FOR $-\infty < F < \infty$ F IN Hz
↑ CAPITAL <u>NOW</u>

PSD IS AGAIN FOURIER TRANSFORM OF ACF

$$P_x(F) = \int_{-\infty}^{\infty} r_x(\tau) e^{-j2\pi F\tau} d\tau$$

<u>PROPERTIES</u>:

17.19 - PSD IS REAL FUNCTION

$$P_x(F) = \int_{-\infty}^{\infty} r_x(\tau) \cos(2\pi F\tau) d\tau$$

17.20 - $P_x(F) \geq 0$

17.21 - $P_x(-F) = P_x(F)$ (EVEN)

17.22 - ACF RECOVERED FROM INVERSE FOURIER TRANSFORM

$$r_x(\tau) = \int_{-\infty}^{\infty} P_x(F) e^{j2\pi F\tau} dF$$

$$= \int_{-\infty}^{\infty} P_x(F) \cos(2\pi F\tau) dF$$

NOTE: PSD IS <u>NOT</u> PERIODIC NOW.

<u>CONTINUOUS-TIME WGN</u>

AS A MODEL FOR PHYSICAL NOISE WE ASSUME $x(t)$ IS GAUSSIAN WITH ZERO MEAN FOR ALL t AND IS WSS WITH

PSD $\quad P_X(F) = \frac{N_0}{2} \quad -\infty < F < \infty$

NOT PHYSICALLY POSSIBLE SINCE

$$r_X(0) = \int_{-\infty}^{\infty} P_X(F)\,dF = \int_{-\infty}^{\infty} \frac{N_0}{2}\,dF \to \infty$$

HOWEVER, ANY SYSTEM WILL HAVE FINITE BANDWIDTH W SO THAT POWER WILL BE FINITE.

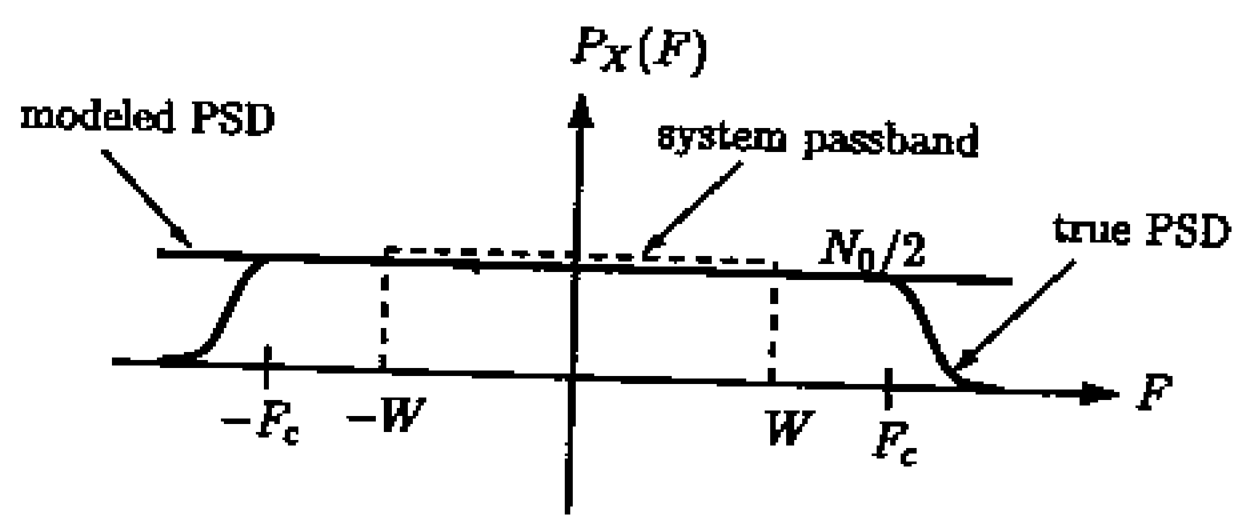


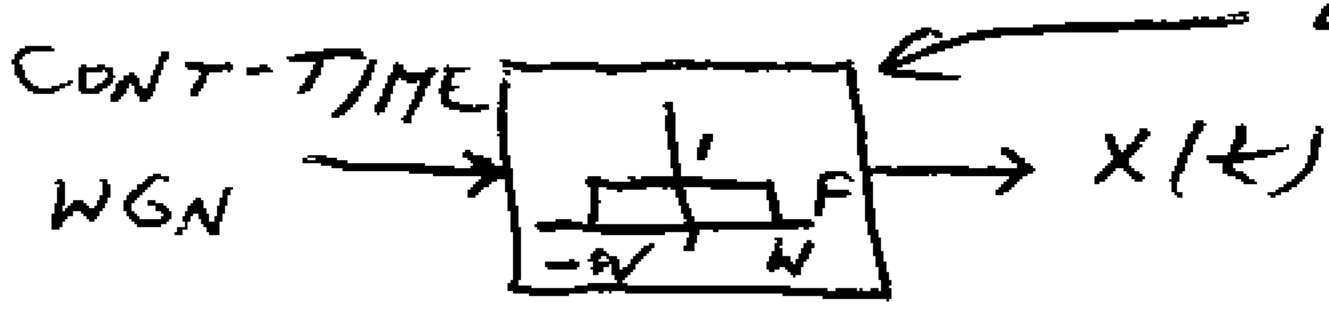
Figure 17.14: True and modeled PSDs for continuous-time white Gaussian noise.

