)$ AND IS SYMMETRIC (EVEN SEQUENCE)

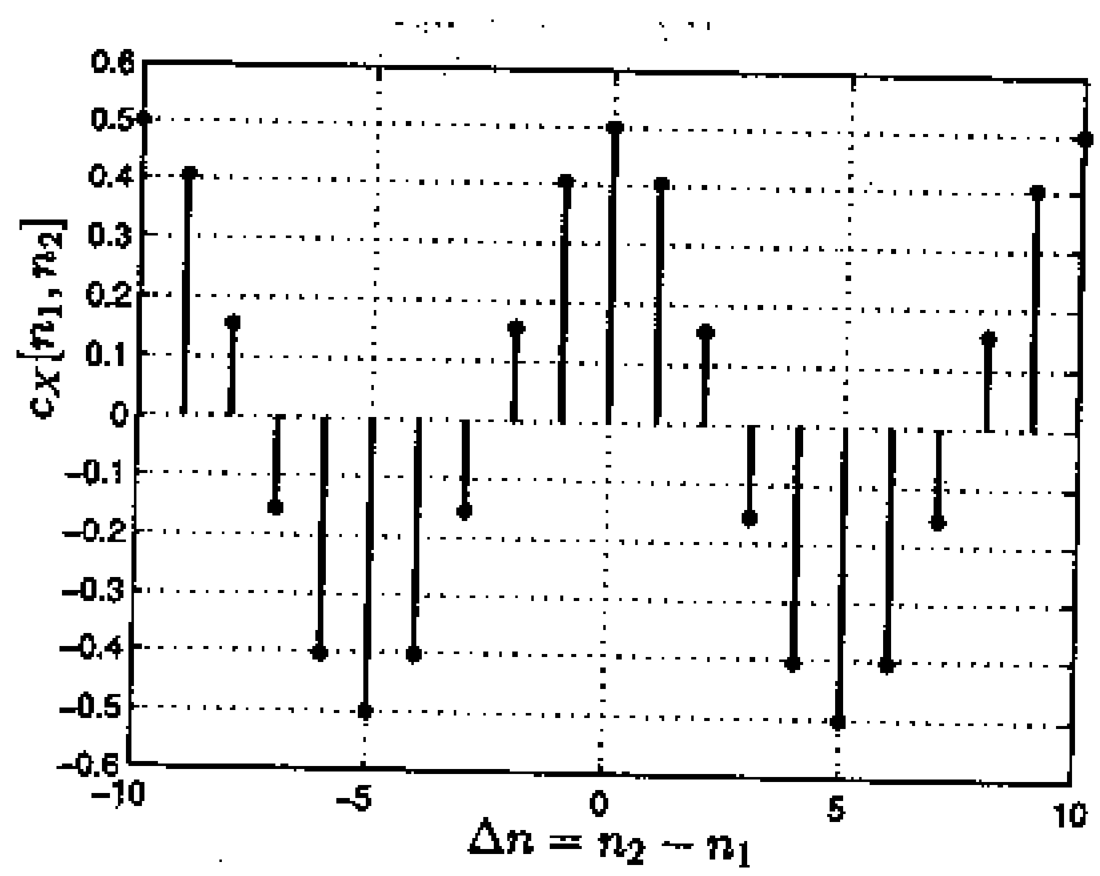
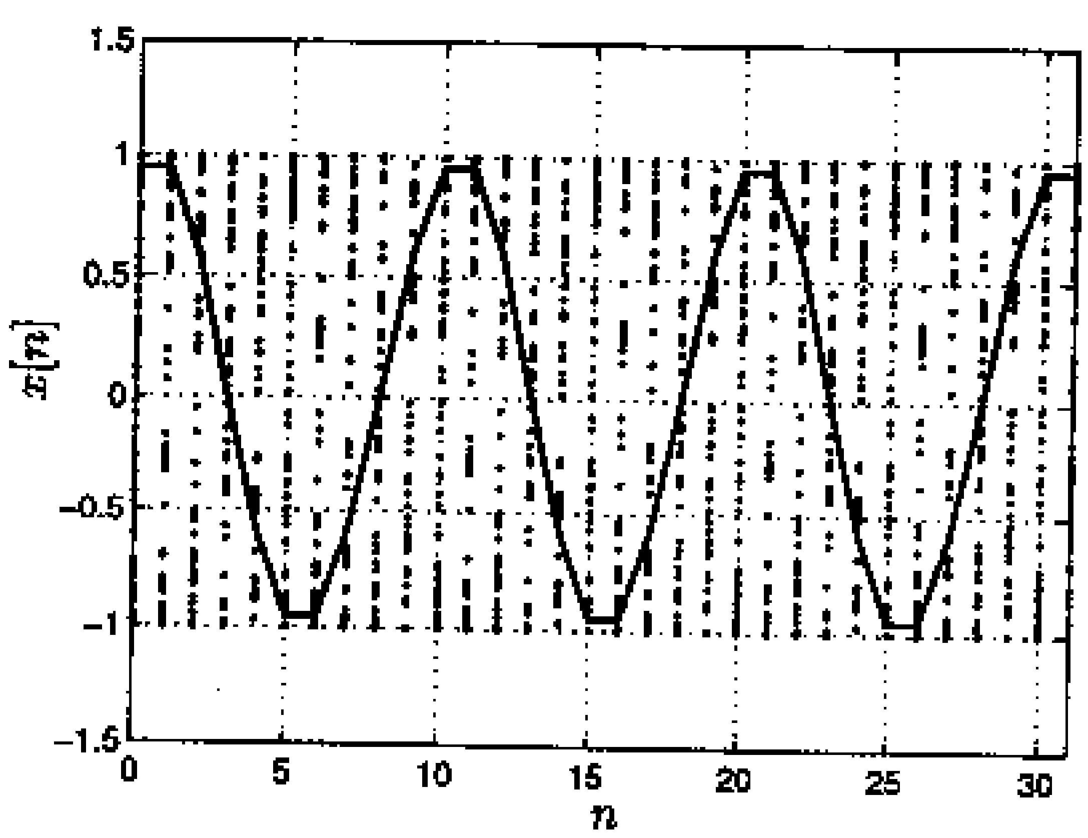


Figure 16.14: Covariance sequence for randomly phased sinusoid.

$f(-\Delta n) = f(\Delta n)$

HOW FAR APART DO SAMPLES HAVE TO BE SO THAT THEY ARE UNCORRELATED?

