FOR $k > 0$

$$E[x(n)u(n+k)] = E\left[\sum_{l=0}^{\infty} a^l u(n-l)u(n+k)\right]$$

$$= \sum_{l=0}^{\infty} a^l E[u(\underset{\uparrow \atop \geq 0}{n-l})u(\underset{\uparrow \atop > 0}{n+k})] \qquad \Rightarrow -l \leq 0, \quad k > 0$$

CANNOT HAVE $-l = k$ $\Rightarrow$

$$E[u[n-l]u[n+k]] = 0$$

$$r_x(k) = a\, r_x(k-1) \qquad k > 0$$

SOLVING $r_x(1) = a\, r_x(0)$

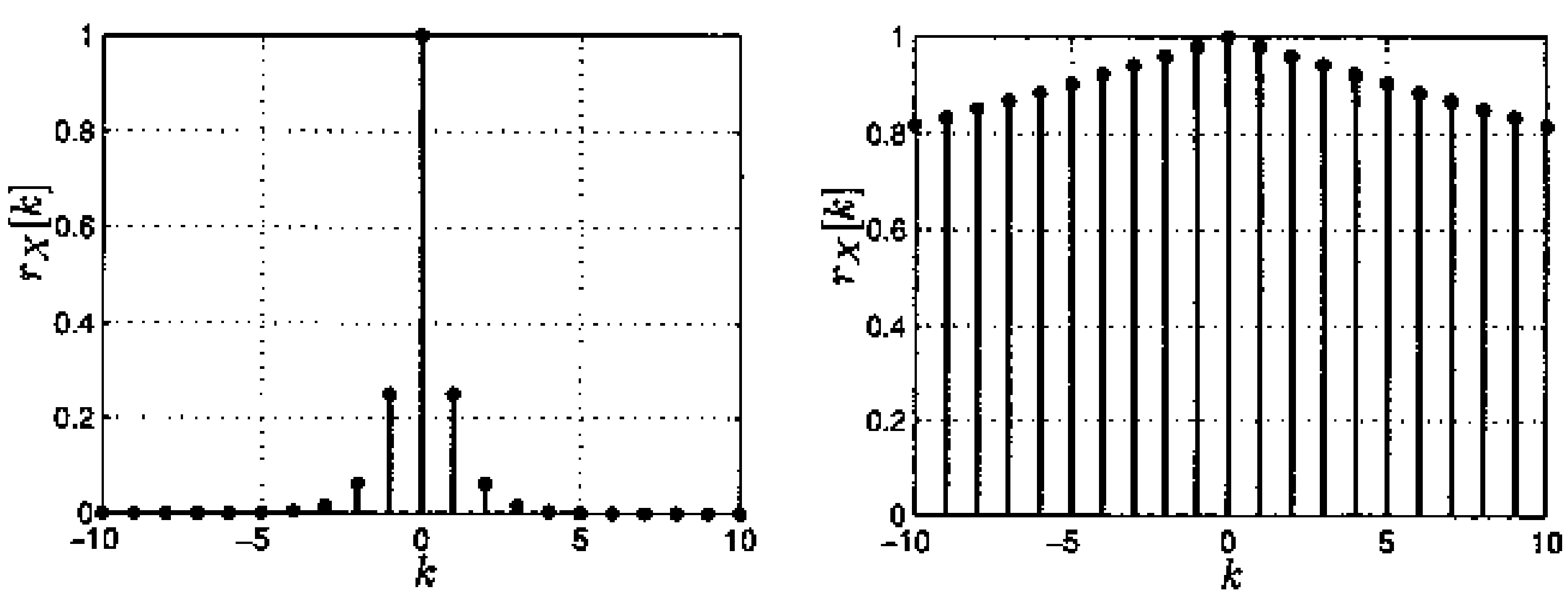
$$r_x(2) = a\, r_x(1) = a(a\, r_x(0)) = a^2 r_x(0)$$

$$\vdots$$

$$r_x(k) = a^k r_x(0)$$

AND CAN BE SHOWN THAT $r_x(0) = \dfrac{\sigma_u^2}{1-a^2}$

$$\therefore \; r_x(k) = \frac{\sigma_u^2}{1-a^2} a^{|k|} \leftarrow$$ ACS MUST BE EVEN



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

AS EXPECTED!

Figure 17.6: The autocorrelation sequence for autoregressive random processes with different parameters.

ONE LAST PROPERTY OF ACS IS POSITIVE DEFINITENESS. RECALL THAT COVARIANCE MATRIX IS P.D. SINCE

$$\mathrm{VAR}(Y) = \mathrm{VAR}(a_0 x[0] + a_1 x[1])$$ SEE SLIDE 110

$$= [a_0 \; a_1] \underbrace{\begin{bmatrix} \mathrm{VAR}(x[0]) & \mathrm{COV}(x[0], x[1]) \\ \mathrm{COV}(x[1], x[0]) & \mathrm{VAR}(x[1]) \end{bmatrix}}_{\underline{C}_x} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} > 0$$

FOR ALL a_0, a_1 (NOT $a_0 = a_1 = 0$ THOUGH)

NOW ASSUME $x[n]$ IS ZERO MEAN

$\Rightarrow$

$$\underline{C}_x = \begin{bmatrix} E[x^2[0]] & E[x[0]x[1]] \\ E(x[1]x[0]) & E[x^2[1]] \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} r_x[0] & r_x[1] \\ r_x[1] & r_x[0] \end{bmatrix}}_{\underline{R}_x \text{ - CALLED THE AUTOCORRELATION MATRIX}}$$

$\Rightarrow$ $\underline{R}_x$ IS P.D., HENCE ALL PRINCIPAL MINORS > 0 OR

$r_x[0] > 0$ ALREADY KNOW THIS

$\mathrm{DET}(\underline{R}_x) = r_x^2[0] - r_x^2[1] > 0$

$\Rightarrow$ $r_x[0] > |r_x[1]|$ ALREADY KNOW THIS

BUT MORE GENERALLY

IF $\underline{x} = [x[0]\ x[1] \ldots x[N-1]]^T$

$$\underline{R}_x = \begin{bmatrix} r_x[0] & r_x[1] & \ldots & r_x[N-1] \\ r_x[1] & r_x[0] & \ldots & r_x[N-2] \\ & & \ddots & \\ r_x[N-1] & r_x[N-2] & & r_x[0] \end{bmatrix}$$

AND IF $\underline{a} = [a_0\ a_1 \ldots a_{N-1}]^T$

$$\underline{a}^T \underline{R}_x \underline{a} > 0 \quad \text{FOR ALL } \underline{a} \neq \underline{0}$$

PLACES LARGE NUMBER OF CONSTRAINTS ON VALID ACS. ACS IS SAID TO BE POSITIVE SEMIDEFINITE SEQUENCE.

↑ MOST GENERAL CONDITION

$\underline{a}^T \underline{R}_x \underline{a} \geq 0$

ERGODICITY AND TEMPORAL AVERAGES

ASSUME $x[n]$ IS WSS WITH $E[x[n]] = \mu$. IF WE OBSERVE A SINGLE REALIZATION, SAY $x_1[n]$, CAN WE DETERMINE μ?