NOTE THAT
$$r_X(\tau) = \int_{-\infty}^{\infty} P_X(F) e^{j2\pi F\tau}\,dF$$
$$= \int_{-\infty}^{\infty} N_0/2\, e^{j2\pi F\tau}\,dF$$
$$\text{"}=\text{"} \; N_0/2\, \delta(\tau)$$

$\Rightarrow$ ALL SAMPLES ARE UNCORRELATED (EVEN IF VERY CLOSE TO EACH OTHER)

NOW CONSIDER OUTPUT OF SYSTEM WITH BANDWIDTH W

LOWPASS FILTER



$$\Rightarrow \quad P_x(F) = \begin{cases} \frac{N_0}{2} & |F| \le W \\ 0 & |F| > W \end{cases}$$

IF WE USE NYQUIST SAMPLING OR $F_s = 2W$ SAMPLES/SEC (TO INPUT DATA INTO A COMPUTER)

$$x(t)\Big|_{t = n\Delta_t} = x[n] \qquad -\infty < n < \infty$$

$F_s = 1/\Delta_t$ $\qquad$ Δ_t = TIME INTERVAL BETWEEN SAMPLES

$x[n]$ = DISCRETE-TIME GAUSSIAN R.P. WITH ZERO MEAN AND ACS

$$\begin{aligned} r_x[k] &= E[x[n]\,x[n+k]] \\ &= E[x(n\Delta_t)\,x((n+k)\Delta_t)] \\ &= r_x(k\Delta_t) \leftarrow \text{SAMPLED VERSION OF } r_x(\tau) \end{aligned}$$

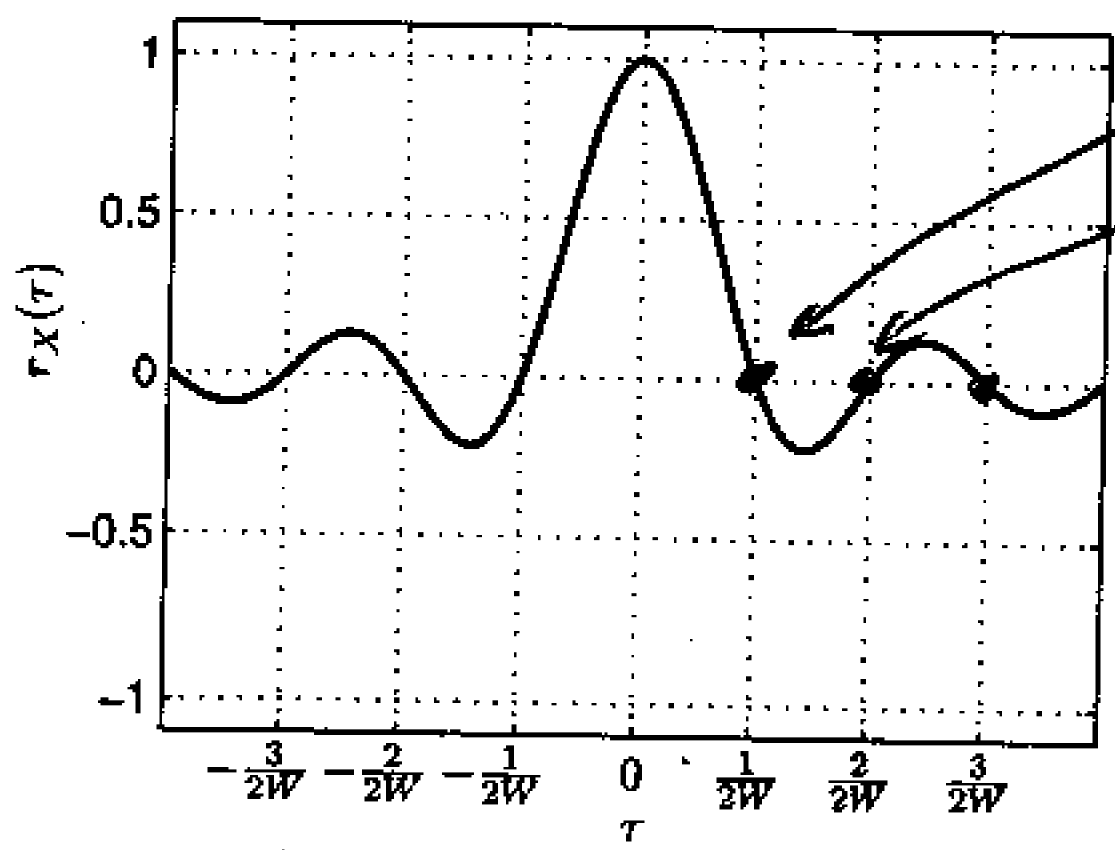
BUT $r_x(\tau) = \mathcal{F}^{-1}\left\{ \begin{matrix} N_0/2 & |F| \le W \\ 0 & |F| > W \end{matrix} \right\}$

$$\begin{aligned} &= \int_{-\infty}^{\infty} P_x(F) e^{j2\pi F\tau}\, dF \\ &= \int_{-W}^{W} \frac{N_0}{2} e^{j2\pi F\tau}\, dF \\ &= \int_{-W}^{W} \frac{N_0}{2} \cos(2\pi F\tau)\, dF \qquad \text{WHY?} \end{aligned}$$

$$= \frac{N_0}{2} \frac{\sin 2\pi F\tau}{2\pi\tau}\Big|_{-W}^{W}$$

$$= \frac{N_0}{2}\left[\frac{\sin(2\pi W\tau) - \sin(-2\pi W\tau)}{2\pi\tau}\right]$$

$$= \frac{N_0 \sin(2\pi W\tau)}{2\pi\tau} = N_0 W \frac{\sin(2\pi W\tau)}{2\pi W\tau}$$



$r_X(1/2W)$

$r_X(2/2W)$

$$r_X[k] = r_X(k\Delta_t) = r_X(k/2W)$$

$$r_X(k/2W) = 0 \quad k = \pm 1, \pm 2, \ldots$$

Figure 17.15: ACF for bandlimited continuous-time WGN with $N_0W = 1$.

$$\Rightarrow \quad r_X[k] = \begin{cases} 0 & k = \pm 1, \pm 2, \ldots \\ r_X(0) & k = 0 \\ N_0 W & k = 0 \end{cases}$$

$$\Rightarrow \quad r_X[k] = N_0 W \delta[k]$$

OR $X[n]$ IS DISCRETE-TIME WGN WITH VARIANCE $\sigma^2 = N_0 W$

$P_X(f)$

$N_0 W$

$-\frac{1}{2}$ $\frac{1}{2}$ f

JUSTIFIES USE OF WGN FOR SYSTEMS ANALYSIS IN DISCRETE-TIME

CHAPTER 18 - LINEAR SYSTEMS AND WSS RPS

CONSIDER ONLY DISCRETE-TIME SYSTEMS (READ APP. D). ASSUME WE INPUT A WSS R.P. $U(n)$ TO A LINEAR SHIFT INVARIANT (LSI) SYSTEM.