WHY IS MEAN ZERO HERE?



SCATTER DIAGRAM

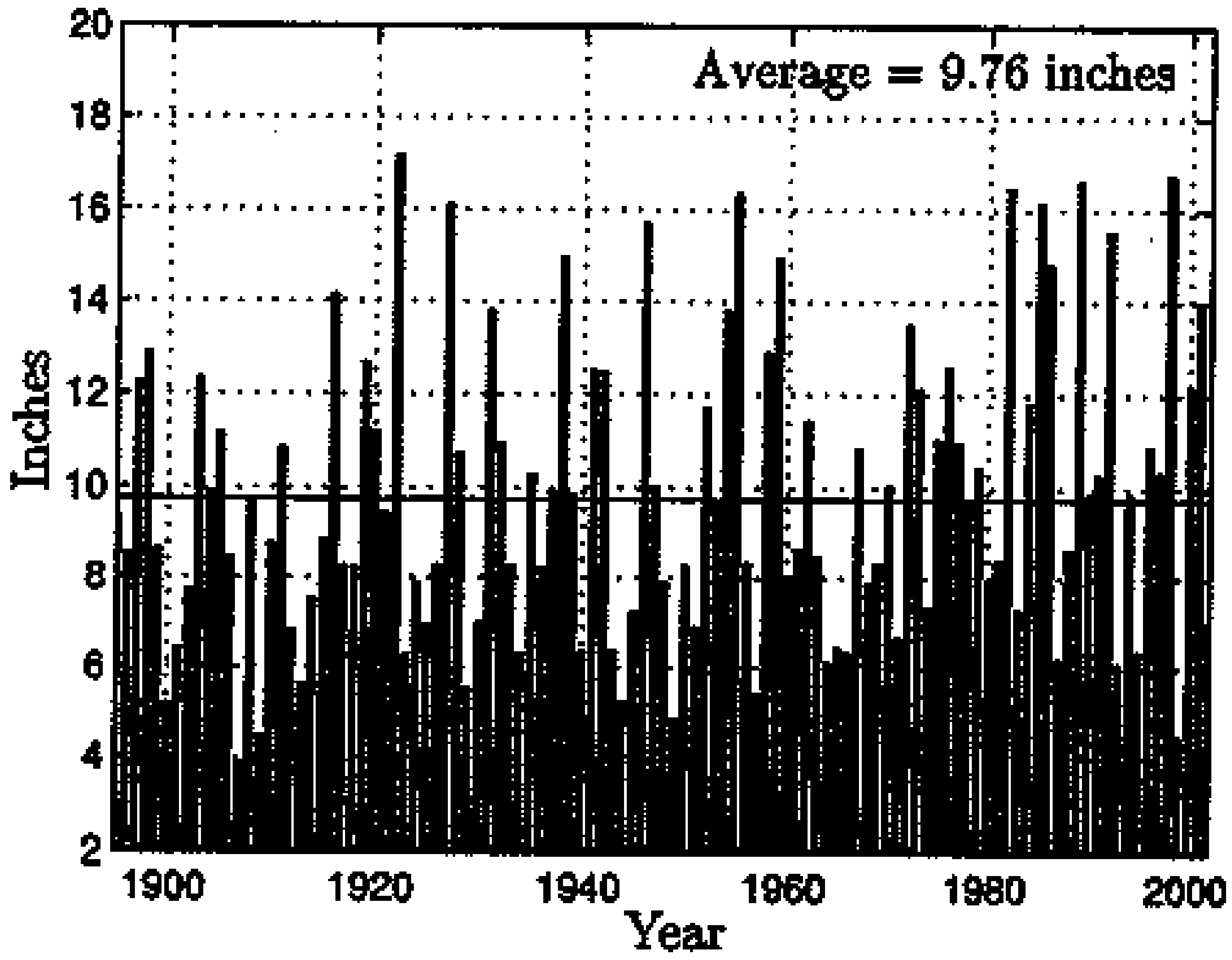
: 16.15: Fifty realizations of randomly phased sinusoid plotted in an overlaid ; with one realization shown with its points connected by straight lines.

REAL WORLD EXAMPLE - RAINFALL IN RI

QUESTION: IS RAINFALL IN RI INCREASING?

$\Rightarrow$ POTENTIAL INDICATOR OF GLOBAL WARMING

ANALYZE LAST 100 YEARS OF ANNUAL SUMMER RAINFALL FOR A TREND



AVERAGE OBTAINED USING

$$\bar{x} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

$n=0 \Rightarrow 1895$

$n=107 \Rightarrow 2002$

Figure 16.1: Annual summer rainfall in Rhode Island from 1895 to 2002.

IS $\mu_X[n] = \mu \approx 9.76$ CORRECT?

MAYBE $\mu_X[n] = an+b$ FOR SOME a, b (NONSTATIONARY R.P.)?

APPROACH: ASSUME $\mu_X[n] = an+b$ AND ESTIMATE a, b. IS $a=0$? IF $a>0$ RAINFALL INCREASING, AND IF $a<0$, RAINFALL DECREASING.

TO ESTIMATE a, b USE LEAST SQUARES (TAKE ELE 661 - ESTIMATION THEORY TO LEARN MORE!)

MINIMIZE LEAST SQUARES ERROR

$$J(a,b) = \sum_{n=0}^{N-1} (x[n]-(an+b))^2$$

OVER $a, b \Rightarrow \hat{a}, \hat{b}$

NOTE: IF WE ASSUMED $\mu_X[n] = \mu = b$
(NO CHANGE IN MEAN),

$$J(b) = \sum_{n=0}^{N-1} (x[n]-b)^2$$

$$\frac{\partial J}{\partial b} = 0 \Rightarrow \hat{b} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] = 9.76$$

NOW,

$$\frac{\partial J}{\partial b} = -2 \sum_n (x[n]-an-b) = 0$$

$$\frac{\partial J}{\partial a} = -2 \sum_n (x[n]-an-b)\, n = 0$$

$$\sum_n an + \sum_n b = \sum_n x[n]$$

$$\sum_n an^2 + \sum_n bn = \sum_n n\, x[n]$$

$$\begin{bmatrix} N & \sum_n n \\ \sum_n n & \sum_n n^2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \sum_n x[n] \\ \sum_n n\, x[n] \end{bmatrix}$$

