EXAMPLE: CONTINUOUS-TIME WIENER RP (BROWNIAN MOTION)

$$X(t) = \int_0^t U(\xi)\, d\xi \qquad t \geq 0$$

$\uparrow$ CONT-TIME WGN WITH $r_U(\tau) = \frac{N_0}{2}\delta(\tau)$

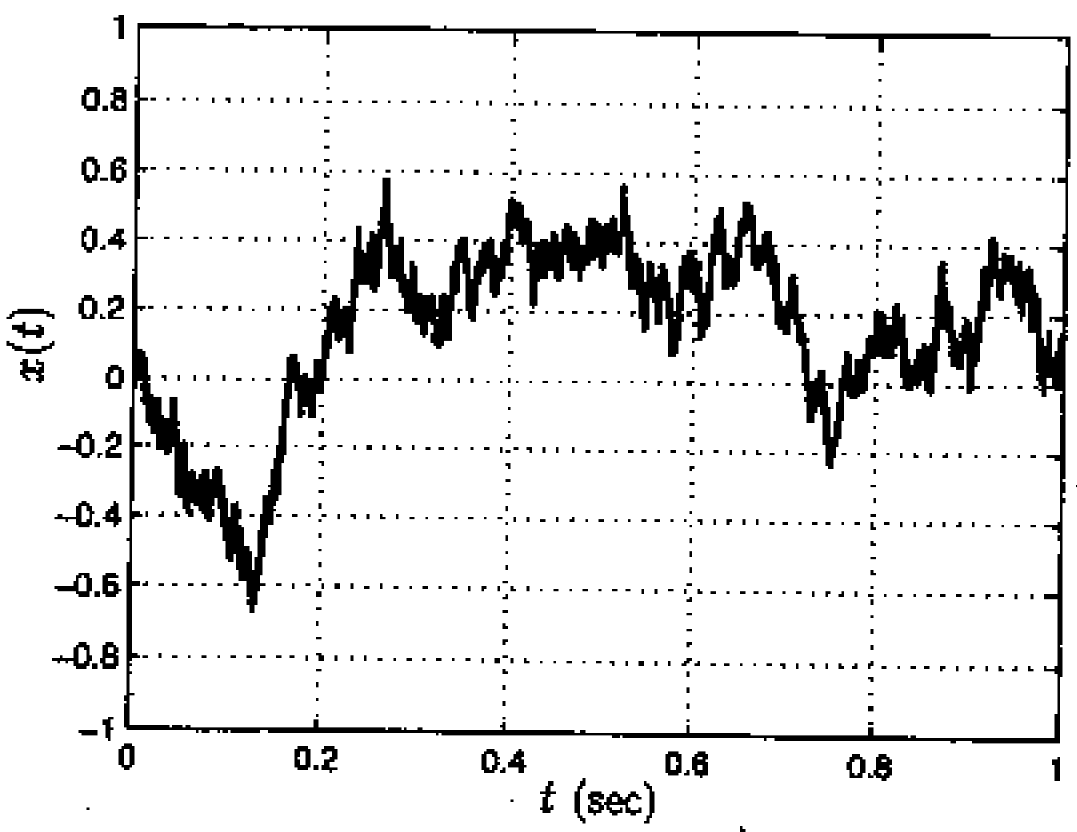


Figure 20.6: Typical realization of the Wiener random process.

NOTE: IT HAS INCREMENTS $X(t_2) - X(t_1)$ THAT ARE INDEPENDENT AND STATIONARY, e.g., CONSIDER $X(4) - X(3)$ AND $X(2) - X(1)$.

$X(t)$ IS GAUSSIAN SINCE IT IS A "SUM" OF GAUSSIAN RVS. (NEED MORE MATH TO PROVE THIS).

$$E(X(t)) = E\left(\int_0^t U(\xi)\, d\xi\right)$$

$$= \int_0^t E(U(\xi))\, d\xi = 0$$

$\uparrow$ WGN

SHOWN IN TEXT THAT

$$E[x(t_1)x(t_2)] = N_0/2 \ \mathrm{MIN}(t_1, t_2)$$

$$\Rightarrow [\underset{\sim}{C}]_{ij} = \frac{N_0}{2} \mathrm{MIN}(t_i, t_j)$$

ALSO, $\mathrm{VAR}(x(t)) = N_0/2 \ t$

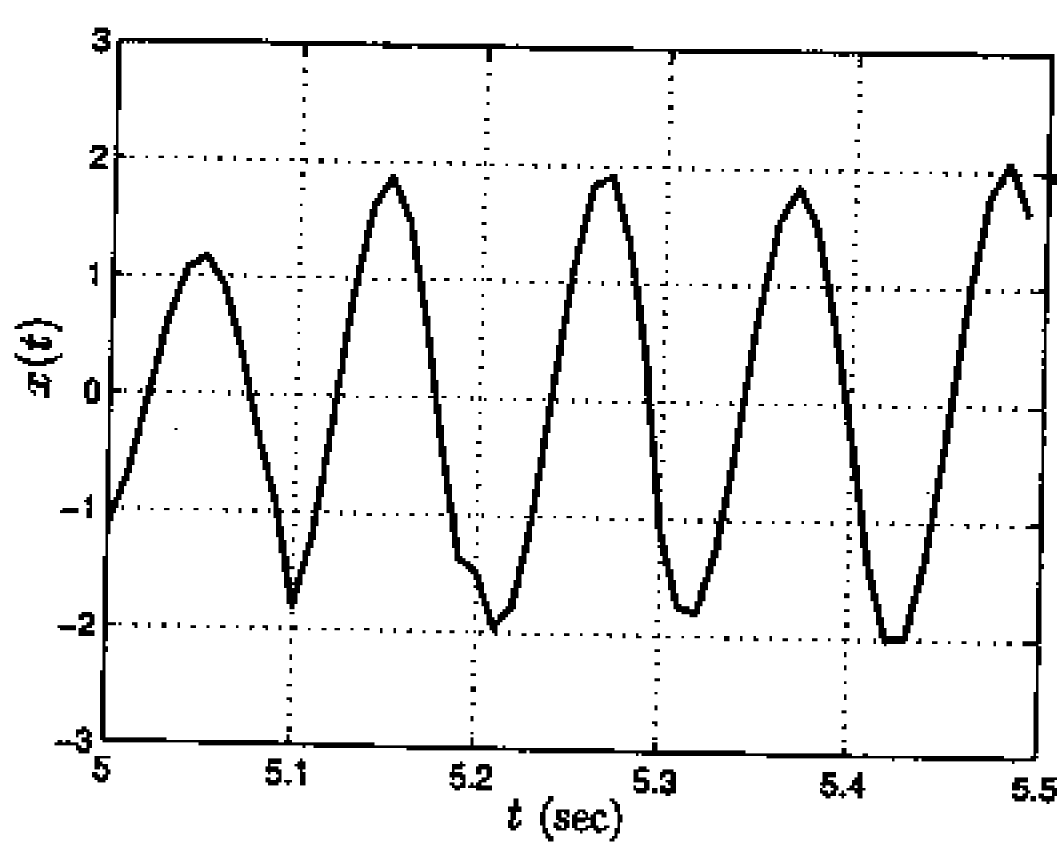
WIENER RP HAS $x(t) \sim \mathcal{N}(0, \frac{N_0}{2} t)$