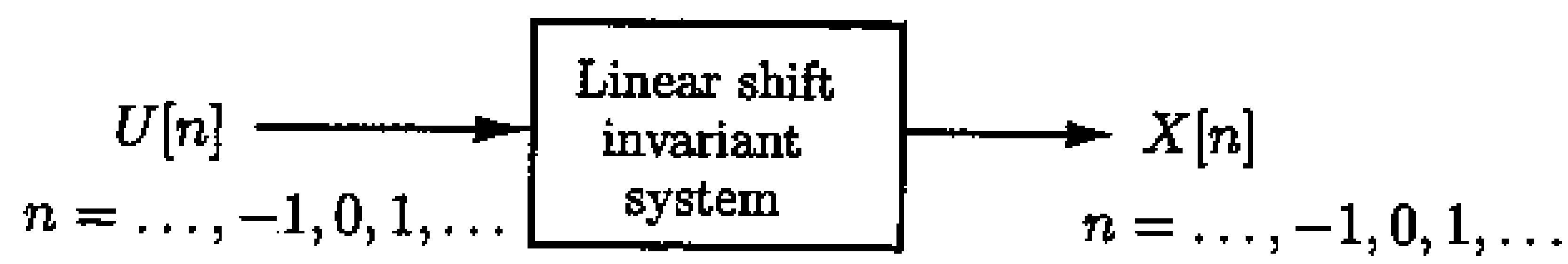


Figure 18.1: Linear shift invariant system with random process input and output.

WHAT CAN BE SAID ABOUT $x(n)$?

EXAMPLE: RECALL MA PROCESS (GENERAL)

$$x[n] = \frac{1}{2}\left(U[n] + U[n-1]\right)$$

ZERO MEAN WSS

$r_U[k] = \sigma_U^2 \delta[k]$

(WHITE NOISE)

NO NEED TO ASSUME $U[n]$ IS GAUSSIAN IN THIS CHAPTER - INTERESTED IN ONLY FIRST TWO MOMENTS OF $x[n]$.

MA PROCESS CAN BE VIEWED AS OUTPUT OF LSI SYSTEM (= FILTER)

WITH IMPULSE RESPONSE

$$h[n] = \begin{cases} \frac{1}{2} & k=0 \\ \frac{1}{2} & k=1 \\ 0 & \text{OTHERWISE} \end{cases}$$

RECALL FOR LSI SYSTEM

$$x[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k] = \underbrace{h[n] \star u[n]}_{\text{CONVOLUTION}}$$

FOR MA RP.

$$x[n] = h[0]u[n] + h[1]u[n-1]$$
$$= \tfrac{1}{2} u[n] + \tfrac{1}{2} u[n-1]$$
$$= \tfrac{1}{2}\left(u[n] + u[n-1] \right)$$

SOME OTHER CHARACTERIZATIONS OF LSI SYSTEM:

1) SYSTEM FUNCTION = $\mathcal{Z}$-TRANSFORM OF IMPULSE RESPONSE

$$\mathcal{H}(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$
$$= \tfrac{1}{2} z^{0} + \tfrac{1}{2} z^{-1} = \tfrac{1}{2} + \tfrac{1}{2} z^{-1} \qquad \text{MA}$$

2) FREQUENCY RESPONSE - FOURIER TRANSFORM OF IMPULSE RESPONSE

$$H(f) = \sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi fk} \qquad |f| \le \tfrac{1}{2}$$

$$= \tfrac{1}{2} e^{-j2\pi f(0)} + \tfrac{1}{2} e^{-j2\pi f(1)}$$

$$= \tfrac{1}{2} + \tfrac{1}{2} e^{-j2\pi f} \qquad \text{MA}$$

ALSO, $H(f) = \mathcal{H}(z)\big|_{z=e^{j2\pi f}}$

RECALL

$$e^{j2\pi f_0 n} \longrightarrow \boxed{LSI} \longrightarrow H(f_0) e^{j2\pi f_0 n}$$

$$-\infty < n < \infty \qquad\qquad -\infty < n < \infty$$

$|H(f_0)|$ = MAGNITUDE RESPONSE AT $f = f_0$

$\measuredangle H(f_0)$ = PHASE RESPONSE AT $f = f_0$

EXAMPLE -

$$u[n] \longrightarrow \boxed{LSI} \longrightarrow x[n]$$

WSS $\mu_u, r_u[k]$ $\qquad \mathcal{H}(z) = h[0] + h[1] z^{-1}$

IS $x[n]$ WSS? IF SO, $\mu_x = ?$, $r_x[k] = ?$

$$x[n] = h[0]u[n] + h[1]u[n-1] \qquad -\infty < n < \infty$$

$$E[x[n]] = h[0]\underbrace{E[u[n]]}_{\mu_u} + h[1]\underbrace{E[u[n-1]]}_{\mu_u}$$

$$= (h[0] + h[1])\mu_u$$

$E[x[n]]$ NOT DEPENDENT ON n

$$\mu_x = (h[0] + h[1])\mu_v$$

IN GENERAL, $\mu_x = \sum_{k=-\infty}^{\infty} h(k) e^{-j2\pi f k}\Big|_{f=0} \mu_v$

$$= H(0)\mu_v \qquad \text{WHY?}$$

$$E[x[n]x[n+k]] = E[(h[0]v[n] + h[1]v[n-1])(h[0]v[n+k] + h[1]v[n+k-1])]$$

$$= h^2[0]E[v[n]v(n+k)] + h[0]h[1]E[v[n]v[n+k-1]] + h[1]h[0]E[v[n-1]v(n+k)] + h^2[1]E[v[n-1]v[n+k-1]]$$