SOLVE $\Rightarrow$ $\hat{a} = 0.0173$, $\hat{b} = 8.8336$

$$\hat{\mu}_X[n] = 0.0173n + 8.8336$$

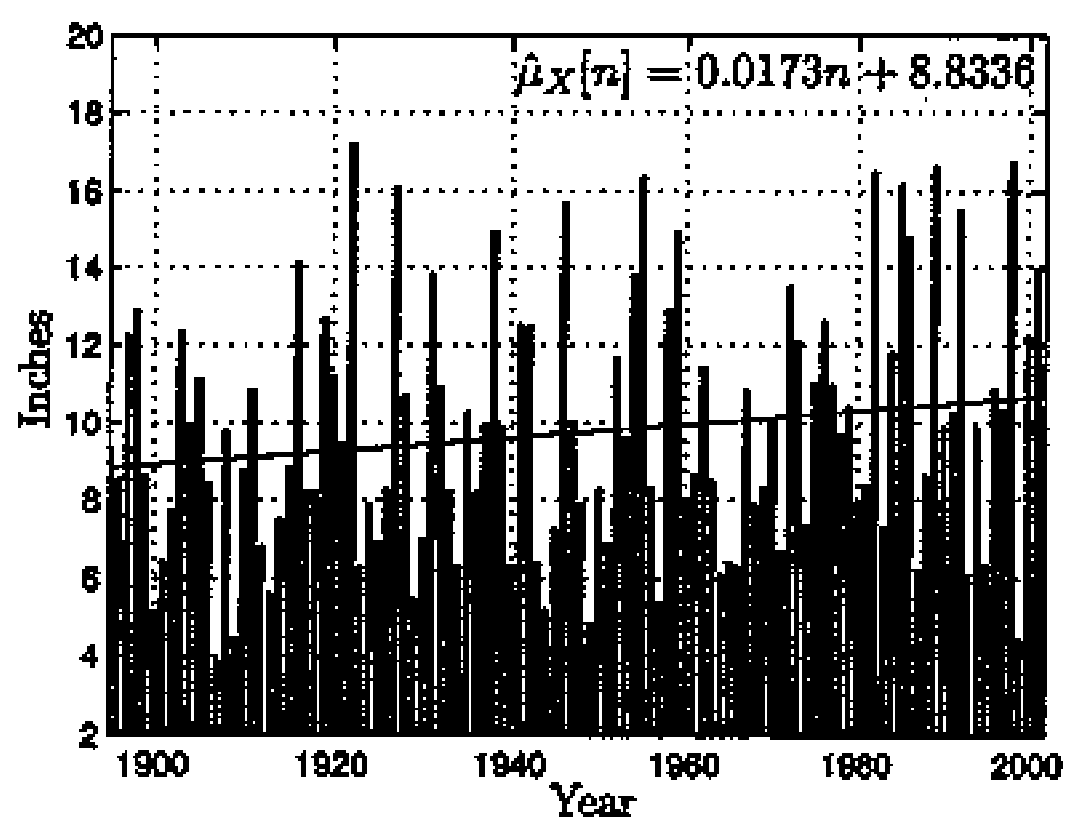


Figure 16.16: Annual summer rainfall in Rhode Island and the estimated mean sequence, $\hat{\mu}_X[n] = 0.0173n + 8.8336$, where $n = 0$ corresponds to the year 1895.

IS $a > 0$?
MAYBE WAIT ANOTHER 100 YEARS TO SEE IF $\hat{a} > 0$ WITH MORE DATA?

TO SEE IF $\hat{a} > 0$ DUE TO "STATISTICAL ERROR", LET $x[n] = \underbrace{an + b}_{\mu_x[n]} + e[n]$

↑ ZERO MEAN ERROR

ASSUME $e[n] \sim N(0, \sigma^2)$

WHAT SHOULD WE USE FOR σ^2?

$$J(\hat{a}, \hat{b}) = \sum_n (x[n] - (\hat{a}n + \hat{b}))^2$$

$$= \sum_n (an + b + e[n] - (\hat{a}n + \hat{b}))^2$$

$$\approx \sum_n e^2[n] \quad \text{IF} \quad \hat{a} \approx a, \ \hat{b} \approx b$$

USE $$\sigma^2 = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{(x[n] - (\hat{a}n + \hat{b}))}_{\approx e[n]}{}^2$$