$\Rightarrow$ NONSTATIONARY GAUSSIAN R.P.

SPECIAL CONT-TIME GAUSSIAN RPS

RAYLEIGH FADING SINUSOID

GOOD MODEL FOR MULTIPATH FADING - OCCURS FOR A TRANSMITTED SINUSOID THAT ARRIVES AT DESTINATION VIA MULTIPLE PATHS. SIMILAR TO FISH COUNTING EXAMPLE, IN WHICH THERE IS INTERFERENCE BETWEEN RETURNS.



AMPLITUDE RANDOM, PHASE RANDOM

Figure 20.8: Segment of waveform shown in Figure 20.2 for $5 \leq t \leq 5.5$ seconds.

AS A MODEL OVER SHORT PERIOD OF TIME

$$X(t) = \underset{\substack{\uparrow \\ R.V.}}{A} \cos(2\pi F_0 t + \underset{\substack{\uparrow \\ R.V.}}{\Theta}) \qquad A > 0, \quad 0 < \Theta < 2\pi$$

WHAT PDFS SHOULD WE ASSIGN TO A AND Θ?

$$X(t) = A \cos 2\pi F_0 t \cos\Theta - A \sin 2\pi F_0 t \sin\Theta$$

$$= \underbrace{A \cos\Theta}_{U} \cos 2\pi F_0 t - \underbrace{A \sin\Theta}_{V} \sin 2\pi F_0 t$$

$$= U \cos 2\pi F_0 t - V \sin 2\pi F_0 t$$

SINCE WE BELIEVE THAT $X(t)$ IS GAUSSIAN DUE TO CLT (LARGE NUMBER OF ADDITIVE RETURNS), WE ASSIGN GAUSSIAN PDFS TO U AND V AS $U \sim N(0, \sigma^2)$, $V \sim N(0, \sigma^2)$

WITH $E(U) = E(V) = 0 \Rightarrow E(X(t)) = 0$

ALSO, ASSUME U AND V ARE INDEPENDENT - AGREES WITH EXPERIMENTS.

$\Rightarrow$ $X(t)$ IS GAUSSIAN RP WITH ZERO MEAN SINCE

$$\underbrace{\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_K) \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} \cos(2\pi F_0 t_1) & -\sin(2\pi F_0 t_1) \\ \vdots & \\ \cos(2\pi F_0 t_K) & -\sin(2\pi F_0 t_K) \end{bmatrix}}_{\underline{G}} \begin{bmatrix} U \\ V \end{bmatrix}$$

$U \sim N(0, \sigma^2)$, $V \sim N(0, \sigma^2)$ AND INDEPENDENT

$\Rightarrow \begin{bmatrix} U \\ V \end{bmatrix}$ HAS MULTIVARIATE GAUSSIAN PDF

$\Rightarrow \underline{X}$ " " " "

$\Rightarrow X(t)$ IS GAUSSIAN RP.

ALSO, $A = \sqrt{U^2+V^2}$

$\Theta = \arctan V/U$

$$\Rightarrow \left.\begin{aligned} A &\sim \text{RAYLEIGH} \\ \Theta &\sim U(0, 2\pi) \end{aligned}\right\} \text{INDEPENDENT}$$

MODEL CALLED RAYLEIGH FADING SINUSOID (OR RAYLEIGH FADING CHANNEL)

FINALLY, WE SHOW $X(t)$ IS WSS.

SINCE $E(U) = E(V) = 0 \Rightarrow E(X(t)) = 0$

$$E(X(t)X(t+\tau)) = E\big[(U\cos(2\pi F_0 t) - V\sin(2\pi F_0 t)) \cdot (U\cos(2\pi F_0(t+\tau)) - V\sin(2\pi F_0(t+\tau)))\big]$$

$$= E(U^2)\cos(2\pi F_0 t)\cos(2\pi F_0(t+\tau))$$
$$+E(V^2)\sin(2\pi F_0 t)\sin(2\pi F_0(t+\tau))$$

SINCE $E(UV) = E(U)E(V) = 0$ U, V IND.

$$= \sigma^2\left(\cos(2\pi F_0 t)\cos(2\pi F_0(t+\tau)) + \sin(2\pi F_0 t)\sin(2\pi F_0(t+\tau))\right)$$