$$= h^2[0]r_v[k] + h[0]h[1]r_v[k-1] + h[1]h[0]r_v[k+1] + h^2[1]r_v[k]$$

$$= (h^2[0] + h^2[1])r_v[k] + h[0]h[1]r_v[k-1] + h[1]h[0]r_v[k+1]$$

$$= r_x[k] \qquad \text{NOT DEPENDENT ON } n$$

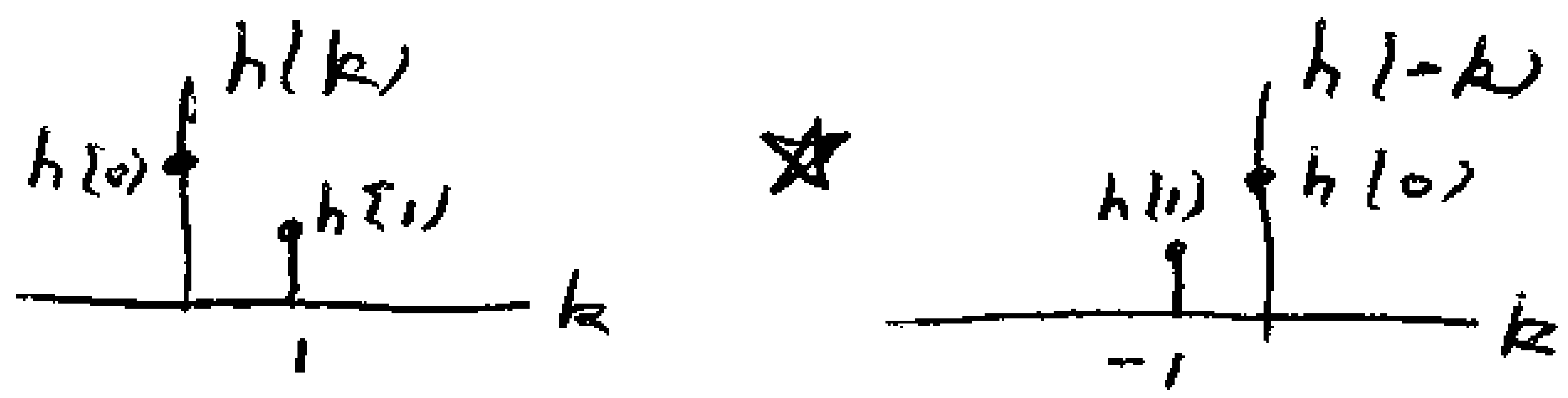
$\Rightarrow x[n]$ IS WSS

ALSO, NOTE THAT IF

$$g[0] = h^2[0] + h^2[1]$$
$$g[1] = h[0]h[1]$$
$$g[-1] = h[1]h[0]$$

$$
\begin{aligned}
r_x[k] &= g[0]\, r_v[k] + g[1]\, r_v[k-1] + g[-1]\, r_v[k+1] \\
&= \sum_{j=-1}^{1} g[j]\, r_v[k-j] \\
&= g[k] \star r_v[k]
\end{aligned}
$$

ALSO, $g[k] = h[k] \star h[-k]$



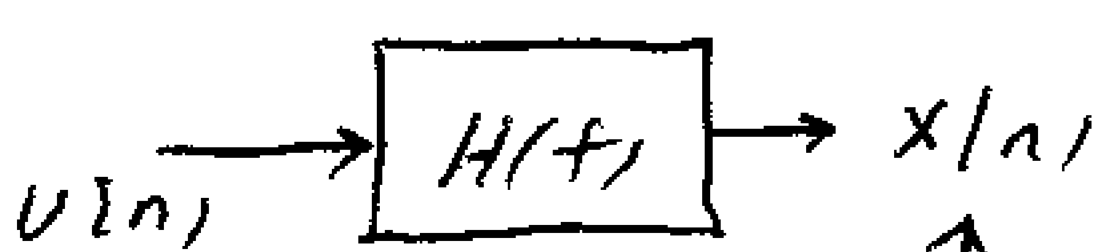
$$
\begin{aligned}
P_x(f) = \mathcal{F}\{r_x[k]\} &= \mathcal{F}\{g[k] \star r_v[k]\} \\
&= G(f)\, P_v(f)
\end{aligned}
$$

BUT $\mathcal{F}\{h[k]\} = H(f)$ FREQUENCY RESPONSE

$\mathcal{F}\{h[-k]\} = H^*(f)$

$$
\begin{aligned}
\Rightarrow \quad P_x(f) &= G(f)\, P_v(f) \\
&= \mathcal{F}\{h[k] \star h[-k]\}\, P_v(f) \\
&= \mathcal{F}\{h[k]\}\, \mathcal{F}\{h[-k]\}\, P_v(f) \\
&= H(f)\, H^*(f)\, P_v(f)
\end{aligned}
$$

$$\therefore \quad P_x(f) = |H(f)|^2 P_v(f)$$



WSS, μ_v, $P_v(f)$

WSS, $H(0)\mu_v = \mu_x$

$|H(f)|^2 P_v(f) = P_x(f)$

SEE GENERAL PROOF - THEOREM 18.3.1

SPECIAL CASE: IF INPUT IS WHITE NOISE, $P_u(f) = \sigma_u^2$

$$\Rightarrow \quad P_x(f) = |H(f)|^2 P_u(f) = |H(f)|^2 \sigma_u^2$$

FILTER "COLORS" THE NOISE - MAKES NOISE CORRELATED AT OUTPUT

<u>EXAMPLE</u>: AR R.P.