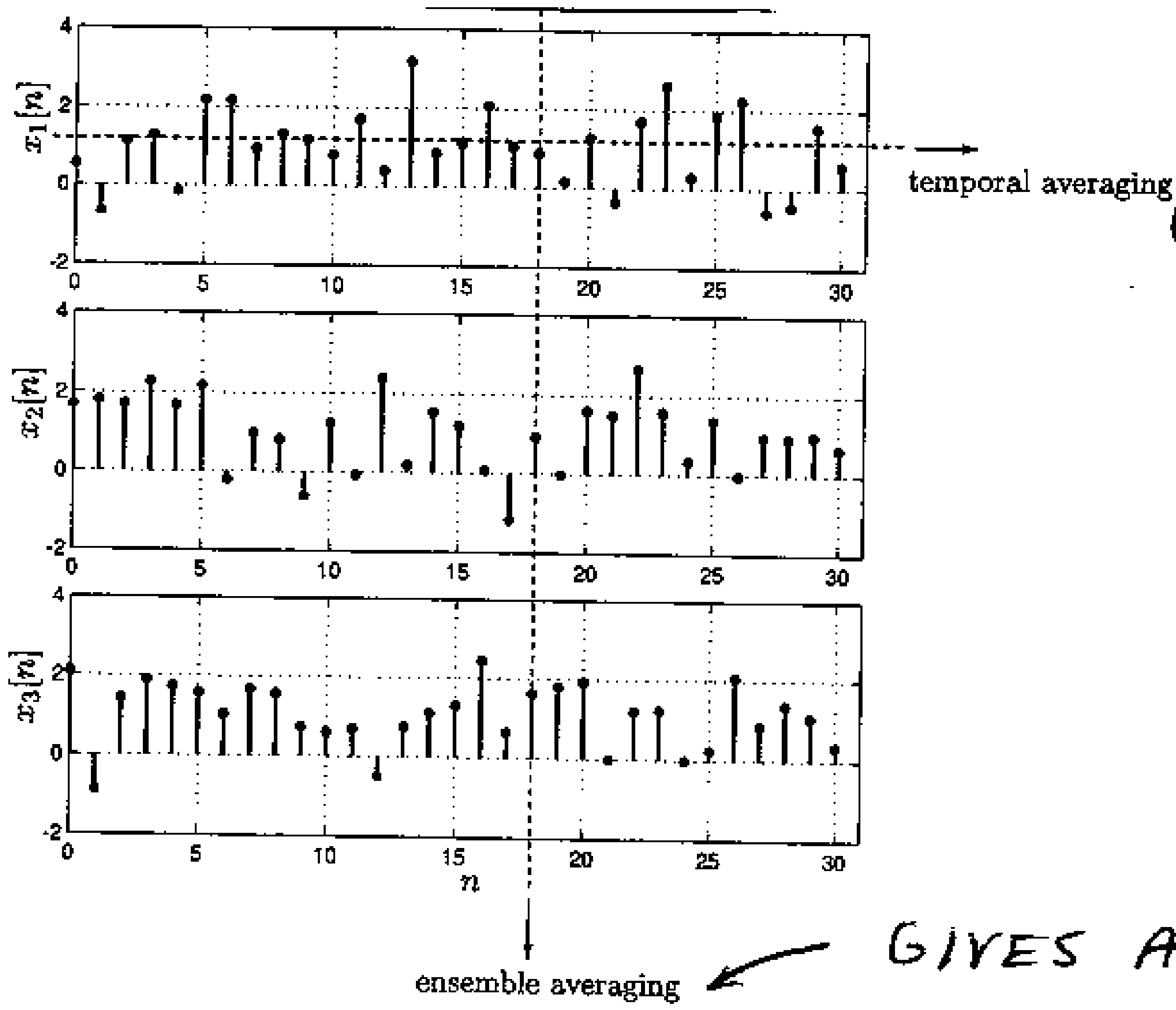
OR $\hat{\mu}_N = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] \rightarrow \mu$ AS $N \rightarrow \infty$?

WHAT IS μ? $\mu = E[x[n]]$ FOR n FIXED $\Rightarrow$ IT IS MEAN OF R.V. $x[n]$, FOR EXAMPLE, $\mu = E[x[18]]$

REPRESENTS AVERAGE VALUE OF $x[18]$
OVER ALL OUTCOMES OF $x[18]$ OR

$$E[x[18]] = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} x_m[18]$$

m^{th} OUTCOME OF $x[18]$



GIVES $\frac{1}{N} \sum_{n=0}^{N-1} x_1[n]$ AS $N \to \infty$

EQUAL ??

GIVES ABOVE LIMIT $= E[x[18]]$

Figure 17.7: Several realizations of WSS random process with $\mu_X[n] = \mu = 1$. Vertical dashed line indicates "ensemble averaging" while horizontal dashed line indicates "temporal averaging."

NOTE THAT IN TEMPORAL AVERAGING WE ONLY USE ONE OUTCOME. ALSO, OBVIOUS THAT FOR $\frac{1}{N} \sum_{n=0}^{N-1} x_1[n] \to \mu$, MUST HAVE

$E[x[n]] = \mu$ FOR ALL n. WHY?

IF TEMPORAL AVERAGE = ENSEMBLE AVERAGE,

THEN WE SAY THAT

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} x_m[18] = \mu$$

ANY S REALIZATION IN GENERAL

ANY n IN GENERAL

WHEN WILL $\hat{\mu}_N = \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} x[n]}_{\text{SAMPLE MEAN}} \to \mu$?

CERTAINLY FOR IID R.P. - WHY?

IN GENERAL, SINCE

$$E(\hat{\mu}_N) = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{E(x(n))}_{\mu} = \mu$$

WE REQUIRE $VAR(\hat{\mu}_N) \to 0$. FOR IID R.P. $VAR(\hat{\mu}_N) = \sigma^2/N \to 0$. IN THIS CASE R.P. IS SAID TO BE <u>ERGODIC IN THE MEAN</u>.

<u>EXAMPLE</u> : GENERAL MA R.P., PDF OF $u[n]$ NOT SPECIFIED, ALSO $E[u(n)] \neq 0$.

$$x[n] = \frac{1}{2}(u[n] + u[n-1])$$

$$\left.\begin{array}{l} E(u[n]) = \mu \\ VAR(u[n]) = \sigma_u^2 \end{array}\right\} -\infty < n < \infty$$

$u[n]$'S UNCORRELATED

CAN EASILY SHOW $x[n]$ IS WSS (SAME CALCULATIONS AS GAUSSIAN CASE).

WISH TO ESTIMATE $E(x[n])$ USING $\hat{\mu}_N$.

$$E(x[n]) = E\left(\tfrac{1}{2}(u[n]+u[n-1])\right) = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu = \mu$$

NEED TO FIND $VAR(\hat{\mu}_N)$ (MORE COMPLICATED SINCE SAMPLES OF $x[n]$ ARE CORRELATED)

$$VAR\Big(\underbrace{\tfrac{1}{N}}_{a_n}\sum_{n=0}^{N-1} x[n]\Big) = \underline{a}^T \underline{C}_x \underline{a}$$

SEE SLIDE 162 FOR $N=2$

$$\underbrace{[\underline{C}_x]_{ij}}_{(i,j)\text{ ELEMENT}} = E\left[(x[i]-E(x[i]))(x[j]-E(x[j]))\right]$$

$i = 0, 1, \ldots, N-1$

$j = 0, 1, \ldots, N-1$

BUT

$$x[n]-E(x[n]) = \tfrac{1}{2}(u[n]+u[n-1]) - \mu$$
$$= \tfrac{1}{2}\big(\underbrace{(u[n]-\mu)}_{\bar{u}[n]} + (u[n-1]-\mu)\big)$$
$$= \tfrac{1}{2}(\bar{u}[n] + \bar{u}[n-1])$$

$$E(\overbrace{\bar{u}[n]}^{\text{WHITE NOISE}}) = 0$$

$$[\underline{C}_x]_{ij} = \tfrac{1}{4}E\left[(\bar{u}[i]+\bar{u}[i-1])(\bar{u}[j]+\bar{u}[j-1])\right]$$

$$= \tfrac{1}{4}\underbrace{E(\bar{u}[i]\bar{u}[j])}_{\sigma_u^2\delta[j-i]} + \tfrac{1}{4}\underbrace{E(\bar{u}[i]\bar{u}[j-1])}_{\sigma_u^2\delta[j-1-i]}$$
$$+ \tfrac{1}{4}\underbrace{E[\bar{u}[i-1]\bar{u}[j])}_{\sigma_u^2\delta[j-i+1]} + \tfrac{1}{4}\underbrace{E[\bar{u}[i-1]\bar{u}[j-1])}_{\sigma_u^2\delta[j-i]}$$

SINCE $\bar{v}[n]$ IS WHITE NOISE.

$$(\underline{C}_x)_{ij} = \begin{cases} \frac{1}{2}\sigma_v^2 & i=j \\ \frac{1}{4}\sigma_v^2 & |i-j|=1 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\mathrm{VAR}(\hat{\mu}_N) = \underline{a}^T \underline{C}_x \underline{a}$$

$$= \left[\tfrac{1}{N}\ \tfrac{1}{N} \cdots \tfrac{1}{N}\right] \begin{bmatrix} \sigma_v^2/2 & \sigma_v^2/4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sigma_v^2/4 & \sigma_v^2/2 & \sigma_v^2/4 & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sigma_v^2}{4} & \frac{\sigma_v^2}{2} & \frac{\sigma_v^2}{4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\sigma_v^2}{4} & \frac{\sigma_v^2}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{N} \\ \frac{1}{N} \\ \vdots \\ \frac{1}{N} \end{bmatrix}$$

$$= \underbrace{\frac{1}{N^2}\sum_{i=0}^{N-1}\frac{\sigma_v^2}{2}}_{\text{DIAGONAL}} + \underbrace{\frac{1}{N^2}\sum_{i=0}^{N-2}\frac{\sigma_v^2}{4}}_{\text{UPPER DIAG.}} + \underbrace{\frac{1}{N^2}\sum_{i=1}^{N-1}\frac{\sigma_v^2}{4}}_{\text{LOWER DIAG.}}$$

$$= \frac{\sigma_v^2}{2N} + \frac{\sigma_v^2}{4}\frac{N-1}{N^2} + \frac{\sigma_v^2}{4}\frac{N-1}{N^2} \to 0 \quad \text{AS } N\to\infty$$

$\Rightarrow$ ERGODIC IN MEAN.