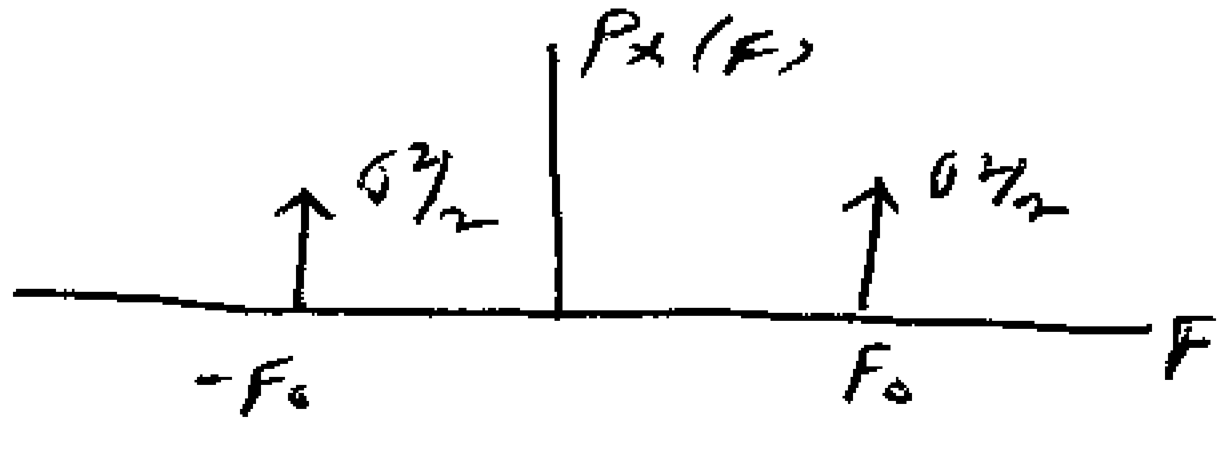
$= \sigma^2 \cos 2\pi F_0 \tau$ NOT DEPENDENT ON t

$\Rightarrow r_X(\tau) = \sigma^2 \cos 2\pi F_0 \tau$

AND $X(t)$ IS WSS

FINALLY, PSD OF RAYLEIGH FADING SINUSOID IS

$$P_X(F) = \mathcal{F}\{r_X(\tau)\}$$
$$= \frac{\sigma^2}{2}\delta(F+F_0) + \frac{\sigma^2}{2}\delta(F-F_0)$$



TOTAL AVERAGE POWER $= r_X(0) = \sigma^2$

BANDPASS GAUSSIAN RP

FOR LONGER OBSERVATION TIMES THE SINUSOIDAL AMPLITUDE CHANGES - SEE FIG. 20.2. BETTER MODEL NOW IS TO LET A, Θ VARY WITH t.

$$x(t) = \underset{\substack{\uparrow \\ \text{AMPLITUDE} \\ \text{"MODULATION"}}}{A(t)} \cos\left(2\pi F_0 t + \underset{\substack{\uparrow \\ \text{PHASE} \\ \text{"MODULATION"}}}{\Theta(t)}\right) \qquad \begin{array}{l} A(t) > 0 \\ 0 < \Theta(t) < 2\pi \end{array}$$

CALLED A BANDPASS RP MODEL (WILL SEE LATER THAT PSD HAS NONZERO BANDWIDTH)

AS BEFORE

$$X(t) = \underbrace{A(t)\cos\Theta(t)}_{U(t)}\cos 2\pi F_0 t - \underbrace{A(t)\sin\Theta(t)}_{V(t)}\sin 2\pi F_0 t$$

$$= U(t)\cos 2\pi F_0 t - V(t)\sin 2\pi F_0 t$$

WE WISH TO USE THIS TO MODEL A GAUSSIAN BANDPASS RP. HENCE, ASSUME THAT $U(t)$, $V(t)$ ARE GAUSSIAN RPS AND INDEPENDENT OF EACH OTHER

$(\, p_{\underline{U},\underline{V}} = p_{\underline{U}}\, p_{\underline{V}}$ FOR ALL $\underline{U}$ AND $\underline{V}\,)$