RECALL $x[n] = a\,x[n-1] + u[n]$

↑ WHITE NOISE

NEED $H(f)$. BUT $H(f) = \mathcal{H}(z)\big|_{z=e^{j2\pi f}}$

AND

$$\mathcal{H}(z) = \frac{\mathcal{X}(z)}{\mathcal{U}(z)}$$

IF $x[n] = a\,x[n-1] + u[n]$ — $u[n]$, $x[n]$ ARE SEQUENCES

$$\mathcal{X}(z) = a z^{-1} \mathcal{X}(z) + \mathcal{U}(z)$$

$$\Rightarrow \quad \mathcal{H}(z) = \frac{1}{1 - a z^{-1}}$$

$$H(f) = \frac{1}{1 - a e^{-j2\pi f}}$$

$$P_x(f) = |H(f)|^2 P_v(f)$$
$$= \frac{1}{|1-ae^{-j2\pi f}|^2}\sigma_v^2$$
$$= \frac{\sigma_v^2}{|1-ae^{-j2\pi f}|^2} \qquad |f| \le \frac{1}{2}$$

SAME RESULT AS BEFORE. TO FIND ACS CAN TAKE INVERSE FOURIER TRANSFORM. EASIER APPROACH IS AS FOLLOWS:

$$P_x(f) = H(f)\,H^*(f)\,P_v(f)$$
$$= H(f)\,H^*(f)\,\sigma_v^2$$
$$\Rightarrow \quad r_x[k] = (h[k] \star h[-k])\,\sigma_v^2$$
$$= \sum_{i=-\infty}^{\infty} h[-i]\,h[k-i]\,\sigma_v^2$$

LET $m=-i$
$$= \underbrace{\sigma_v^2 \sum_{m=-\infty}^{\infty} h[m]\,h[m+k]}_{\text{CORRELATION OF } h[k] \text{ WITH ITSELF}}$$

FOR AR R.P. WE ASSUME LSI SYSTEM IS CAUSAL $\Rightarrow$ $h[m]=0 \quad m<0$

$$r_x[k] = \begin{cases} \sigma_v^2 \sum_{m=0}^{\infty} h[m]\,h[m+k] & k \ge 0 \\ r_x[-k] & k<0 \end{cases}$$

WHAT IS $h[m]$?

RECALL FROM SLIDE 160

$$x[n] = \sum_{l=0}^{\infty} a^l v[n-l]$$

(OR $x[n] = a x[n-1] + \delta[n]$ AND LET $x[-1] = 0$ WHY?, RECURSE FORWARD TO FIND $x[n] = h[n] = a^n \quad n \geq 0$)

$$\Rightarrow \quad h[l] = \left.\begin{matrix} a^l & l \geq 0 \\ 0 & l < 0 \end{matrix}\right\} = a^l u_s[l]$$

↑ UNIT STEP

$$r_x[k] = \sigma_v^2 \sum_{m=0}^{\infty} a^m u_s[m]\, a^{m+k} u_s[m+k]$$

FOR $k \geq 0$ AND $m \geq 0$ $\quad u_s[m] = u_s[m+k] = 1$

$$r_x[k] = \sigma_v^2 \sum_{m=0}^{\infty} a^{2m+k} = \sigma_v^2 a^k \sum_{m=0}^{\infty} (a^2)^m$$

$$= \frac{\sigma_v^2}{1-a^2} a^k \qquad k \geq 0$$

$$= \frac{\sigma_v^2}{1-a^2} a^{|k|} \qquad -\infty < k < \infty$$

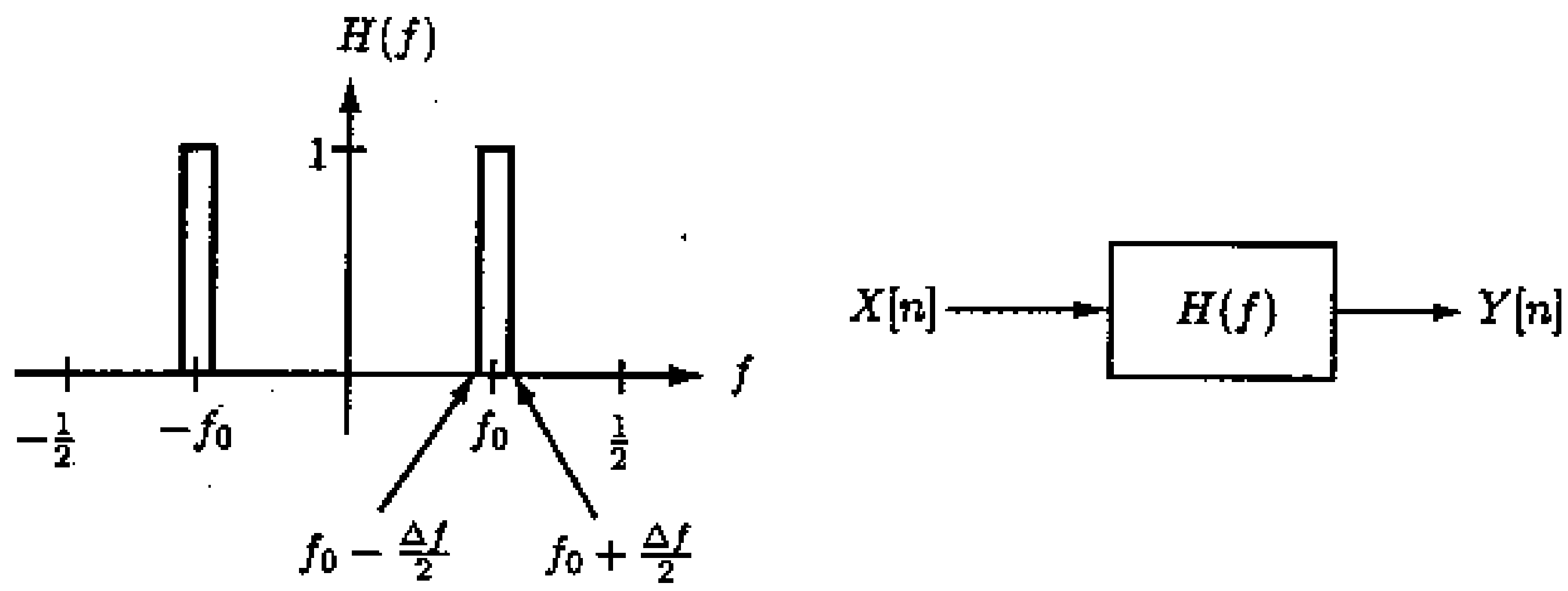
SEE ALSO MA EXAMPLE 18.3

INTERPRETATION OF PSD

CAN FINALLY PROVE THAT

$$\int_{f_1}^{f_2} P_x(f)\, df = \text{AVERAGE POWER IN } [f_1, f_2]$$

TO MEASURE AVERAGE POWER WE FILTER $x[n]$ BY LSI FILTER WITH A NARROW BAND AND DETERMINE POWER AT FILTER OUTPUT.