<u>COUNTEREXAMPLE</u> : RANDOM DC LEVEL

$$x[n] = A \quad -\infty < n < \infty$$

WHERE $A \sim N(0,1)$ WSS ?

$$\mu_x[n] = E[x[n]] = E[A] = 0 = \mu \quad -\infty < n < \infty$$

$$r_x[k] = E[x[n]x[n+k]]$$

$$= E[A^2] = 1 \quad \text{NOT DEPENDENT ON } n$$

$\Rightarrow$ WSS

$$\mathrm{VAR}(\hat{\mu}_N) = \underline{a}^T \underline{C}_x \underline{a} = \underline{a}^T \underline{R}_x \underline{a} \quad \text{WHY?}$$

$$= \left[\tfrac{1}{N}\ \tfrac{1}{N} \cdots \tfrac{1}{N}\right] \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \tfrac{1}{N} \\ \tfrac{1}{N} \\ \vdots \\ \tfrac{1}{N} \end{bmatrix} =$$

$$\left[\tfrac{1}{N}\ \tfrac{1}{N} \cdots \tfrac{1}{N}\right] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 1 \not\to 0 \quad \text{AS } N \to \infty$$

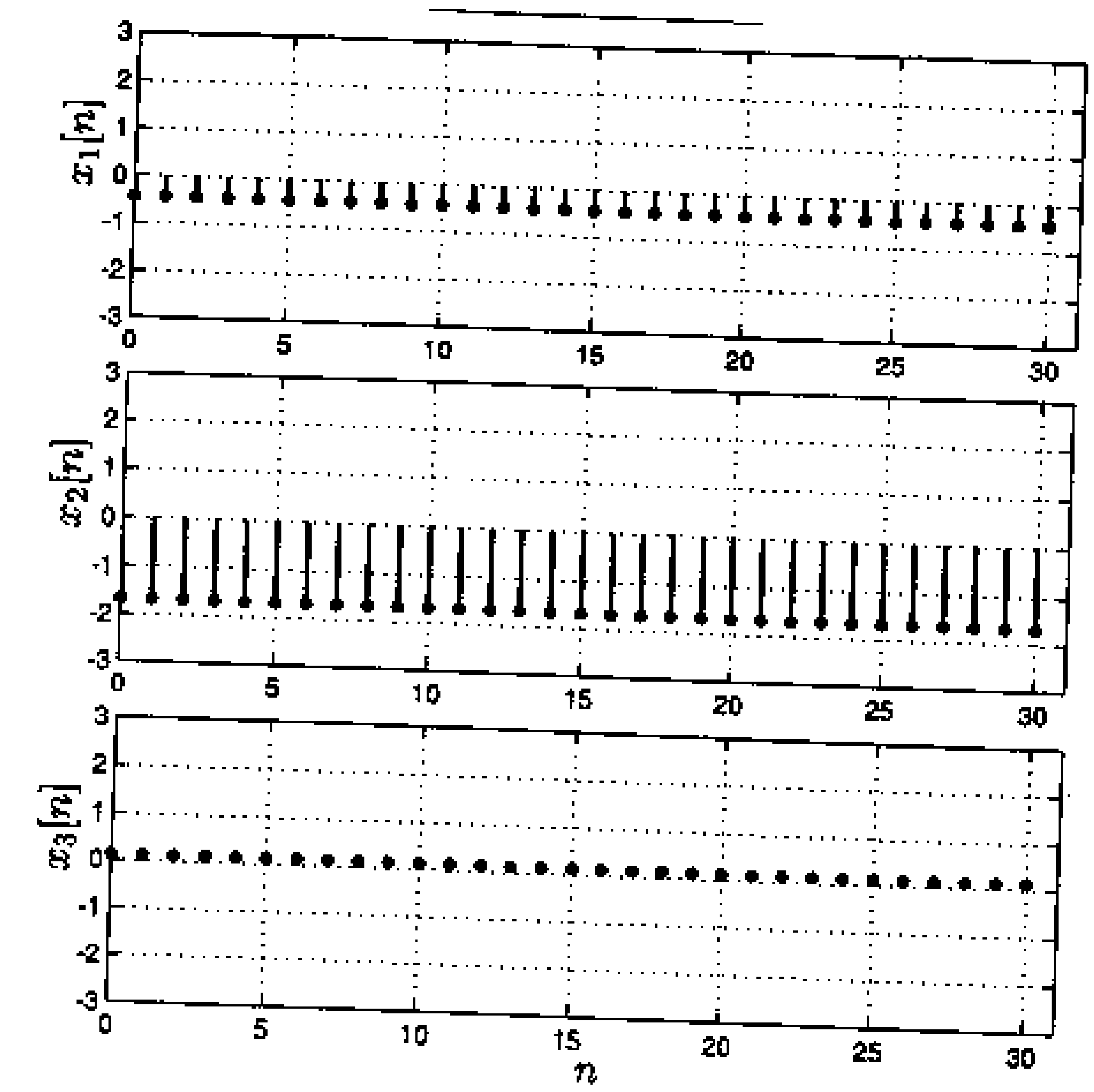


Figure 17.8: Several realizations of the random DC level process.

$\hat{\mu}_N = -0.43$

FOR ANY N

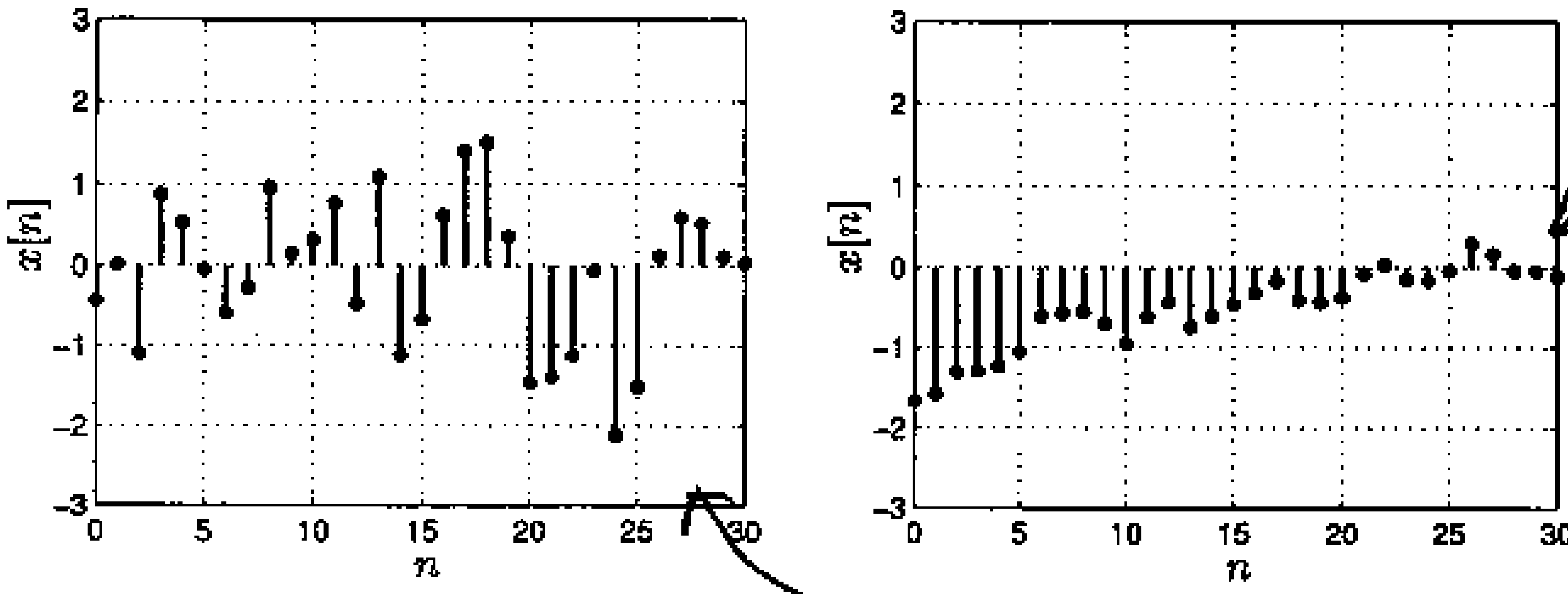
$\hat{\mu}_N \not\to 0 = \mu$

NOTE DIFFERENCE IN COVARIANCE MATRICES BETWEEN THIS AND MA EXAMPLE.