FOR $E[x(t)] = 0$ ASSUME $E[U(t)] =$
$E[V(t)] = 0$ ALL t

FOR $X(t)$ TO BE WSS WE ASSUME $U(t)$, $V(t)$ ARE EACH WSS AND HAVE SAME ACF $r_U(\tau) = r_V(\tau)$. CAN NOW SHOW THAT

$$r_X(\tau) = r_U(\tau) \cos 2\pi F_0 \tau$$

SEE BOOK PG 692

$\Rightarrow$ $X(t)$ IS WSS AND

$$P_X(F) = \mathcal{F}\{r_X(\tau)\}$$

$$= \frac{1}{2} P_U(F+F_0) + \frac{1}{2} P_U(F-F_0)$$

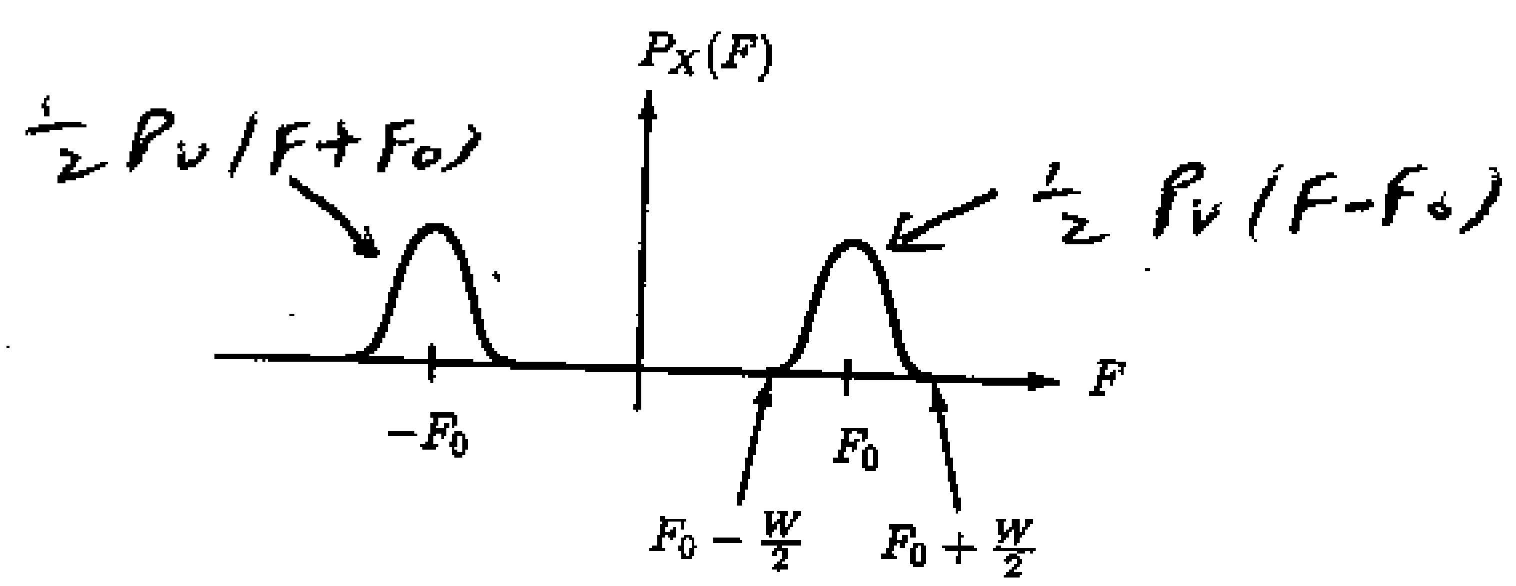
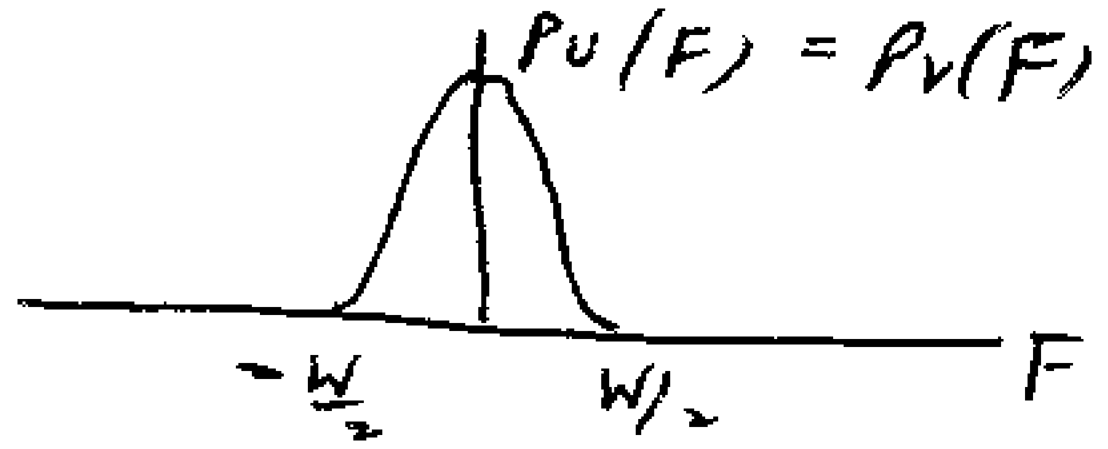


Figure 20.9: Typical PSD for bandpass random process. The PSD is assumed to be symmetric about $F = F_0$ and also that $F_0 > W/2$.

TO MODEL THIS WE REQUIRE $U(t)$ AND $V(t)$ TO BE LOWPASS RPS, WHOSE BANDWIDTH IS $W/2$



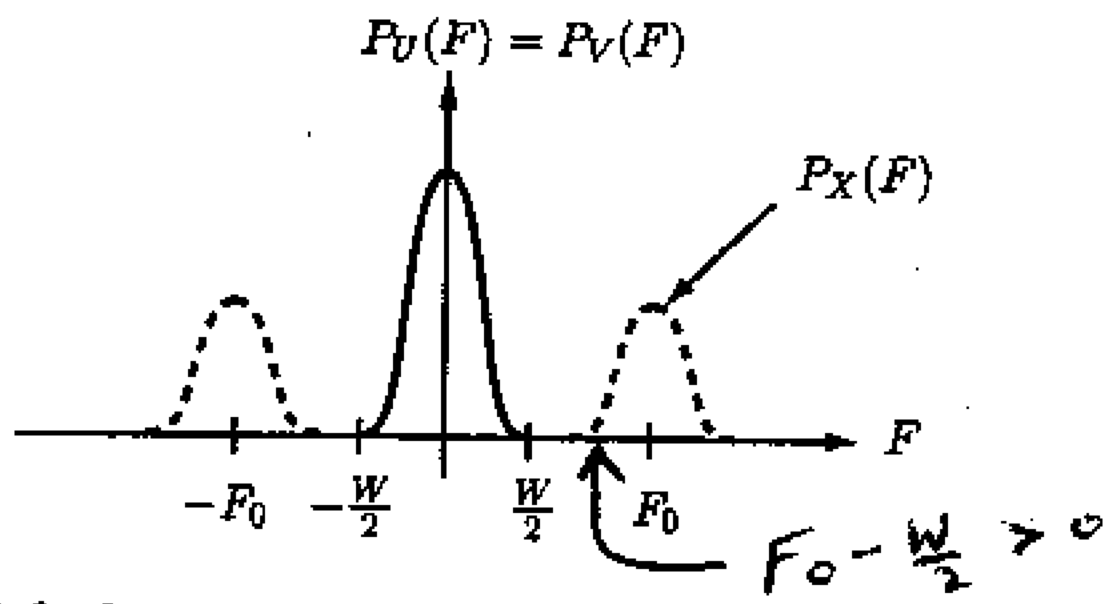


Figure 20.10: PSD for lowpass random processes $U(t)$ and $V(t)$. The PSD for the bandpass random process $X(t)$ is shown as the dashed curve.

NEED
$F_0 > W/2$

HENCE, IT IS SEEN THAT WE REPRESENT A BANDPASS RP BY MODULATING TWO INDEPENDENT LOWPASS RPs TO BE CENTERED ABOUT $F = F_0$.

CAN ONLY MODEL $P_X(F)$ FOR WHICH $P_X(F)$ IS SYMMETRIC ABOUT $F = F_0$ (SINCE $P_U(F) = P_V(F)$ ARE SYMMETRIC ABOUT $F = 0$).