(a) (b)

Figure 18.3: Narrowband filtering of random process to measure power within a band of frequencies.

AVERAGE POWER OUT OF FILTER $= E[y^2[n]]$

$$= r_y[0]$$

BUT $$r_y[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_y(f)\, e^{j2\pi f k}\, df$$

$$\Rightarrow \quad r_y[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_y(f)\, df$$

$$E[y^2[n]] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_y(f)\, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P_x(f)\, df$$

$$= \int_{-f_0-\frac{\Delta f}{2}}^{-f_0+\Delta f/2} 1 \cdot P_x(f)\, df$$

$$+ \int_{f_0-\Delta f/2}^{f_0+\Delta f/2} 1 \cdot P_x(f)\, df$$

$$= \quad 2\int_{f_0-\Delta f/2}^{f_0+\Delta f/2} P_X(f)\,df \qquad \text{WHY ?}$$

TOTAL AVERAGE POWER IN BAND $[f_0 - \frac{\Delta f}{2}, f_0 + \frac{\Delta f}{2}]$

$$= \int_{f_0-\Delta f/2}^{f_0+\Delta f/2} P_X(f)\,df$$

VALID FOR ANY Δf (NOT NECESSARILY SMALL)

OR FOR $f_1 \le f \le f_2$

$$\text{TOTAL AVERAGE POWER} = \int_{f_1}^{f_2} P_X(f)\,df$$

(SIMILAR TO PDF !)

NOW LET $\Delta f \to 0$,

$$r_Y[0] = 2\int_{f_0-\Delta f/2}^{f_0+\Delta f/2} P_X(f)\,df$$

$$\to 2P_X(f_0)\int_{f_0-\Delta f/2}^{f_0+\Delta f/2} df$$

$$= 2P_X(f_0)\Delta f$$

$$P_X(f_0) = \frac{1}{2}\,\frac{r_Y[0]}{\Delta f}$$

$$= \frac{1}{2}\,\frac{\text{TOTAL AVERAGE POWER OUT OF FILTER}}{\Delta f}$$

$$= \frac{1}{2} \; \frac{2 \times \text{TOTAL AVERAGE POWER IN } [f_0 - \Delta f/2,\ f_0 + \Delta f/2)}{\Delta f}$$

$$= \frac{\text{TOTAL AVERAGE POWER IN } [f_0 - \frac{\Delta f}{2},\ f_0 + \frac{\Delta f}{2})}{\Delta f}$$

= POWER PER UNIT FREQUENCY

= POWER SPECTRAL DENSITY !!

ORTHOGONALITY PRINCIPLE AGAIN

SEE SLIDES 92, 93 – WE NOW GENERALIZE THESE RESULTS. SEE PAGES 471-475 IN TEXT.

RECALL THAT FOR X, Y ZERO MEAN RVS WE CAN PREDICT Y BASED ON $X = x$ USING

$$\hat{Y} = a_{OPT}\, x \qquad \text{WHERE} \quad a_{OPT} = \frac{COV(X,Y)}{VAR(X)}$$

THIS CHOICE OF a MINIMIZES THE MSE

$$MSE(a) = E[(Y - aX)^2]$$

OVER a. ALTERNATIVELY, a FOUND USING ORTHOGONALITY PRINCIPLE

$$E[(Y - aX)X] = 0 \quad \Rightarrow \quad a_{OPT} = \frac{E(XY)}{E(X^2)}$$

$\uparrow$ ERROR $\quad \uparrow$ "DATA"