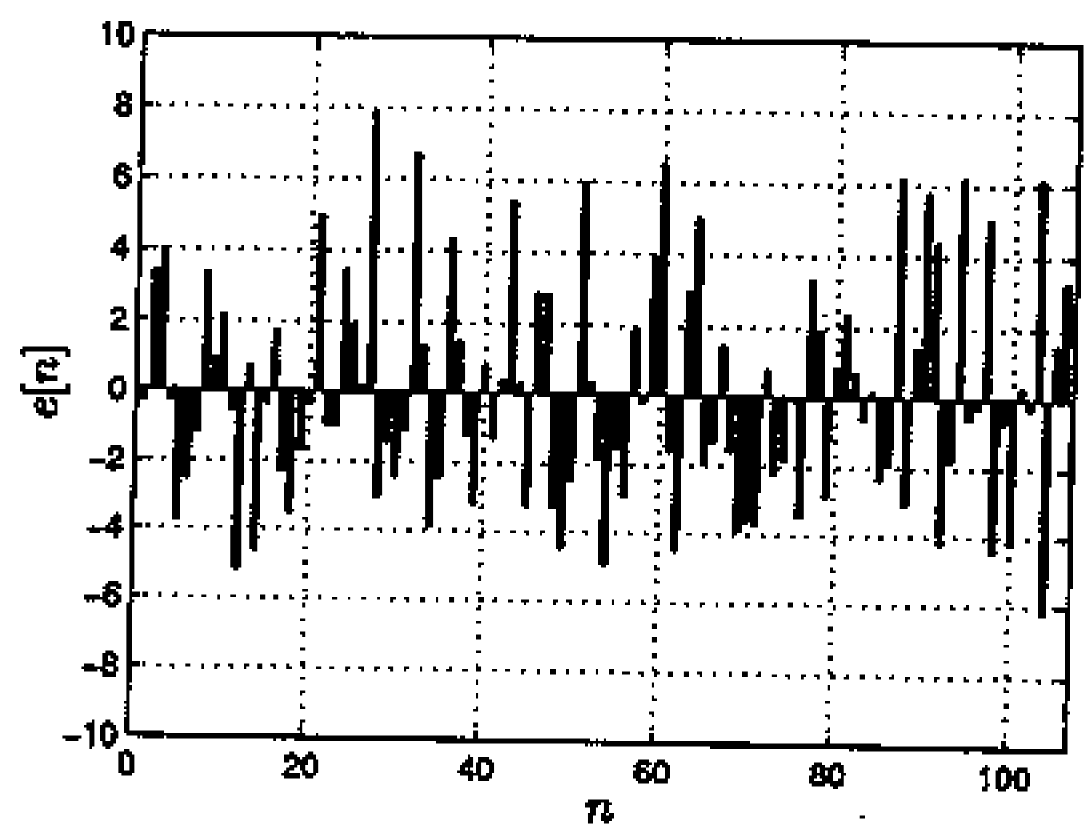


Figure 16.17: Least squares error sequence for annual summer rainfall in Rhode Island fitted with a straight line.

$\sigma^2 = 10.05$

$\Rightarrow$ WE ASSUME NOW THAT

$x[n] = \overset{9.76}{b} + u[n]$

$u[n] \sim N(0, 10.05)$

OR CONSTRAIN $a = 0$

COULD WE STILL OBSERVE INCREASE IN MEAN RAINFALL BASED ON <u>ESTIMATED</u> $\hat{a}, \hat{b}$ EVEN THOUGH $a = 0$?

OR IS $\hat{a} = 0.0173$ DUE TO ERROR IN ESTIMATING a BECAUSE OF $u[n]$?

$$\begin{bmatrix} N & \sum_n n \\ \sum_n n & \sum_n n^2 \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} \sum_n (b + u[n]) \\ \sum_n n(b + u[n]) \end{bmatrix}$$

IF $u[n] = 0$
$\Rightarrow \hat{b} = b$
$\hat{a} = 0$

IF $u[n] \neq 0$, $\hat{a} \neq 0$.

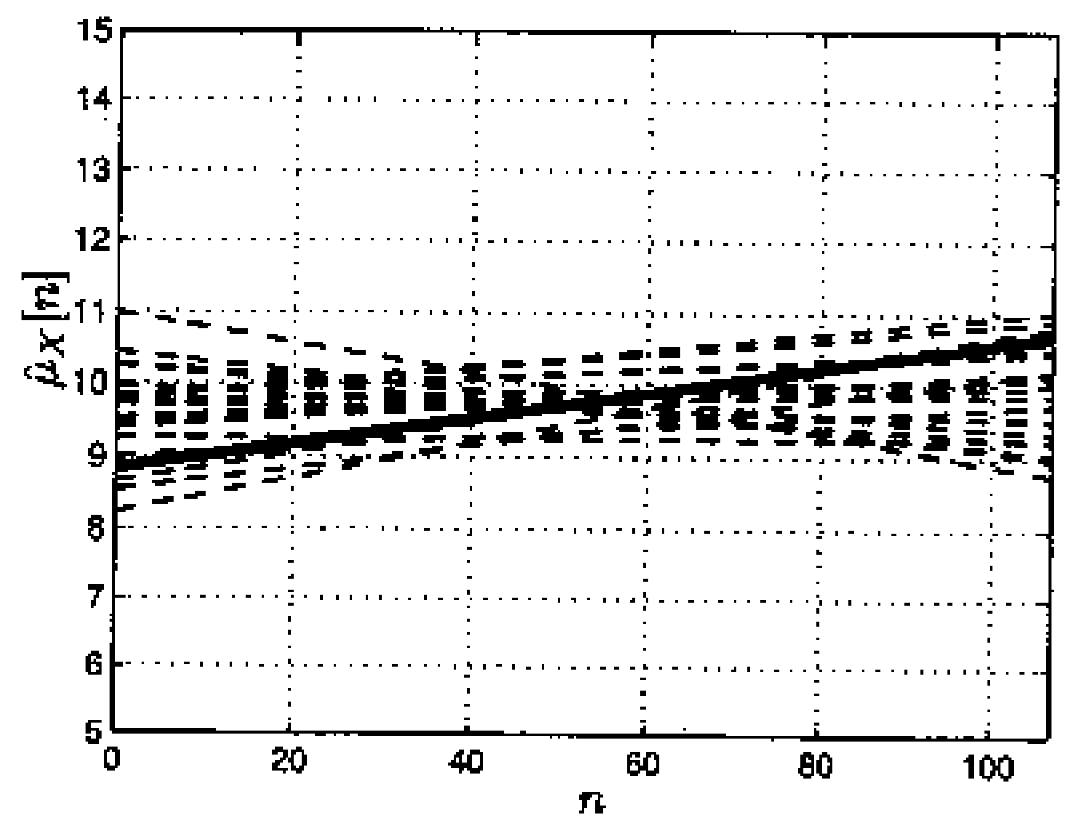


Figure 16.18: Twenty realizations of the estimated mean sequence $\hat{\mu}_X[n] = \hat{a}n + \hat{b}$ based on the random process $X[n] = 9.76 + U[n]$ with $U[n]$ being WGN with $\sigma^2 = 10.05$. The realizations are shown as dashed lines. The estimated mean sequence from Figure 16.16 is shown as the solid line.

TRUE a IS <u>ZERO</u>!

NOT EVIDENT FROM DATA, THOUGH.

CHAPTER 17 - WIDE SENSE STAT. (WSS) R.P.

WILL ALLOW US TO ANALYZE <u>FREQUENCY CONTENT OF R.P.</u> - SIMILAR TO FOURIER TRANSFORM OF DETERMINISTIC SIGNAL

HAVE ALREADY ENCOUNTERED WSS R.P.s, WGN, MA PROCESS, RANDOMLY PHASED SINUSOID

A WSS R.P. IS DEFINED AS ONE WHOSE

$\mu_x[n] = \mu$ (CONSTANT) $\quad -\infty < n < \infty$

$C_x[n_1, n_2] = g(|n_2 - n_1|) \quad -\infty < n_1 < \infty$

$-\infty < n_2 < \infty$

$\Rightarrow$ 1) MEAN CONSTANT

2) COVARIANCE ONLY DEPENDS ON SPACING BETWEEN R.P. SAMPLES

EXAMPLE : MA PROCESS

$\mu_x[n] = 0 \quad (\mu = 0)$

$$C_x[n_1, n_2] = \begin{cases} \frac{1}{2}\sigma_v^2 & |n_2 - n_1| = 0 \\ \frac{1}{4}\sigma_v^2 & |n_2 - n_1| = 1 \\ 0 & |n_2 - n_1| > 1 \end{cases}$$

NOTE THAT $VAR(x[n]) = \sigma_x^2[n]$

$= C_x[n, n] = \frac{1}{2}\sigma_v^2$ OR

VARIANCE IS ALSO CONSTANT

A WSS R.P. IS A SPECIAL CASE OF STATIONARY R.P. OR

STAT. R.P $\Rightarrow$ WSS

WSS $\not\Rightarrow$ STAT R.P. (IN GENERAL)

WSS MUCH MORE USEFUL - ONLY INVOLVES CONSTRAINTS ON FIRST TWO MOMENTS.