<u>SUMMARY</u>: FOR $X(t)$ A WSS GAUSSIAN RP WITH ZERO MEAN AND A BANDPASS PSD GIVEN AS $P_X(F) = \frac{1}{2} P_U(F + F_0) + \frac{1}{2} P_U(F - F_0)$ FOR $P_U(F) = 0 \quad |F| > W/2$, USE

$$X(t) = U(t) \cos 2\pi F_0 t - V(t) \sin 2\pi F_0 t$$

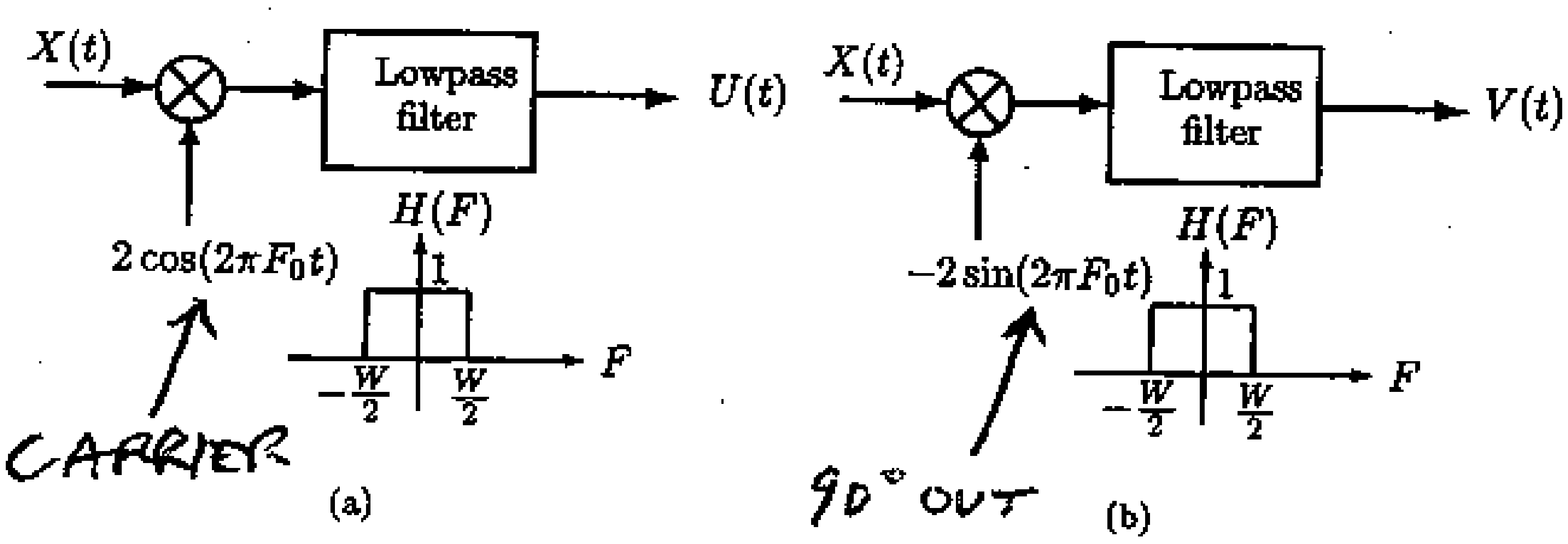
WHERE 1) $U(t)$, $V(t)$ ARE WSS GAUSSIAN RPs

2) $E[U(t)] = E[V(t)] = 0$

3) $r_U(\tau) = r_V(\tau) = \mathcal{F}^{-1}\{P_U(F)\}$

4) $U(t)$, $V(t)$ ARE INDEPENDENT OF EACH OTHER

$U(t)$, $V(t)$ CALLED <u>IN-PHASE</u> AND <u>QUADRATURE</u> COMPONENTS OF $X(t)$



CARRIER

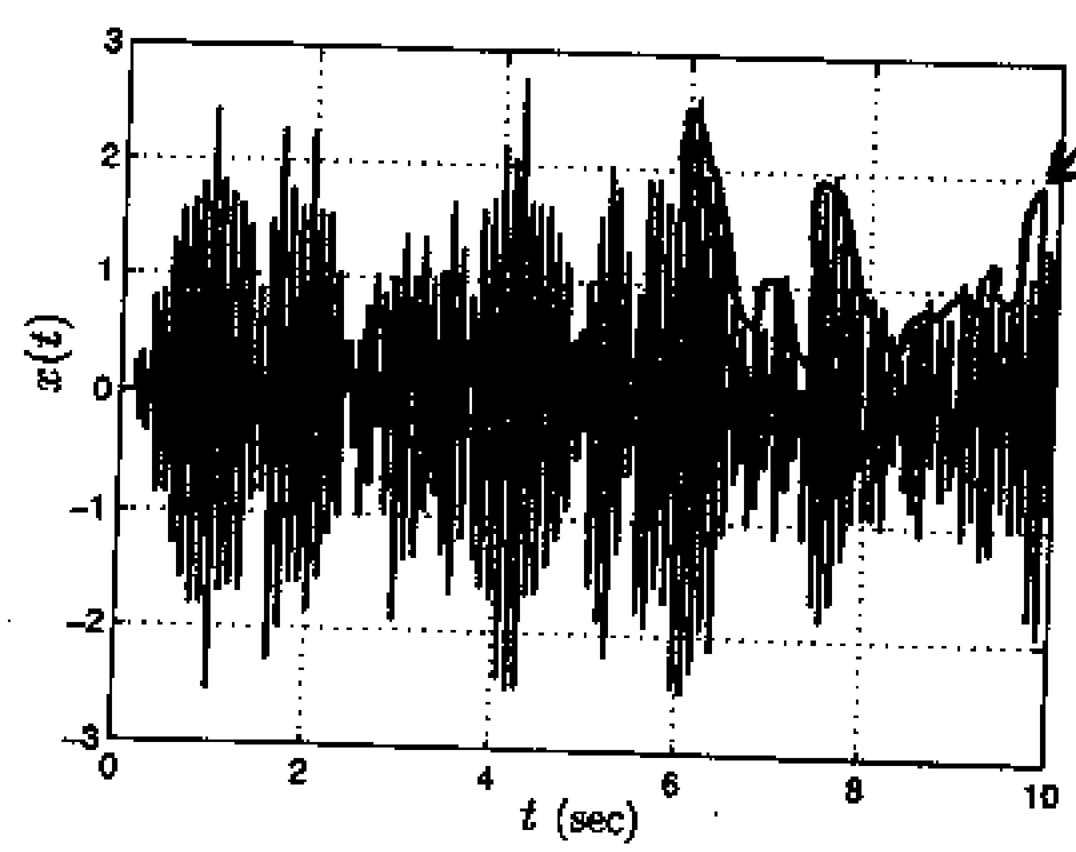
90° OUT OF PHASE WITH CARRIER

Figure 20.16: Extraction of bandpass random process lowpass components for Problem 20.24.