$$\text{NEED } \underbrace{\mathrm{cov}(x[n], x[n+k])}_{r_x[k] - \mu^2} \to 0 \quad \text{AS } k \to \infty.$$

POWER SPECTRAL DENSITY

RECALL AR PROCESS EXAMPLES

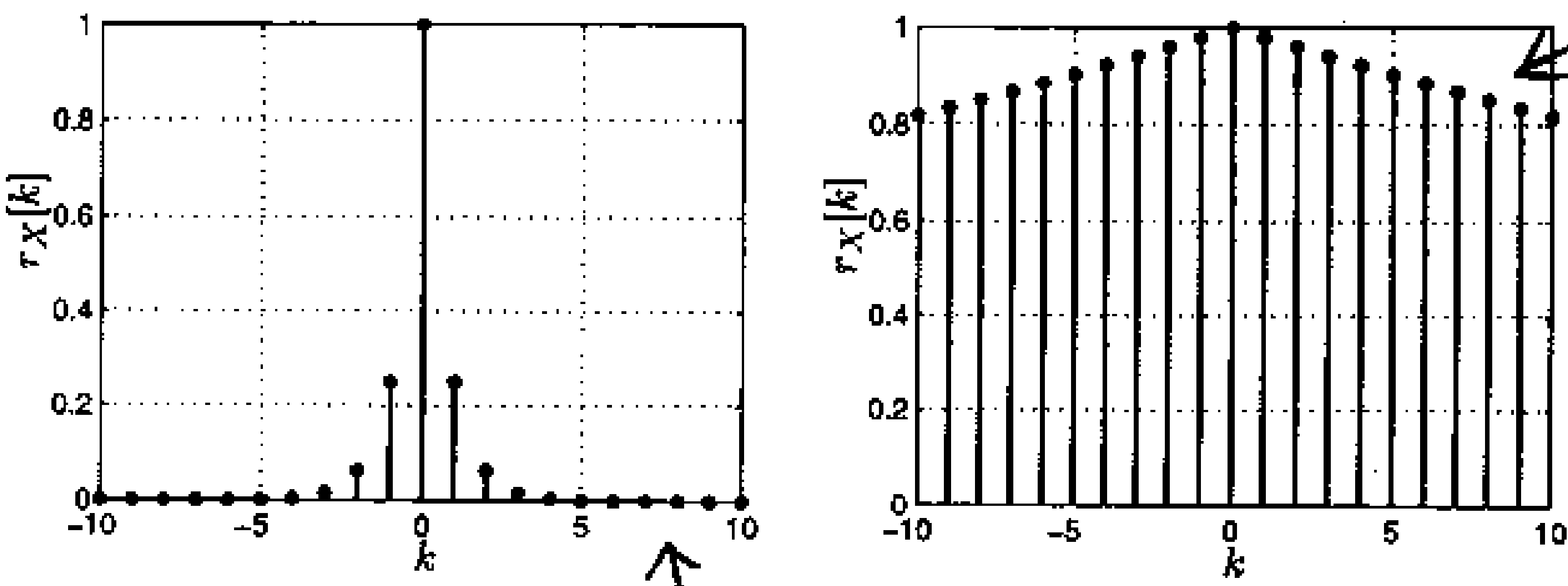


(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

CHANGES SLOWLY (COMPOSED OF LOW FREQUENCIES)

CHANGES RAPIDLY (HIGH FREQUENCIES)

Figure 17.5: Typical realizations of autoregressive random process with different parameters.



HEAVILY CORRELATED

(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

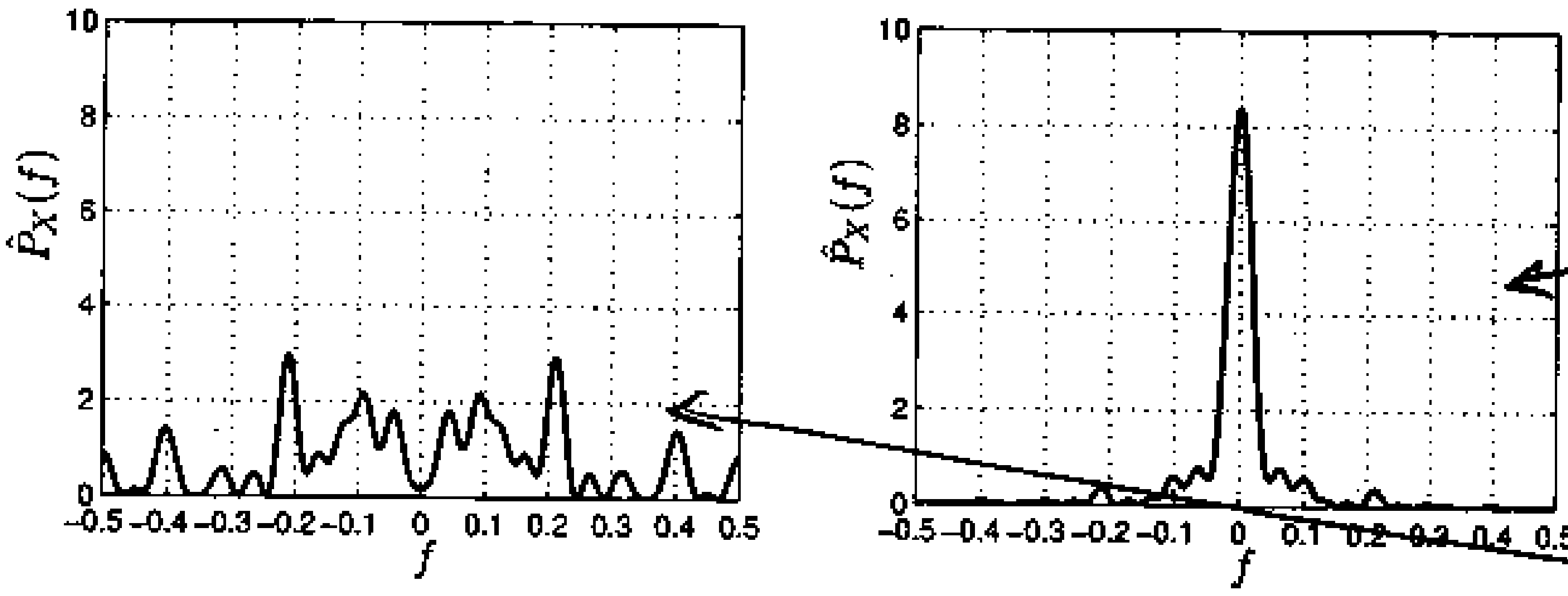
Figure 17.6: The autocorrelation sequence for autoregressive random processes with different parameters.

LITTLE CORRELATION

FOR DETERMINISTIC SIGNALS THE FOURIER TRANSFORM MEASURES FREQUENCY CONTENT. CAN WE USE IT HERE? CONSIDER <u>POWER</u> CONTENT AND USE (CALLED A <u>PERIODOGRAM</u>)

$$\hat{P}_X(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi f n}\right|^2 \qquad -\frac{1}{2} \le f \le \frac{1}{2}$$

(READ APPENDIX D!)



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$

(b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.9: Periodogram for autoregressive random process with different parameters. The realizations shown in Figure 17.5 were used to generate these estimates.

CURVES APPEAR SOMEWHAT RANDOM AS THEY SHOULD

ESTIMATE $\rightarrow \hat{P}_X(f) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] e^{-j2\pi f n} \right|^2$

RANDOM VARIABLES

$$= g(x[0], x[1], \ldots, x[N-1])$$

NEED AVERAGE POWER DISTRIBUTION.

ALSO, SINCE FOURIER TRANSFORM OF A SEGMENT OF WAVEFORM (OR TIME WINDOWED) PRODUCES SMOOTHED RESULT, NEED $x[n]$, $-\infty < n < \infty$.

$$P_X(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left| \sum_{n=-M}^{M} x[n] e^{-j2\pi f n} \right|^2 \right]$$

IS DEFINED AS THE POWER SPECTRAL DENSITY (PSD) OF $x[n]$

$x[n]$ MUST BE WSS FOR THIS TO APPLY.

(WILL SEE WHY SHORTLY).

NOTE: PSD DOES NOT PROVIDE ANY PHASE INFORMATION. DUE TO $|\ |^2$.