PROOF : ASSUME $x[n]$ IS STATIONARY OR

$$p_{x[n_1+n_0], \ldots, x[n_N+n_0]} = p_{x[n_1], \ldots, x[n_N]}$$

FOR ALL $n_1, n_2, \ldots, n_N$, ALL N, AND ALL n_0.

NOW LET $N=1$ AND $n_1 = n$

$\Rightarrow \quad p_{x(n+n_0)} = p_{x(n)}$

LET $n=0 \Rightarrow p_{x(n_0)} = p_{x(0)}$ ALL n_0

PDF CAN'T DEPEND ON TIME $\Rightarrow \mu_x(n) = \mu$ FOR $-\infty < n < \infty$.

NEXT LET $N=2$

$$p_{x(n_1+n_0),\, x(n_2+n_0)} = p_{x(n_1),\, x(n_2)}$$

LET $n_0 = -n_1$

$\Rightarrow \quad p_{x(0),\, x(n_2-n_1)} = p_{x(n_1),\, x(n_2)}$

$\Rightarrow E[x(n_1)x(n_2)] = E[x(0)x(n_2-n_1)]$

LET $n_0 = -n_2$

$\Rightarrow \quad p_{x(n_1-n_2),\, x(0)} = p_{x(n_1),\, x(n_2)}$

$\Rightarrow E(x(n_1)x(n_2)) = E[x(0)x(n_1-n_2)]$

OR COMBINING RESULTS

$$E(x(n_1)x(n_2)) = E[x(0)x(|n_2-n_1|)]$$

SO $c_x(n_1, n_2) = E(x(n_1)x(n_2)) - E\{x[n_1]\}E\{x(n_2)\}$

$$= E(x(0)x(|n_2-n_1|)) - \mu^2$$

$$= g(|n_2-n_1|)$$

AUTOCORRELATION SEQUENCE

ASSUME HENCEFORTH THAT $x(n)$ IS WSS.
$E(x(n_1)x(n_2))$ DEPENDS ONLY ON $|n_2-n_1|$.
LET $n_1 = n$, $n_2 = n+k$

$E(x(n_1)x(n_2)) = E(x(n)x(n+k))$ DEPENDS ONLY ON k (AND MORE GENERALLY ON $|k|$)

$\Rightarrow$ DEFINE $r_x(k) = E(x[n]x(n+k))$ $\quad -\infty < k < \infty$

CALLED THE AUTOCORRELATION SEQUENCE (ACS)

NOTE: NOT DEPENDENT ON n, ALTHOUGH USED IN DEFINITION

k = TIME DIFFERENCE BETWEEN SAMPLES
CALLED THE LAG

ACS MEASURES CORRELATION (WILL SEE LATER) BETWEEN WSS R.P. SAMPLES

EXAMPLE: DIFFERENCER

$$x[n] = v[n] - v[n-1]$$

$$v[n] \sim N(\mu, \sigma_v^2) \quad \text{IID}$$

$$r_x[k] = E[x[n]x[n+k]]$$
$$= E[(v[n]-v[n-1])(v[n+k]-v[n+k-1])]$$
$$= E[v[n]v[n+k]] - E[v[n]v[n+k-1]]$$
$$- E[v[n-1]v[n+k]] + E[v[n-1]v[n+k-1]]$$

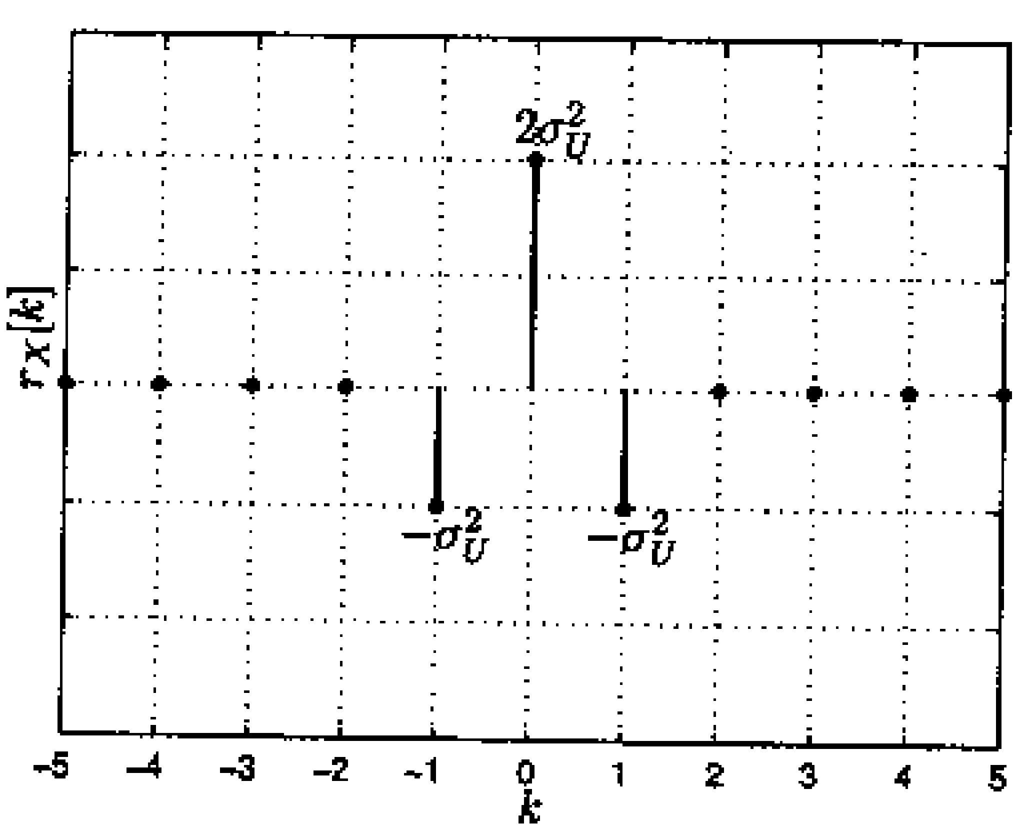
FOR $n_1 \neq n_2$, $E[v[n_1]v[n_2]] = E[v[n_1]]E[v[n_2]]$
$= \mu^2$ WHY?