CAN ALSO WRITE $X(t)$ AS

$$X(t) = \underbrace{\sqrt{U^2(t) + V^2(t)}}_{A(t)} \cos\left(2\pi F_0 t + \text{ARCTAN} \frac{V(t)}{U(t)}\right)$$

CALLED THE ENVELOPE OF $X(t)$



$A(t)$, SLOWLY VARYING

Figure 20.2: Received waveform consisting of many randomly overlapped and random amplitude echos.

EXAMPLE : "BANDPASS" WGN

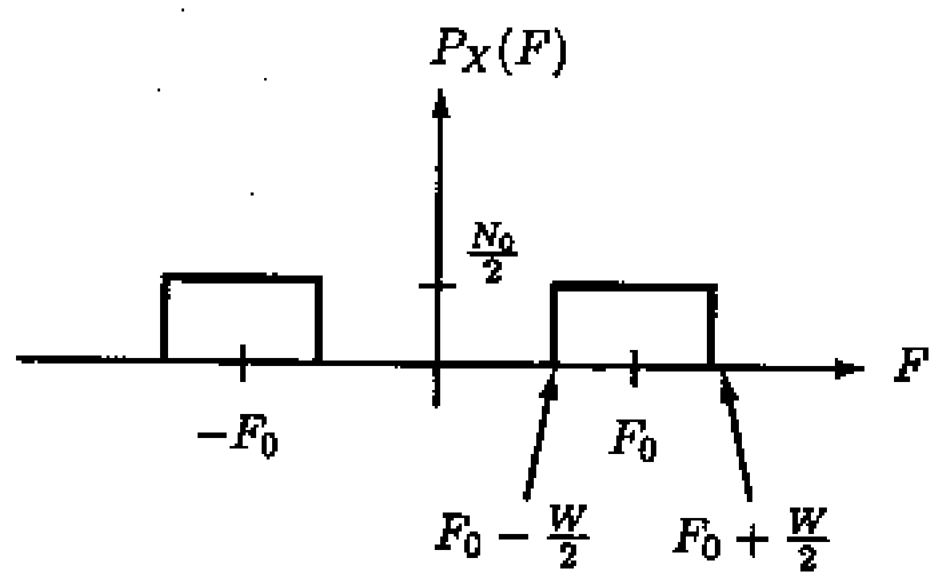


Figure 20.12: PSD for bandpass "white" Gaussian noise.

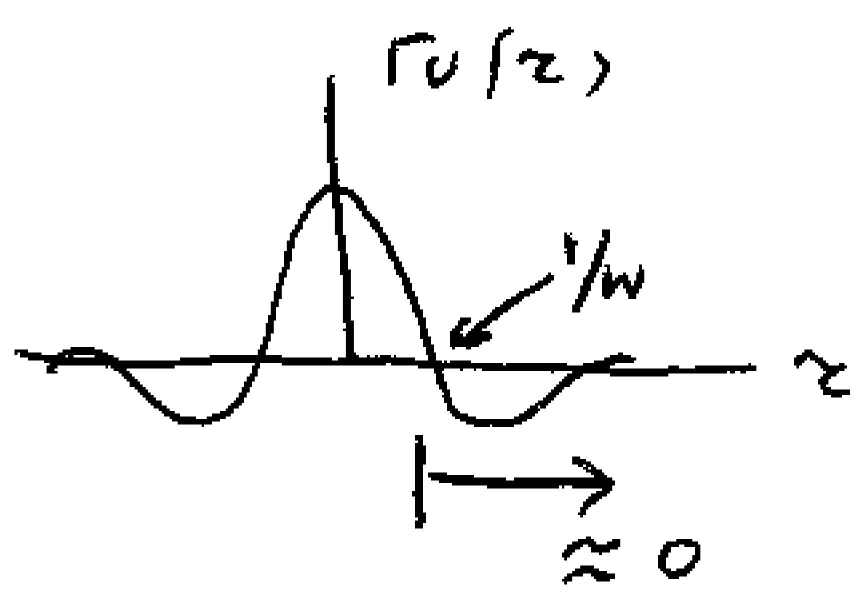
OBTAINED AFTER BANDPASS FILTERING TO EXCLUDE NOISE OUTSIDE OF SIGNAL BAND $|F-F_0| \le W/2$