EXAMPLE: WHITE NOISE

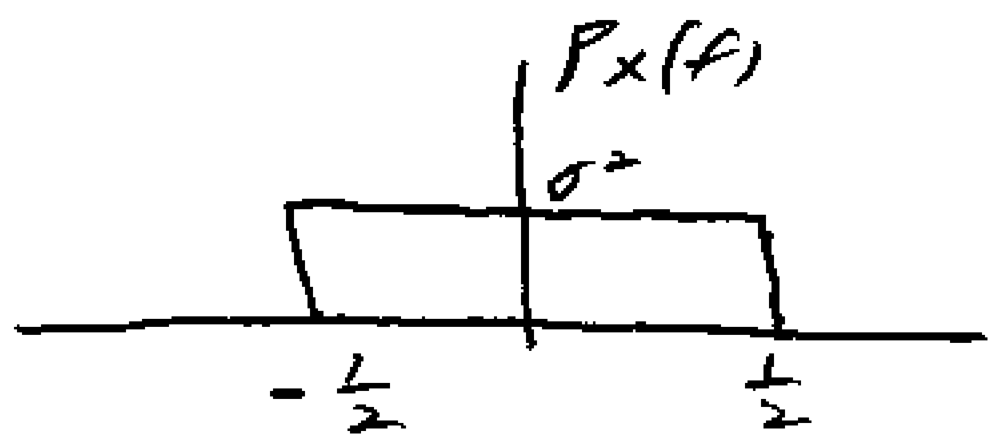
$\mu = 0, \quad r_x[k] = \sigma^2 \delta[k]$

$$P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left(\sum_{n=-M}^{M} x[n] e^{-j2\pi f n}\right)^* \left(\sum_{m=-M}^{M} x[m] e^{-j2\pi f m}\right)\right]$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_n \sum_m \underbrace{E[x[n]x[m]]}_{r_x[m-n]} e^{-j2\pi f(m-n)} \quad \leftarrow \text{NEED WSS HERE!}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_n \sum_m \sigma^2 \delta[m-n] e^{-j2\pi f(m-n)}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \sigma^2 = \lim_{M\to\infty} \sigma^2 = \sigma^2 \quad -\frac{1}{2} \le f \le \frac{1}{2}$$



HAS EQUAL CONTRIBUTIONS OF POWER AT ALL FREQS. $\Rightarrow$ WHITE

IN GENERAL,

$$P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} r_x[m-n] e^{-j2\pi f(m-n)}$$

CAN SIMPLIFY USING

$$\sum_{n=-1}^{1}\sum_{m=-1}^{1} g[m-n] = \text{SUM OF ELEMENTS IN}$$

$$\begin{array}{cc|ccc} & & -1 & 0 & 1 \leftarrow n \\ \hline & -1 & g[0] & g[-1] & g[-2] \\ m \rightarrow & 0 & g[1] & g[0] & g[-1] \\ & 1 & g[2] & g[1] & g[0] \end{array}$$

$$= g[-2] + 2g[-1] + 3g[0] + 2g[1] + g[2]$$

$$= \sum_{k=-2}^{2} (3-|k|)\, g[k]$$

OR $$\sum_{n=-M}^{M}\sum_{m=-M}^{M} g[m-n] = \sum_{k=-2M}^{2M} (2M+1-|k|)\, g[k]$$

$$\Rightarrow \quad P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{k=-2M}^{2M} (2M+1-|k|)\, r_x[k]\, e^{-j2\pi f k}$$

$$= \lim_{M\to\infty} \sum_{k=-2M}^{2M} \left(1 - \frac{|k|}{2M+1}\right) r_x[k]\, e^{-j2\pi f k}$$

$$= \sum_{k=-\infty}^{\infty} r_x[k]\, e^{-j2\pi f k}$$

$\therefore$ PSD IS DISCRETE-TIME FOURIER TRANSFORM OF ACS. $\Rightarrow$ WIENER-KHINCHINE THEOREM

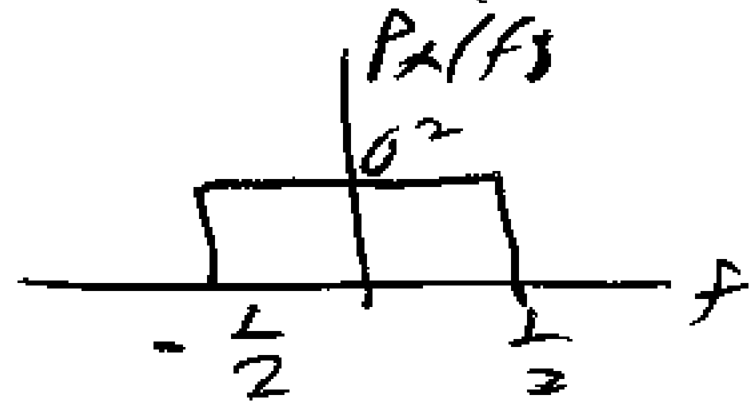
EXAMPLE : WHITE NOISE AGAIN

$$r_x[k] = \sigma^2 \delta[k]$$

$$P_x(f) = \sum_{k=-\infty}^{\infty} r_x[k]\, e^{-j2\pi f k}$$

$$= \sum_{k=-\infty}^{\infty} \sigma^2 \delta(k) e^{-j2\pi f k} = \sigma^2$$

NOTE ALSO THAT TOTAL AVERAGE POWER OR $E[x^2(n)] = r_x(0) = \sigma^2$ IS EQUAL TO AREA UNDER PSD CURVE.



EXAMPLE: AR R.P.

$$r_x(k) = \frac{\sigma_v^2}{1-a^2} a^{|k|} \qquad -\infty < k < \infty$$

$$P_x(f) = \sum_{k=-\infty}^{\infty} \frac{\sigma_v^2}{1-a^2} a^{|k|} e^{-j2\pi f k} = \frac{\sigma_v^2}{1-a^2}\left[\sum_{k=-\infty}^{-1} a^{-k} e^{-j2\pi f k}\right.$$

$$\left.+ \sum_{k=0}^{\infty} a^k e^{-j2\pi f k}\right] = \frac{\sigma_v^2}{1-a^2}\left[\sum_{k=1}^{\infty} a^k e^{j2\pi f k} + \sum_{k=0}^{\infty} a^k e^{-j2\pi f k}\right]$$

$$= \frac{\sigma_v^2}{1-a^2}\left[\sum_{k=1}^{\infty} (a e^{j2\pi f})^k + \sum_{k=0}^{\infty} (a e^{-j2\pi f})^k\right]$$

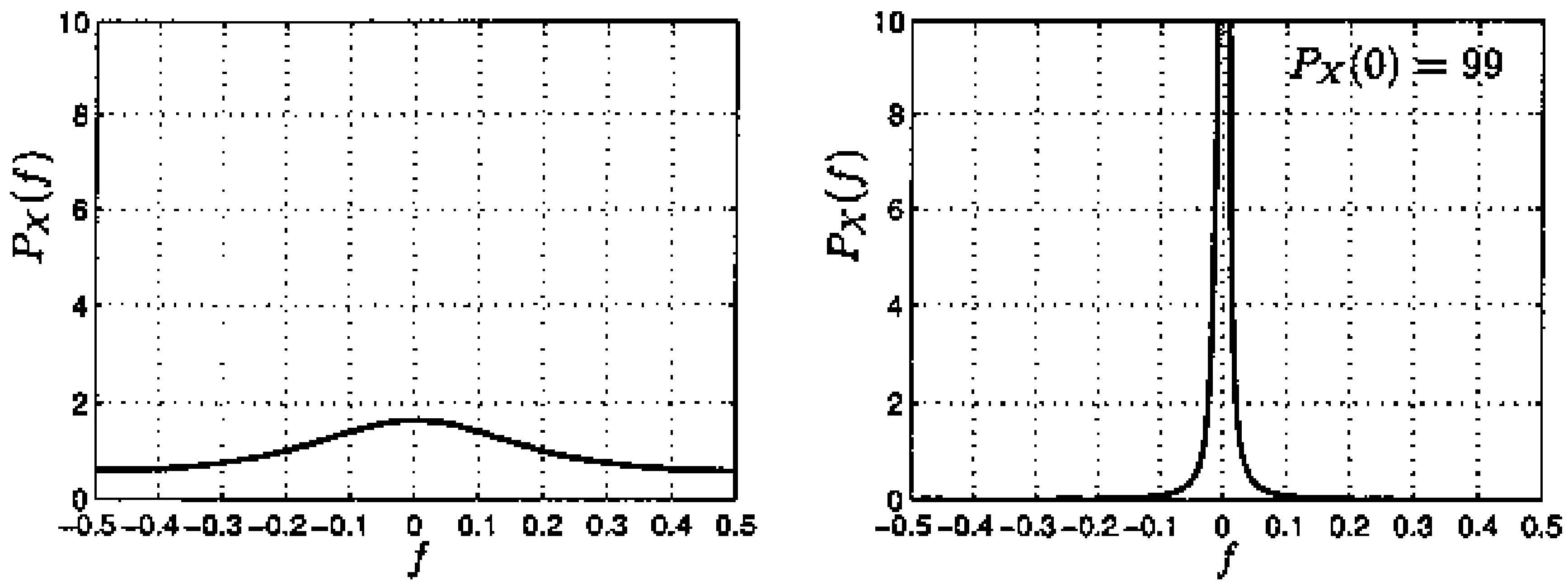
BUT $\displaystyle\sum_{k=k_0}^{\infty} z^k = \frac{z^{k_0}}{1-z}$ IF $|z| < 1$ (Z COMPLEX)