FOR $n_1 = n_2 = n$, $E[v[n_1]v[n_2]] = E[v^2[n]]$
$= E[v^2[0]] = \sigma_v^2 + \mu^2$
WHY?

$$\Rightarrow E[v[n_1]v[n_2]] = \mu^2 + \sigma_v^2\,\delta(n_2 - n_1)$$

$$r_x[k] = \mu^2 + \sigma_v^2\delta[k] - [\mu^2 + \sigma_v^2\delta(k-1)]$$
$$- [\mu^2 + \sigma_v^2\delta(k+1)] + \mu^2 + \sigma_v^2\delta[k]$$

$$= 2\sigma_v^2\,\delta[k] - \sigma_v^2\,\delta[k-1] - \sigma_v^2\,\delta[k+1]$$



WHAT IS CORRELATION COEFFICIENT BETWEEN 2 SUCCESSIVE SAMPLES?

Figure 17.4: Autocorrelation sequence for differenced random process.

BY DEFINITION

$$\rho_{x[n],x[n+1]} = \frac{\operatorname{cov}(x[n],x[n+1])}{\sqrt{\operatorname{VAR}(x[n])\operatorname{VAR}(x[n+1])}}$$

SINCE $x[n]$ HAS ZERO MEAN (WHY?),

$$\rho_{x[n],x[n+1]} = \frac{E[x[n]x[n+1]]}{\sqrt{E[x^2[n]]\,E[x^2[n+1]]}}$$

$$= \frac{r_x[1]}{\sqrt{r_x[0]\,r_x[0]}} = \frac{r_x[1]}{r_x[0]}$$

(ONLY DEPENDS ON SPACING BETWEEN R.P. SAMPLES!)

HERE, $\rho_{x[n],x[n+1]} = \frac{-\sigma_U^2}{2\sigma_U^2} = -\frac{1}{2}$

$\Rightarrow$ NEGATIVE CORRELATION

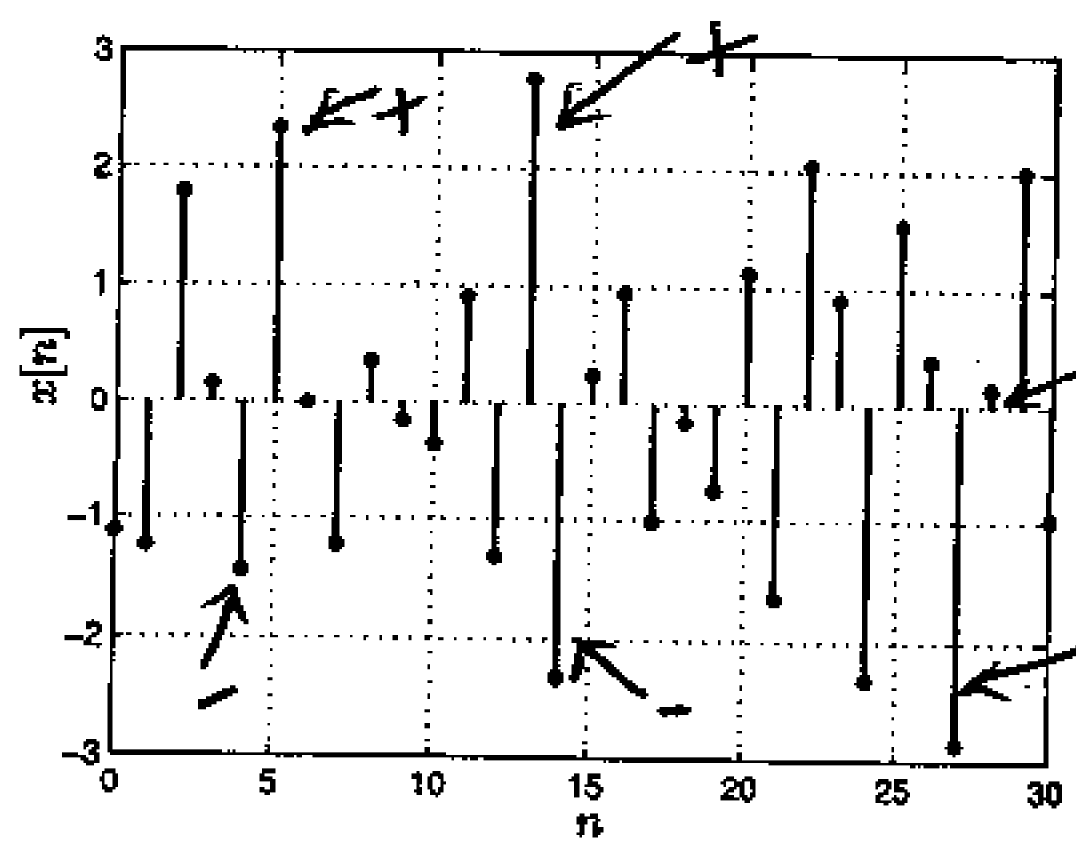


Figure 17.3: Typical realization of a differenced IID Gaussian random process with $U[n] \sim \mathcal{N}(1,1)$.

HENCE, IF STOCK PRICE GOES DOWN ON MON., THEN CALL BROKER ON TUES. AND ____?