FROM FIGURE 20.10 MUST HAVE

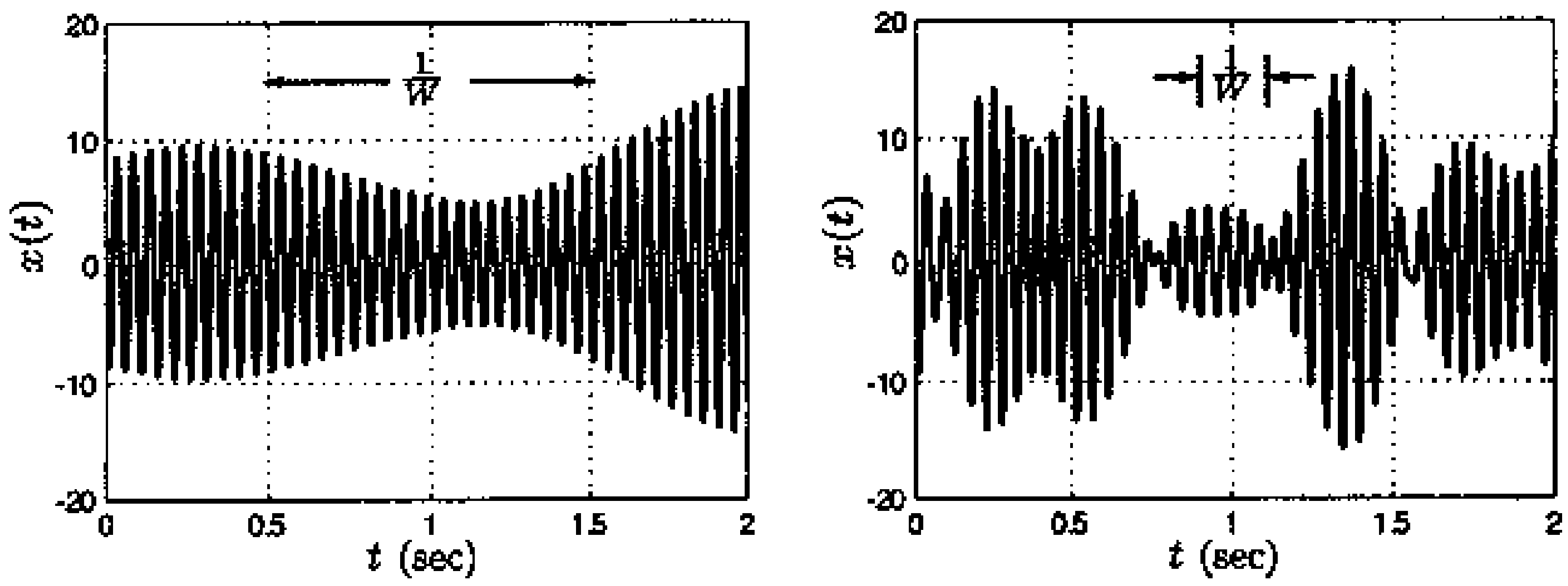
$$P_U(F) = P_V(F) = \begin{cases} N_0 & |F| \le W/2 \\ 0 & |F| > W/2 \end{cases}$$

NOTE THAT

$$\begin{aligned} r_U(\tau) &= r_V(\tau) \\ &= \mathcal{F}^{-1}\{P_U(F)\} \\ &= N_0 W \frac{\sin \pi W \tau}{\pi W \tau} \quad -\infty < \tau < \infty \\ &\approx 0 \quad \text{FOR } \tau > 1/W \end{aligned}$$



SINCE $A(t) = \sqrt{U^2(t) + V^2(t)}$
WE EXPECT ENVELOPE TO BE UNCORRELATED FOR $\tau = \Delta_t > 1/W$
(JUST A "RULE OF THUMB" - APPROXIMATE)



(a) $F_0 = 20$ Hz, $W = 1$ Hz

(b) $F_0 = 20$ Hz, $W = 4$ Hz

Figure 20.13: Typical realizations of bandpass "white" Gaussian noise. The PSD is given in Figure 20.12.

COMPUTER SIMULATION

CONSIDER GENERATING SEGMENT OF REALIZATION OF DISCRETE-TIME WSS GAUSSIAN R.P.
WE ARE GIVEN $\mu = 0$ (OTHERWISE JUST ADD μ) AND TYPICALLY PSD $P_X(f)$.

APPROACH: USE $P_X(f) = |H(f)|^2 P_U(f)$

$\uparrow$ WGN

LET $P_U(f) = \sigma_U^2 = 1 \Rightarrow$

$$P_X(f) = |H(f)|^2 \Rightarrow H(f) = \sqrt{P_X(f)}\, e^{j\theta(f)}$$

WHERE $\theta(f)$ IS ARBITRARY. CALLED A <u>SPECTRAL FACTORIZATION</u>. TYPICALLY $\theta(f)$ IS CHOSEN TO PRODUCE A CAUSAL FILTER.

EXAMPLE: $P_X(f) = \frac{1}{2}(1 + \cos(4\pi f))$

$$|H(f)| = \sqrt{P_X(f)} = \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}$$

$$H(f) = \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}\; e^{j\theta(f)} \qquad |f| \le \tfrac{1}{2}$$

ASSUME WE USE TIME DOMAIN IMPLEMENTATION,

$$x(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

NEED IMPULSE RESPONSE $h(k)$

BUT $h(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(f)\, e^{j2\pi f n}\, df$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\tfrac{1}{2}(1+\cos(4\pi f))}\; e^{j\theta(f)} e^{j2\pi f n}\, df$$

$$= \sqrt{\tfrac{1}{2}(1 + \cos^2 2\pi f - \sin^2 2\pi f)} \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

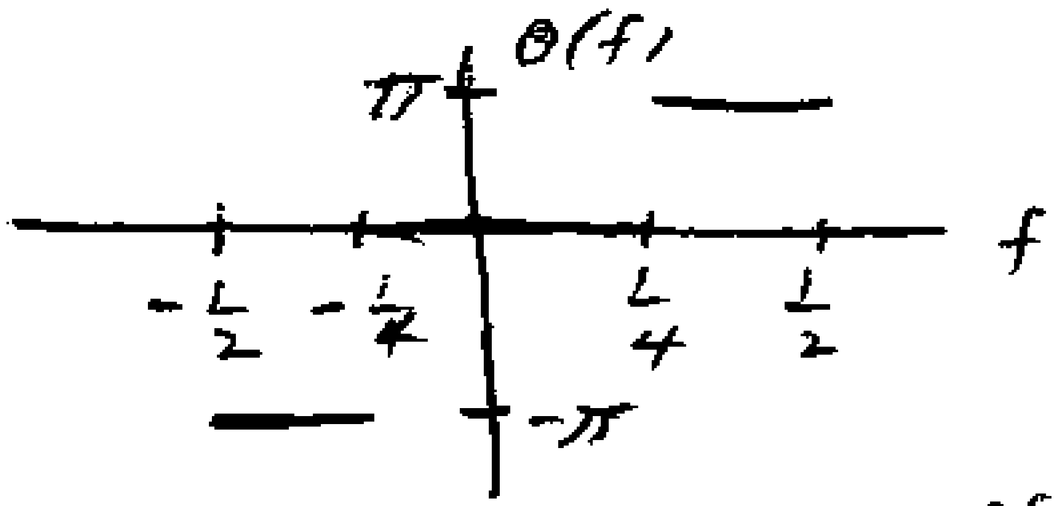
$$= \sqrt{\tfrac{1}{2}(\sin^2 2\pi f + \cos^2 2\pi f + \cos^2 2\pi f - \sin^2 2\pi f)}$$

$$= \sqrt{\cos^2 2\pi f} = |\cos 2\pi f|$$

$$h(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\cos 2\pi f|\; e^{j\theta(f)} e^{j2\pi f n}\, df$$