SINCE $|a e^{\pm j2\pi f}| = |a| < 1$, THIS CAN BE USED.

$$P_x(f) = \frac{\sigma_v^2}{1-a^2}\left(\frac{a e^{j2\pi f}}{1-a e^{j2\pi f}} + \frac{1}{1-a e^{-j2\pi f}}\right)$$

$$= \frac{\sigma_v^2}{|1-a e^{-j2\pi f}|^2} \qquad \text{SEE BOOK}$$

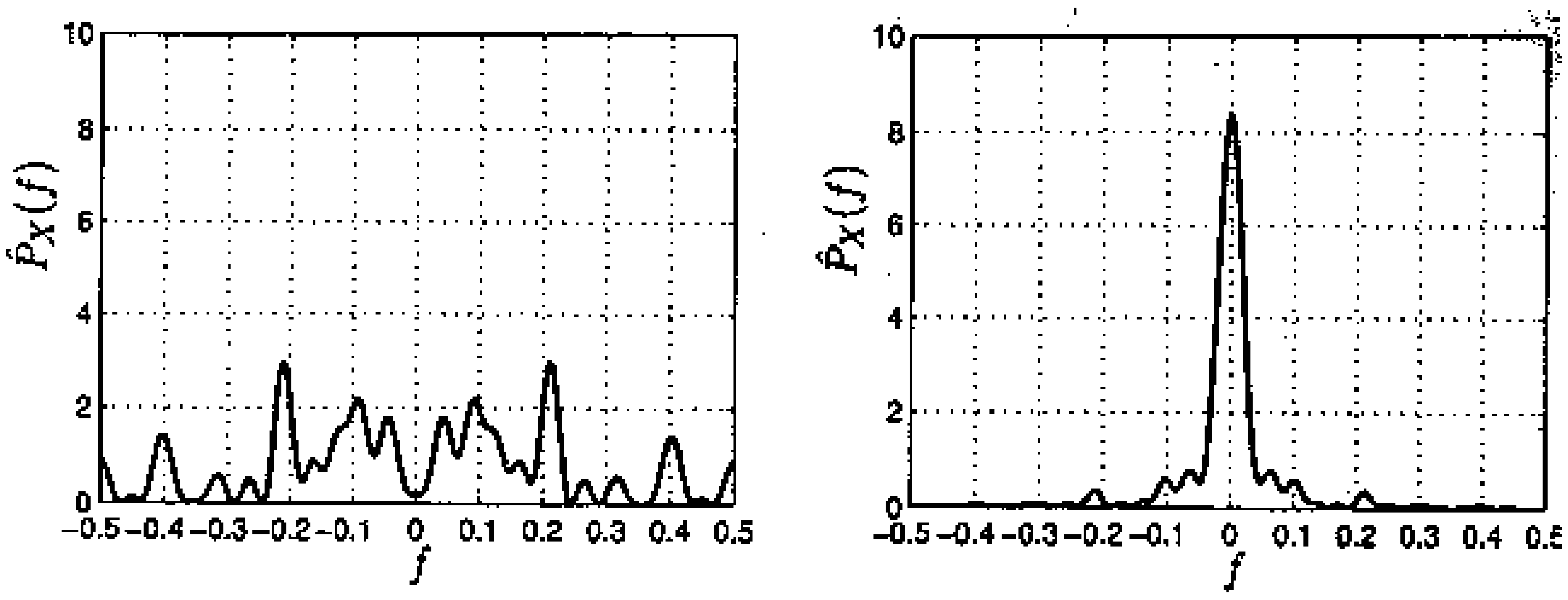
$$= \frac{\sigma_v^2}{1+a^2-2a\cos(2\pi f)} \qquad -\tfrac{1}{2} \le f \le \tfrac{1}{2}$$



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.11: Power spectral densities for autoregressive random process with different parameters. The periodograms, which are estimated PSDs, were given in Figure 17.9.

TRUE PSDS



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.9: Periodogram for autoregressive random process with different parameters. The realizations shown in Figure 17.5 were used to generate these estimates.

PSD ESTIMATES (NO E[ ], NO $M \to \infty$)

PROPERTIES OF PSD

17.7 - PSD IS REAL FUNCTION $P_X(f) = \sum_{k=-\infty}^{\infty} r_X[k] \cos(2\pi f k)$

SEE BOOK

17.8 - PSD IS NONNEGATIVE

WILL PROVE LATER (ALSO SEE DEFINITION)

17.9 - PSD IS SYMMETRIC ABOUT $f = 0$

(EVEN FUNCTION)

17.10 - PSD IS PERIODIC WITH PERIOD 1

$P_X(f+1) = P_X(f)$ WE USE $-\frac{1}{2} \le f \le \frac{1}{2}$

SEE BOOK

17.11 - ACS CAN BE RECOVERED FROM PSD SINCE $P_X(f) = \sum_{k=-\infty}^{\infty} r_X[k] e^{-j2\pi fk}$

$$\Rightarrow r_X[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f) e^{j2\pi fk} \, df$$ INVERSE FOURIER TRANSFORM

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f) \cos(2\pi fk) \, df$$ WHY?

17.12 - PSD YIELDS AVERAGE POWER OVER BAND OF FREQUENCIES

AVERAGE PHYSICAL POWER IN $[f_1, f_2]$ =

$$2 \int_{f_1}^{f_2} P_X(f) df$$

PROOF GIVEN LATER

NOTE HOWEVER THAT FOR ENTIRE BAND

$$= 2 \int_0^{\frac{1}{2}} P_X(f) df = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f) df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f) e^{j2\pi f(0)} df = r_X[0]$$ WHY?

TOTAL AVERAGE POWER $= r_X[0]$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f) df$$

ESTIMATION OF ACS

IN PRACTICE WE HAVE AVAILABLE A SEGMENT OF ONE REALIZATION. SIMILAR TO MEAN ESTIMATION IN WHICH WE USED $\hat{\mu}_N = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$ AS AN ESTIMATOR FOR $E(x(n))$, NOW MUST ESTIMATE

$$r_x(k) = E(x(n)x(n+k))$$

CONSIDER $k = k_0$

$$r_x(k_0) = E(x(n)x(n+k_0))$$

LET $y(n) = x(n)x(n+k_0) \Rightarrow$

$$r_x(k_0) = \underbrace{E(y(n))}_{\uparrow \text{NOT DEPENDENT ON } n} \qquad -\infty < n < \infty$$

SAME SETUP AS BEFORE $\Rightarrow$

USE $\hat{r}_x(k_0) = \frac{1}{N}\sum_{n=0}^{N-1} y(n)$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)x(n+k_0) \qquad k_0 \geq 0$$

$(r_x(-k) = r_x(k) \Rightarrow$ ONLY NEED $k \geq 0)$

SLIGHT PROBLEM IS:

GIVEN $x[0], x[1], \ldots, x[N-1]$ MUST HAVE

$n + k_0 \leq N-1 \Rightarrow n \leq N-1-k_0$

$$\hat{r}_x(k) = \frac{1}{N-k_0}\sum_{n=0}^{N-1-k_0} x(n)x[n+k_0]$$

$\uparrow$ NUMBER OF TERMS AVERAGED

OR FINALLY, IF $x[0], x[1], \ldots, x[N-1]$ GIVEN,

$$\hat{r}_x(k) = \frac{1}{N-k} \sum_{n=0}^{N-1-k} x[n]\,x[n+k] \qquad k = 0, 1, \ldots, N-1$$