SOME OBSERVATIONS ABOUT ACS:

1) $r_x[0] > 0$

2) $r_x[-k] = r_x[k]$ EVEN SEQUENCE

3) $|r_x[k]| \leq r_x[0]$

PROOF: 1) $r_x[0] > 0$

$r_x[k] = E\{x[n]x[n+k]\}$

$r_x[0] = E\{x^2[n]\} > 0$

= AVERAGE POWER ACROSS 1Ω RESISTOR IF $x[n]$ = VOLTAGE

DOESN'T CHANGE WITH TIME

2) $r_x[-k] = r_x[k]$ EVEN SEQUENCE

PROOF: $r_x[k] = E\{x[n]x[n+k]\}$

$r_x[-k] = E\{x[n]x[n-k]\}$

LET $n = m+k$, n IS ARBITRARY

$$\begin{aligned} \Rightarrow r_x[-k] &= E\{x[m+k]x[m]\} \\ &= E\{x[m]x[m+k]\} \quad \text{NOT DEPENDENT ON } m. \\ &= E\{x[n]x[n+k]\} \\ &= r_x[k] \end{aligned}$$

3) MAXIMUM ABSOLUTE VALUE AT $k=0$, $|r_x[k]| \leq r_x[0]$

CAUCHY-SCHWARZ INEQUALITY SAYS:

$$|E_{V,W}[VW]| \leq \sqrt{E_V[V^2]E_W[W^2]}$$

EQUALITY HOLDS IF AND ONLY IF
$W = cV$ c A CONSTANT
(LIKE $|\underline{v} \cdot \underline{w}| \leq \|\underline{v}\| \|\underline{w}\|$ EUCLIDEAN VECTORS)

PROOF: LET $V = x[n]$, $W = x[n+k]$

$$|E[x[n]x[n+k]]| \leq \sqrt{E[x^2(n)]E[x^2[n+k]]}$$

$$|r_x[k]| \leq \sqrt{r_x[0]\, r_x[0]} = |r_x[0]|$$

$$= r_x[0]$$

ALREADY SHOWED THAT FOR ZERO MEAN WSS R.P.

$$\rho_{x[n],x[n+k]} = \frac{r_x[k]}{r_x[0]}$$

NOW KNOW THAT AS EXPECTED

$$|\rho_{x[n],x[n+k]}| \leq 1 \quad \text{WHY?}$$

ACS EXAMPLES

1) WHITE NOISE - DEFINED AS WSS R.P. WITH ZERO MEAN, IDENTICAL VARIANCE σ^2, AND UNCORRELATED SAMPLES

NO MENTION OF PDFS HERE ⇒ NOT NEC-

ESSARILY WGN. $x[n]$'S DO NOT HAVE TO BE INDEPENDENT, ONLY UNCORRELATED.

$$r_x[k] = E[x[n]x[n+k]]$$
$$= E[x[n]]E[x[n+k]] \quad \text{FOR } k \neq 0$$

DUE TO <u>UNCORRELATED</u> ASSUMPTION

$(\text{cov}(x,y) = 0 \Rightarrow E[xy] - E[x]E[y] = 0)$

$r_x[k] = 0 \qquad k \neq 0$ SINCE $E[x[n]] = 0$ ALL n

FOR $k = 0$

$$r_x[k] = E[x^2[n]] = \text{VAR}(x[n]) = \sigma^2$$

$$\therefore \quad r_x[k] = \sigma^2 \delta[k]$$

<u>ASIDE</u>: $C_x[n_1, n_2]$ FOUND FROM $r_x[k]$ AND μ AS

$$C_x[n_1, n_2] = E[x[n_1]x[n_2]] - E[x[n_1]]E[x[n_2]]$$
$$= r_x[n_2 - n_1] - \mu^2$$

ALSO, THEN IF $n_1 = n$, $n_2 = n+k$

$$C_x[n, n+k] = r_x[k] - \mu^2$$

THUS, IF TWO SAMPLES ARE UNCORRELATED AS $k \to \infty$ THEN $C_x[n, n+k] \to 0$ AS $k \to \infty$, AND $r_x[k] \to \mu^2$ AS $k \to \infty$.

MOST PHYSICAL PROCESSES HAVE THIS PROPERTY.

2) MA R.P.

RECALL $C_x[n_1, n_2) = \begin{cases} \frac{\sigma_U^2}{2} & n_1 = n_2 \\ \frac{\sigma_U^2}{4} & |n_2 - n_1| = 1 \\ 0 & |n_2 - n_1| > 1 \end{cases}$

FROM ASIDE: $r_x(k) = C_x(n, n+k) + \mu^2$

BUT ~~HERE~~ $\mu = 0$

$$r_x(k) = \begin{cases} \frac{\sigma_U^2}{2} & k = 0 \\ \frac{\sigma_U^2}{4} & k = \pm 1 \\ 0 & |k| > 1 \end{cases}$$

NOTE: $r_x(k)$ HAS $r_x[0) > 0$, $|r_x[k)| \leq r_x(0)$

$r_x[-k) = r_x(k]$.

3) RANDOMLY PHASED SINUSOID

AGAIN $\mu = 0$, AND

$$C_x(n_1, n_2) = \frac{1}{2}\cos[2\pi(0.1)(n_2 - n_1))$$

$$\Rightarrow \quad r_x(k) = \frac{1}{2}\cos[2\pi(0.1)k]$$

4) AUTOREGRESSIVE R.P.

USED EXTENSIVELY AS MODEL FOR PHYSICAL PROCESSES

A SPECIAL CASE OF AUTOREGRESSIVE (AR) R.P. DEFINED AS

$$x(n) = a\,x(n-1) + u(n) \qquad -\infty < n < \infty$$

WHERE $|a| < 1$ AND $u(n)$ IS WGN WITH VARIANCE σ_u^2.

NOTE THAT THE EQUATION IS A <u>FIRST-ORDER RECURSIVE DIFFERENCE EQUATION.</u>

RP EVOLVES AS:

$$\vdots$$
$$x[0] = a\,x[-1] + u[0]$$
$$x[1] = a\,x[0] + u[1]$$
$$x[2] = a\,x[1] + u[2]$$
$$\vdots$$