$$= \frac{COV(XY)}{VAR(X)}$$

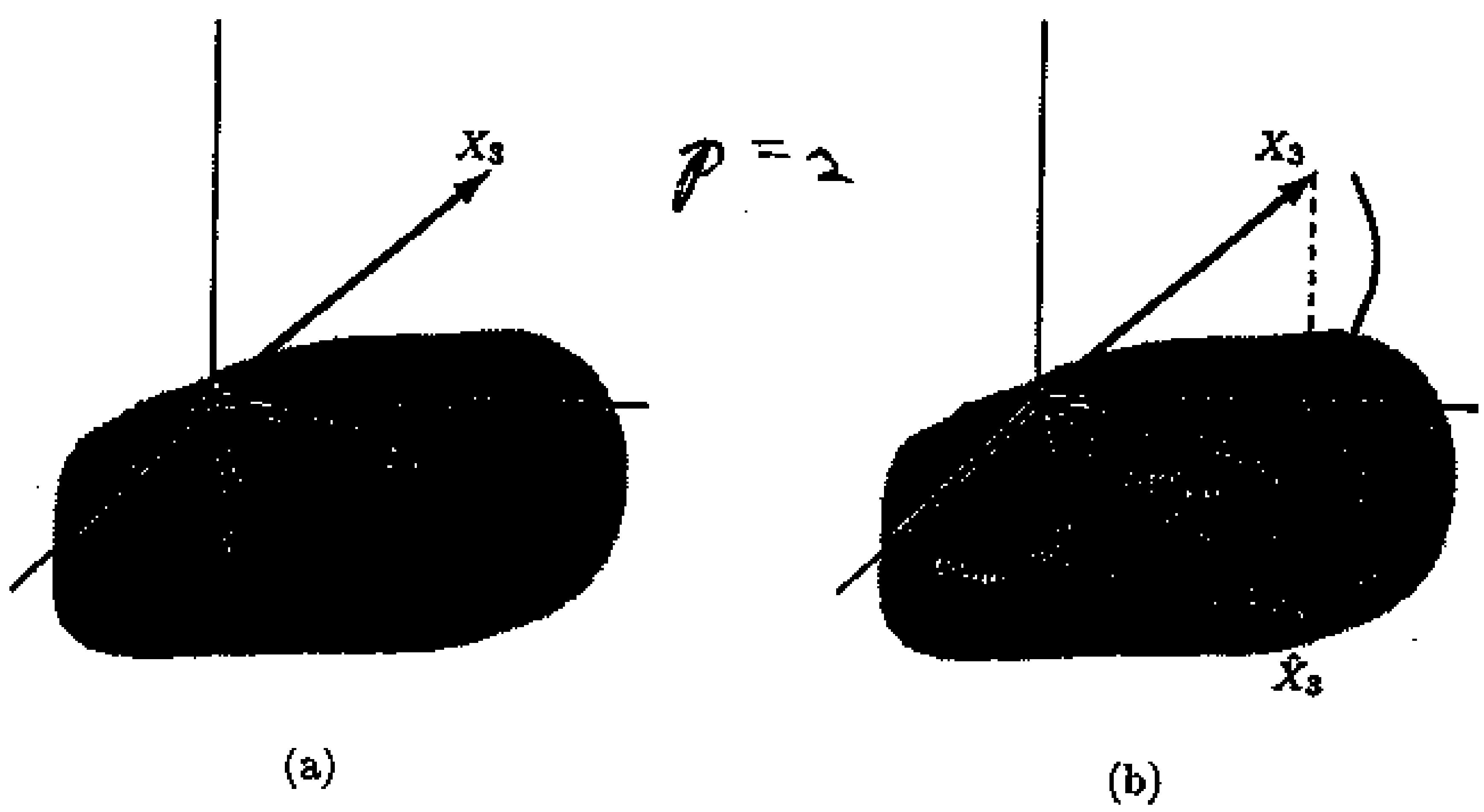
NOW ASSUME WE HAVE RVS

$X_1, X_2, \ldots, X_p, X_{p+1}$ AND WE WISH TO PREDICT X_{p+1} BASED ON OBSERVING $X_1 = x_1, X_2 = x_2, \ldots, X_p = x_p$ AS

$$\hat{X}_{p+1} = \sum_{i=1}^{p} a_i x_i$$

CHOOSE $a_1, a_2, \ldots, a_p$ TO MINIMIZE

$$MSE(a_1, a_2, \ldots, a_p) = E_{X_1, \ldots, X_{p+1}}\left[\left(X_{p+1} - \sum_{i=1}^{p} a_i X_i\right)^2\right]$$



$p = 2$

Figure 14.3: Geometrical interpretation of linear prediction.

WANT $\epsilon \perp X_1$
$\epsilon \perp X_2$

$\Rightarrow E[\epsilon X_1] = 0$
$E[\epsilon X_2] = 0$

GENERAL ORTHOGONALITY PRINCIPLE

$$\Rightarrow \quad E\left[\left(X_{p+1} - \sum_{i=1}^{p} a_i X_i\right) X_k\right] = 0 \qquad k = 1, 2, \ldots, p$$

$$E[X_{p+1}X_k] = \sum_{i=1}^{p} a_i E[X_i X_k]$$

LET $c_{ij} = \mathrm{COV}(X_i, X_j) = E[X_i X_j] = c_{ji}$

$$\sum_{i=1}^{p} a_i c_{ik} = c_{p+1,k} \qquad k = 1, 2, \ldots, p$$

OR

$$\sum_{i=1}^{p} c_{ki} a_i = c_{k,p+1} \qquad k = 1, 2, \ldots, p$$

SET OF p SIMULTANEOUS LINEAR EQUATIONS

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ & & \ddots & \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix}}_{\underline{C}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}_{\underline{a}} = \underbrace{\begin{bmatrix} c_{1,p+1} \\ c_{2,p+1} \\ \vdots \\ c_{p,p+1} \end{bmatrix}}_{\underline{c}}$$

$$\underline{a}_{OPT} = \underline{C}^{-1}\underline{c}$$

$\underline{C}$ IS A COVARIANCE MATRIX $\Rightarrow$ POSITIVE DEFINITE $\Rightarrow$ INVERTIBLE

EXAMPLE : LET $p = 2$

$$\underline{C}_x = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 1 & 2/3 \\ 1/3 & 2/3 & 1 \end{bmatrix} = E[\underline{X}\,\underline{X}^T]$$

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

PREDICT X_3 BASED ON $X_1 = x_1$, $X_2 = x_2$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2/3 \\ 2/3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\frac{1}{1-(2/3)^2} \begin{bmatrix} 1 & -2/3 \\ -2/3 & 1 \end{bmatrix}}_{\frac{\tilde{C}^T}{DET(\underline{C})}} \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$\tilde{\underline{C}} \equiv$ COFACTOR MATRIX

$$= \begin{pmatrix} -4/5 \\ 4/5 \end{pmatrix}$$

WHAT DO WE GET IF $\underline{C}_x = \sigma^2 \underline{I}$?

FINALLY ORTHOGONALITY PRINCIPLE HOLDS IF $p \to \infty$.