FOR A GOOD ESTIMATE WE REQUIRE $N \gg k_{MAX}$ SINCE FOR EXAMPLE, IF $k = N-1$

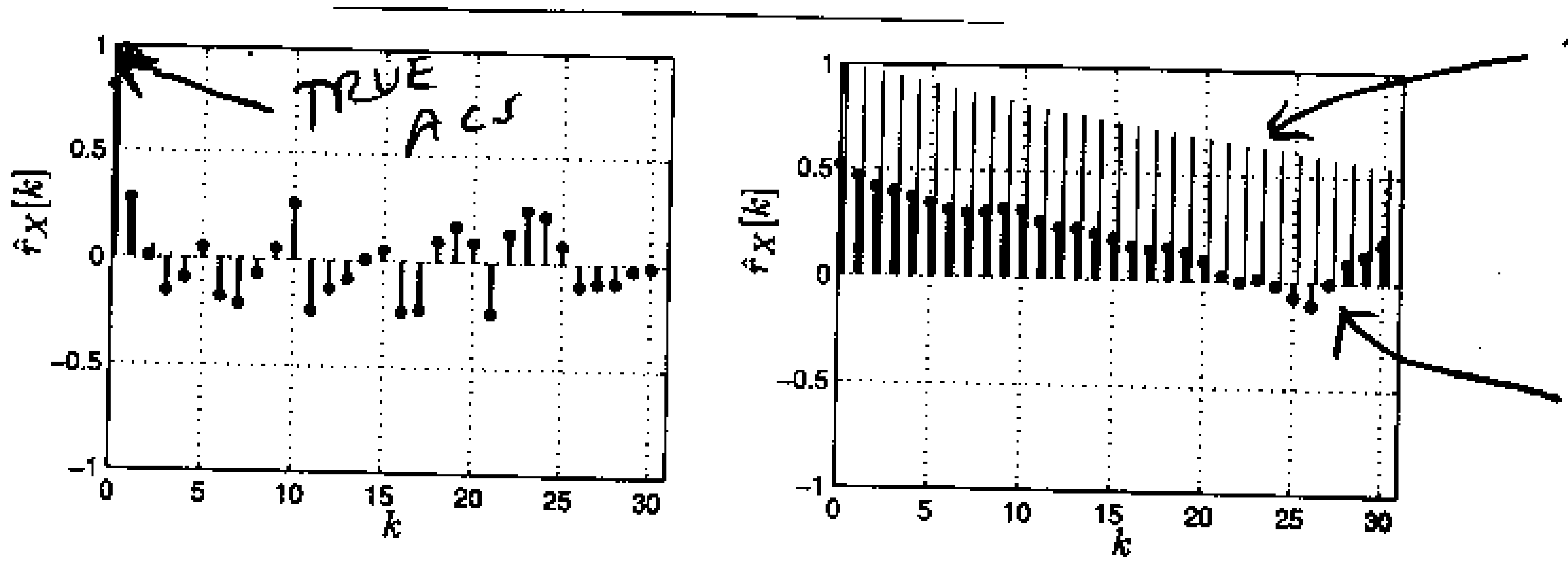
$$\hat{r}_x(N-1) = \frac{1}{N-(N-1)} \sum_{n=0}^{N-1-(N-1)} x[n]\,x[n+N-1]$$

$$= 1 \cdot \sum_{n=0}^{0} x[n]\,x[n+N-1]$$

$$= x[0]\,x[N-1]$$

NUMBER OF TERMS IN AVERAGE = $N-k$

NOTE THAT $E[\hat{r}_x[k]] = r_x(k)$ AND FOR MOST RPS AS $N \to \infty$, $\hat{r}_x(k) \to r_x(k)$.



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$

(b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

$N = 31$

VERY POOR HERE
$N-k$ TERMS IN AVERAGE

Figure 17.12: Estimated ACSs (dark lines) and the true ACSs given in Figure 17.6 (light lines) for the AR random process realizations shown in Figure 17.5.

```
n=[0:30]';N=length(n);
a1=0.25;a2=0.98;
varu1=1-a1^2;varu2=1-a2^2;
r1true=(varu1/(1-a1^2))*a1.^n; % see (17.21)
r2true=(varu2/(1-a2^2))*a2.^n;
for k=0:N-1
   r1est(k+1,1)=(1/(N-k))*sum(x1(1:N-k).*x1(1+k:N));
   r2est(k+1,1)=(1/(N-k))*sum(x2(1:N-k).*x2(1+k:N));
end
```

$x_1(n)$, $x_2(n)$ ALREADY GENERATED

$\hat{r}_X[k]$

ESTIMATION OF PSD

RECALL DEFINITION

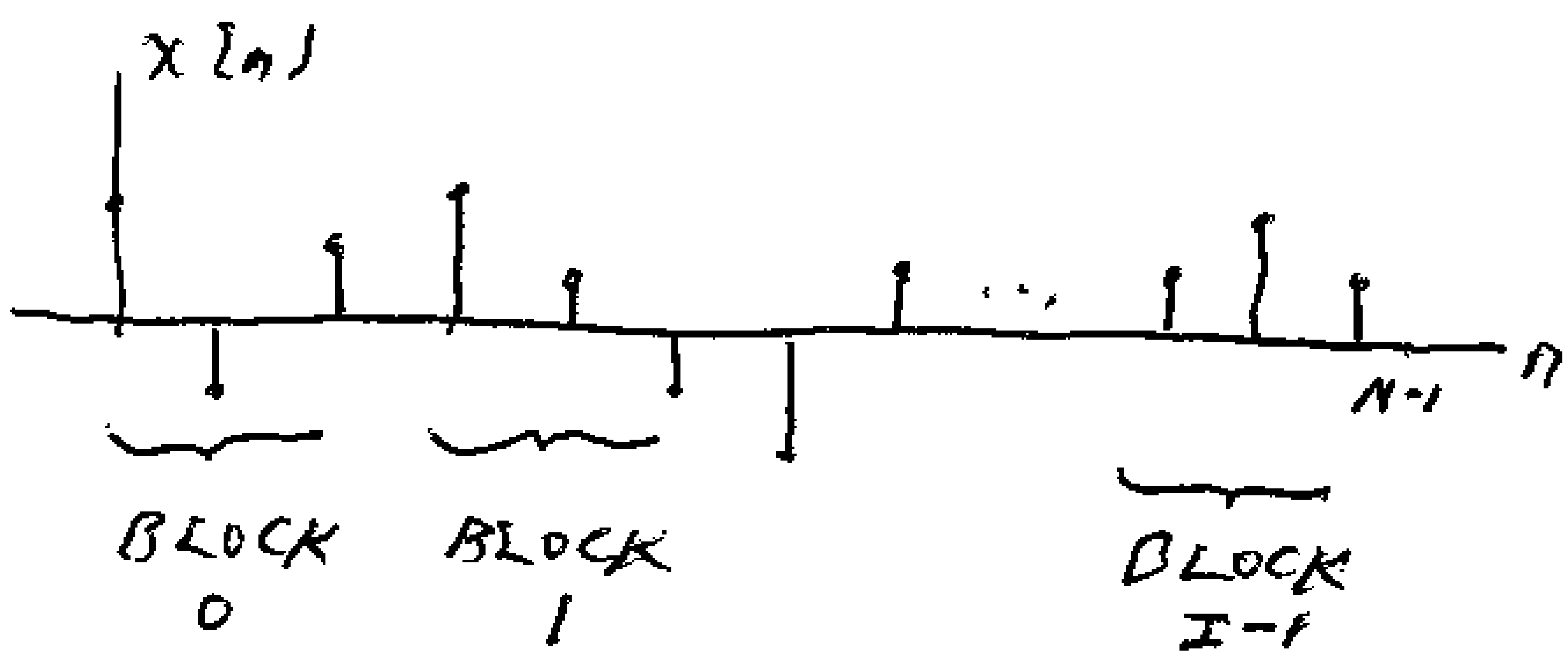
$$P_X(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left| \sum_{n=-M}^{M} x[n] e^{-j2\pi f n} \right|^2 \right]$$

SEGMENT

ONLY HAVE THE REALIZATION SEGMENT $\{x[0], x[1], \ldots, x[N-1]\}$

$\Rightarrow$ CAN'T TAKE LIMIT AND NO $E(\ )$.