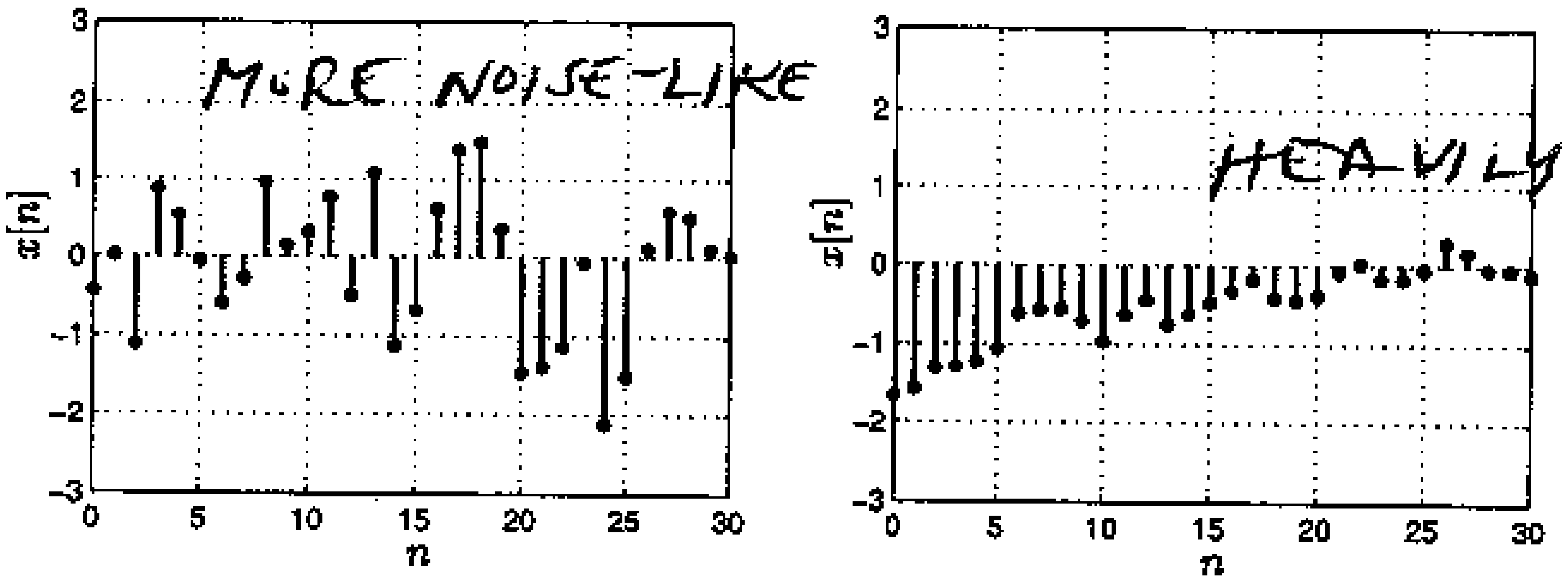
AND $x[n]$ DEPENDS ONLY ON PRESENT AND PAST SAMPLES OF $u[n]$. FOR EXAMPLE,

$$x[2] = a\,x[1] + u[2]$$
$$= a(a\,x[0] + u[1]) + u[2] = a^2 x[0] + a u[1] + u[2]$$

$$= a^2(a\,x[-1] + u[0]) + a\,u[1] + u[2]$$
$$= a^3 x[-1] + a^2 u[0] + a u[1] + u[2]$$
$$= \sum_{k=0}^{\infty} a^k u[2-k]$$

LEADING TERM $a^3 x(-1)$, ETC $\to 0$ SINCE $|a| < 1$.

CLEARLY, WE MUST HAVE $|a|<1$ FOR A STABLE RECURSION.

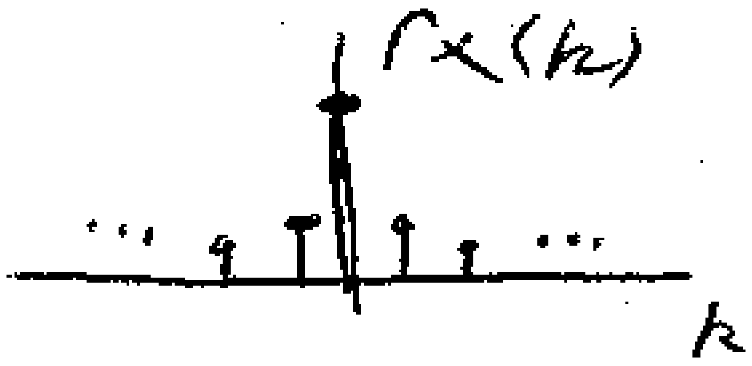


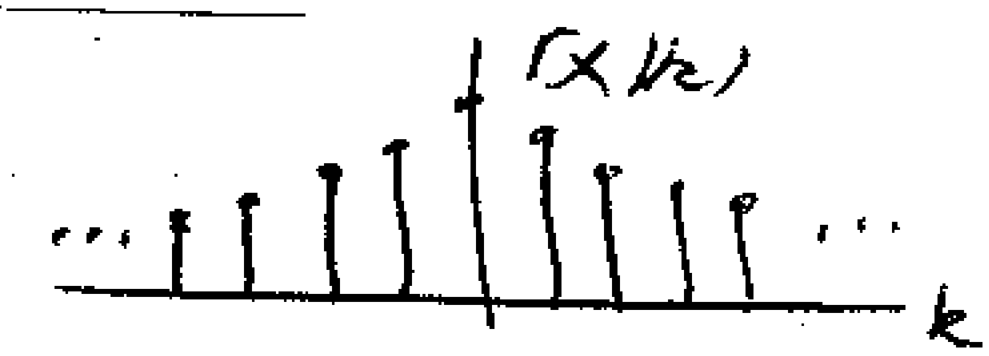
(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$

(b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.5: Typical realizations of autoregressive random process with different parameters.

EXPECT





WHY?

```
clear all
randn('state',0)
a1=0.25;a2=0.98;
varu1=1-a1^2;varu2=1-a2^2;
varx1=varu1/(1-a1^2);varx2=varu2/(1-a2^2); % this is r_X[0]
x1(1,1)=sqrt(varx1)*randn(1,1); % set initial condition X[-1]
                                % see Problems 17.17, 17.18
x2(1,1)=sqrt(varx2)*randn(1,1);
for n=2:31
   x1(n,1)=a1*x1(n-1)+sqrt(varu1)*randn(1,1);
   x2(n,1)=a2*x2(n-1)+sqrt(varu2)*randn(1,1);
end
```

TO DETERMINE ACS:

$$r_X[k] = E(X[n]X[n+k])$$
$$= E(X[n](aX[n+k-1]+U[n+k]))$$
$$= aE[X[n]X[n+k-1]] + E[X[n]U[n+k]]$$
$$= a\,r_X[k-1] + E[X[n]U[n+k]]$$

BUT $X[n] = \sum_{l=0}^{\infty} a^l U[n-l]$