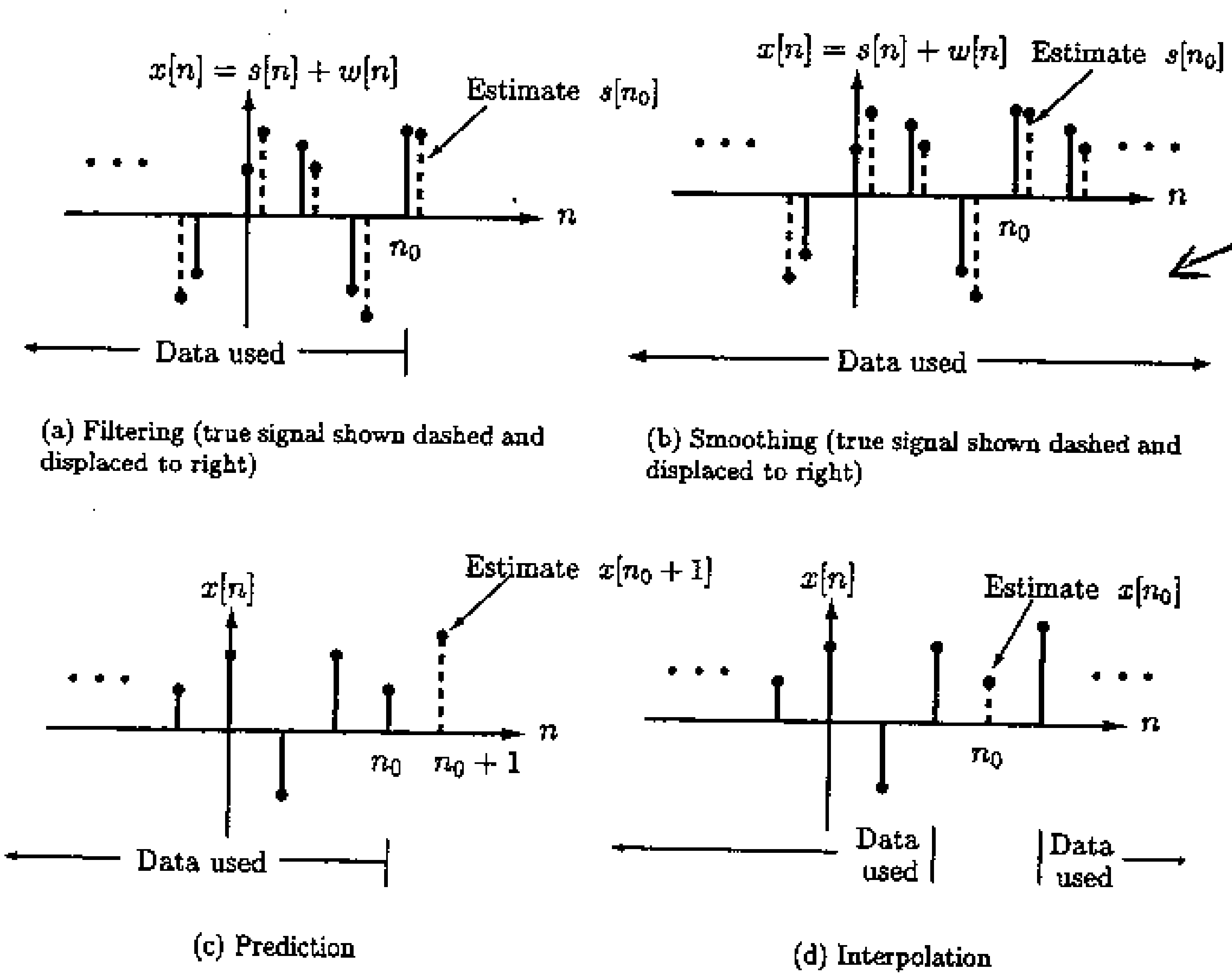
PREDICT $x[n_0]$ BASED ON $\{x[n_0-1], x[n_0-2], \ldots\}$ AS

$$\hat{x}[n_0] = \sum_{k=1}^{\infty} a_k x[n_0-k] \quad \text{AS AN EXAMPLE}$$

$$\Rightarrow E\Big[\underbrace{\Big(x[n_0] - \sum_{k=1}^{\infty} a_k x[n_0-k]\Big)}_{\text{ERROR}} \underbrace{x[n_0-\ell]}_{\text{"DATA"}}\Big] = 0 \qquad \ell = 1, 2, \ldots$$

WIENER FILTERING

WIENER "FILTERING" REFERS TO 4 SEPARATE BUT RELATED PROBLEMS



(a) Filtering (true signal shown dashed and displaced to right)

(b) Smoothing (true signal shown dashed and displaced to right)

WILL SOLVE THIS ONE.

(c) Prediction

(d) Interpolation

Figure 18.4: Definition of Wiener "filtering" problems.

SMOOTHING – NONREAL-TIME PROBLEM

WE OBSERVE $x[n] = s[n] + w[n] \quad -\infty < n < \infty$

ASSUMPTIONS: $s[n]$ IS ZERO MEAN WSS RP WITH PSD $P_s(f)$

$w[n]$ IS ZERO MEAN WSS RP (WITH PSD $P_w(f)$),

WISH TO ESTIMATE $s[n]$ FOR SOME $n = n_0$ USING A LINEAR FILTER

AS

$$\hat{s}[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0-k]$$

$\uparrow$ NONCAUSAL SINCE $h[k] \neq 0$ FOR $k < 0$

ALSO, ESTIMATE AT $n = n_0$ BASED ON FUTURE SAMPLES $x[n_0+1]$, $x[n_0+2]$, ...

$\uparrow h[-1]$ $\uparrow h[-2]$

CALLED A WIENER SMOOTHER.

NOTE: JUST LIKE $\hat{x}_{p+1} = \sum_{i=1}^{p} a_i x_i$ BUT INFINITE NUMBER OF a_i's.

TO DETERMINE $h[k]$ WE MINIMIZE MSE OR

$$MSE = E\left[(s[n_0] - \hat{s}[n_0])^2\right]$$

USING ORTHOGONALITY PRINCIPLE

$$E\left[\left(s[n_0] - \sum_{k=-\infty}^{\infty} h[k]x[n_0-k]\right)x[n_0-l]\right] = 0 \qquad -\infty < l < \infty$$

$$E[s[n_0]x[n_0-l]] = \sum_{k=-\infty}^{\infty} h[k]E[x[n_0-k]x[n_0-l]]$$

BUT IT IS REASONABLE TO ASSUME THAT $s[n]$ AND $w[n]$ ARE UNCORRELATED. SINCE THEY ARE ZERO MEAN $E(s[n]w[m]) = 0$ FOR ALL m, n.