TO APPROXIMATE $E(\ )$ OPERATION, WHICH IS AN AVERAGE "DOWN THE ENSEMBLE" OF REALIZATIONS :



I BLOCKS

$L = N/I$ SAMPLES PER BLOCK

$$y_i(n) = x[n+iL] \qquad n = 0, 1, \ldots, L-1$$
$$\qquad\qquad\qquad\qquad i = 0, 1, \ldots, I-1$$

$$\hat{P}_X^{(i)}(f) = \frac{1}{L}\left|\sum_{n=0}^{L-1} y_i[n]\, e^{-j2\pi f n}\right|^2$$

ESTIMATE BASED ON BLOCK i

$\Rightarrow$ AVERAGED PERIODOGRAM ESTIMATOR IS

$$\hat{P}_{AV}(f) = \frac{1}{I}\sum_{i=0}^{I-1} \hat{P}_X^{(i)}(f)$$

CAN SHOW THAT AS $N \to \infty$ SO THAT $I \to \infty$ AND $L \to \infty$, $\hat{P}_{AV}(f) \to P_X(f)$

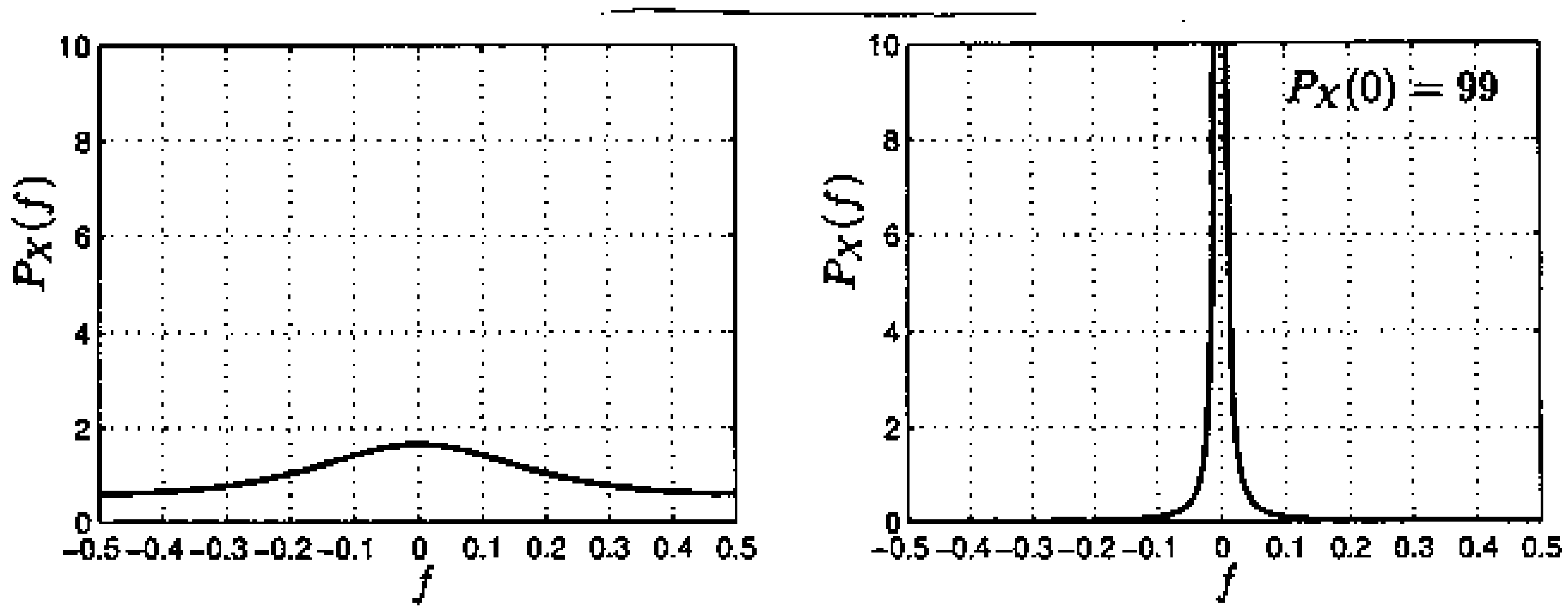
EXAMPLE :

$N = 310$

$I = 10$

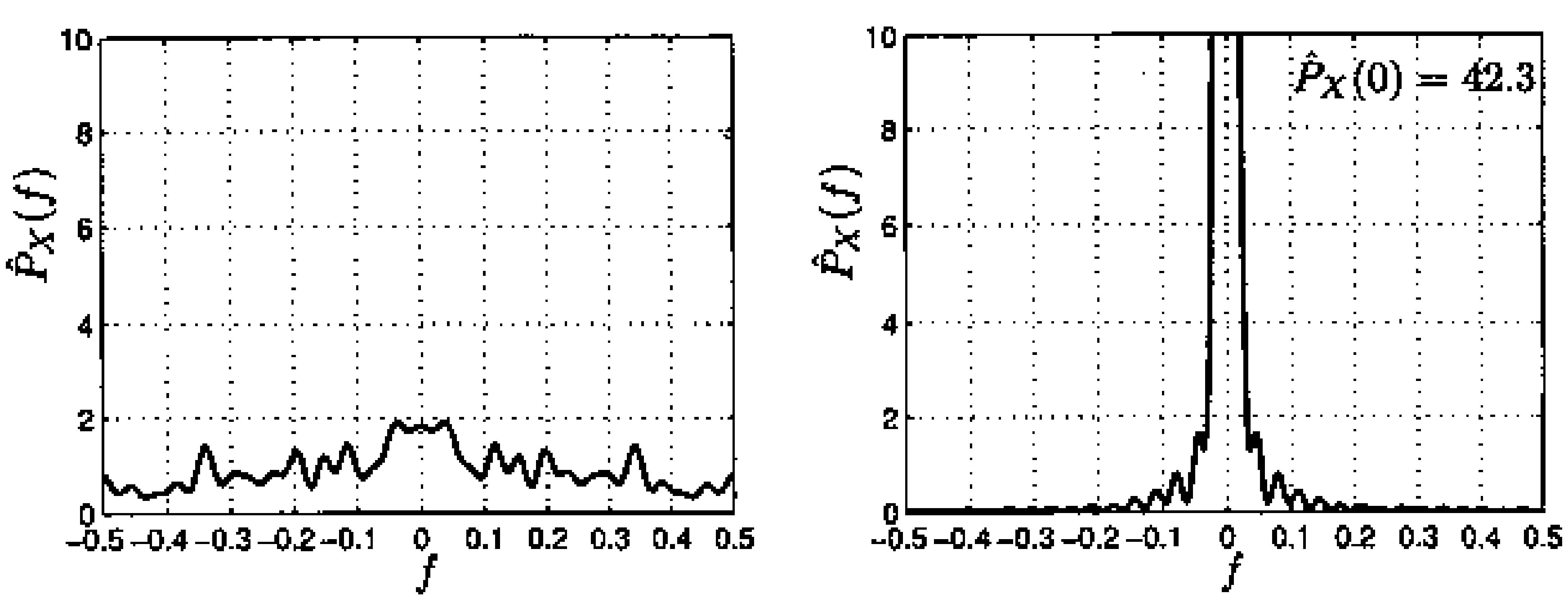
$L = 31$

SEE MATLAB CODE ON PG 580



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.11: Power spectral densities for autoregressive random process with different parameters. The periodograms, which are estimated PSDs, were given in Figure 17.9.



(a) $a = 0.25$, $\sigma_U^2 = 1 - a^2$ (b) $a = 0.98$, $\sigma_U^2 = 1 - a^2$

Figure 17.13: Power spectral density estimates using the averaged periodogram method for autoregressive processes with different parameters. The true PSDs are shown in Figure 17.11.

CONTINUOUS-TIME WSS RPs

$X(t) \quad -\infty < t < \infty$

IF WSS, THEN $\mu_X(t) = E[X(t)] = \mu \quad -\infty < t < \infty$

$r_X(\tau) = E[X(t)X(t+\tau)]$

$-\infty < \tau < \infty$

$r_X(\tau)$ IS CALLED AUTOCORRELATION FUNCTION (ACF) $\quad (\tau = \text{LAG})$

PROPERTIES:

17.13 - $r_X(0) > 0$ AND $r_X(0) = E[X^2(t)]$ IS TOTAL AVERAGE POWER

17.14 - $r_X(-\tau) = r_X(\tau)$ (EVEN)

17.15 - $|r_X(\tau)| \leq r_X(0)$

17.16 -

$$\rho_{X(t), X(t+\tau)} = \frac{r_X(\tau)}{r_X(0)} \quad \left(\begin{matrix}\text{ASSUMES} \\ \mu = 0\end{matrix}\right)$$

17.17 - $r_X(\tau) \to \mu^2$ AS $\tau \to \infty$

17.18 - ACF IS POSITIVE SEMIDEFINITE FUNCTION

PSD IS DEFINED AS

$$P_X(F) = \lim_{T \to \infty} \frac{1}{T} E\left[\left| \int_{-T/2}^{T/2} X(t) e^{-j2\pi F t} dt \right|^2 \right]$$