LET $\theta(f) = 0$ FOR $\cos 2\pi f > 0$

$= \pm \pi$ FOR $\cos 2\pi f < 0$



$$\Rightarrow \quad |\cos 2\pi f| \, e^{j\theta(f)} = \cos 2\pi f \qquad \text{ALL } f$$

$$h[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2\pi f \, e^{j2\pi f n} \, df$$

$$= \begin{cases} \frac{1}{2} & n = \pm 1 \\ 0 & \text{OTHERWISE} \end{cases} \qquad \text{WHY?}$$

$$x[n] = \sum_{k=-\infty}^{\infty} h[k] \, v[n-k]$$

$$= h[-1] \, v[n+1] + h[1] \, v[n-1]$$

$$= \frac{1}{2} v[n+1] + \frac{1}{2} v[n-1]$$

OR SINCE $|H(f)| = \left| \frac{1}{2} + \frac{1}{2} e^{-j4\pi f} \right|$

$$= \left| \frac{1}{2} e^{j2\pi f} + \frac{1}{2} e^{-j2\pi f} \right|$$

WE CAN ALSO USE

$$x[n] = \frac{1}{2} v[n] + \frac{1}{2} v[n-2]$$

IF WE WANT A CAUSAL FILTER.

OTHER METHODS:

1) ARMA

$$x[n] = \sum_{k=1}^{p} a[k]\, x[n-k] + u[n] - \sum_{k=1}^{q} b[k]\, u[n-k]$$

NEED TO FIND $a[k]$'s, $b[k]$'s, σ_u^2 BASED ON GIVEN $P_x(f)$ $\Rightarrow$ FILTER DESIGN PROBLEM

2) BRUTE FORCE BLOCK METHOD

FOR N SAMPLES $x[0], x[1], \ldots, x[N-1]$ WITH ZERO MEAN

$$\underline{x} = \underline{G}\,\underline{u} \qquad \underline{u} = \begin{bmatrix} u[0] \\ \vdots \\ u[N-1] \end{bmatrix} \sim \mathcal{N}(\underline{0}, \underline{I})$$

USE RANDN(N, 1)

$$\underline{C}_x = \underline{G}\,\underline{G}^T$$

WHERE $(\underline{C}_x)_{ij} = r_x[i-j] = \mathcal{F}^{-1}\{P_x(f)\}\big|_{k=i-j}$ CAN BE FOUND FROM PSD

AND USE CHOLESKY DECOMPOSITION TO FIND $\underline{G}$.

REAL-WORLD EXAMPLE - ESTIMATING FISH POPULATION (FIGURE 20.14)

EACH FISH REFLECTS INCOMING SINUSOIDAL PULSE. AT RECEIVER i^{th} FISH CONTRIBUTES

$$x_i(t) = A_i \cos(2\pi F_0(t-\tau_i) + \Theta_i)$$

F_0 = TRANSMITTED FREQUENCY

$\tau_i = 2R_i/c$ $\quad$ R_i = RANGE

c = SPEED OF SOUND

A_i, Θ_i DEPEND ON POSITION OF FISH, ORIENTATION, MOTION, ETC - TOO COMPLICATED TO MODEL - ASSUME A_i, Θ_i ARE RVS. ALSO, DO NOT KNOW $R_i \Rightarrow \tau_i$ UNKNOWN AS WELL $\Rightarrow$

$$x_i(t) = A_i \cos(2\pi F_0 t + \underbrace{\Theta_i - 2\pi F_0 \tau_i})$$

Θ_i' - MODEL AS NEW RANDOM PHASE - REDUCE TO $(0, 2\pi)$

FOR N REFLECTIONS

$$x(t) = \sum_{i=1}^{N} x_i(t)$$

$$= \sum_{i=1}^{N} A_i \cos(2\pi F_0 t + \Theta_i')$$

LET $U_i = A_i \cos \Theta_i'$

$V_i = A_i \sin \Theta_i'$

$$X(t) = \sum_{i=1}^{N} (U_i \cos 2\pi F_0 t - V_i \sin 2\pi F_0 t)$$

$$= \underbrace{\left(\sum_{i=1}^{N} U_i\right)}_{U} \cos 2\pi F_0 t - \underbrace{\left(\sum_{i=1}^{N} V_i\right)}_{V} \sin 2\pi F_0 t$$

NOW ASSUME FISH ARE ABOUT SAME SIZE AND HENCE U_i, V_i ARE IDENTICALLY DISTRIBUTED. ALSO, REFLECTIONS ARE INDEPENDENT (VALID?)

BY CLT THEN U, V ARE GAUSSIAN. ALSO, ASSUME $E(U_i) = E(V_i) = 0$ (HYDROSTATIC WATER PRESSURE NOT MEASURED) AND $\text{VAR}(U_i) = \text{VAR}(V_i) = \sigma^2$

$\Rightarrow$ $U_i \sim N(0, \sigma^2)$ $\quad V_i \sim N(0, \sigma^2)$

AND ALL U_i'S, V_i'S ARE INDEPENDENT

FINALLY, $\left.\begin{array}{l} U \sim N(0, N\sigma^2) \\ V \sim N(0, N\sigma^2) \end{array}\right\}$ IND

$\Rightarrow$ RAYLEIGH FADING SINUSOID MODEL

$$X(t) = U \cos 2\pi F_0 t - V \sin 2\pi F_0 t$$

NEED TO FIND N. ASSUME WE KNOW σ^2 (MAYBE DO EXPERIMENTS IN LAB)

POSSIBLE APPROACH: LET $A = \sqrt{U^2+V^2}$

$$\Rightarrow \quad p_A(a) = \begin{cases} \dfrac{a}{N\sigma^2} e^{-\frac{1}{2} a^2/N\sigma^2} & a \geq 0 \\ 0 & a < 0 \end{cases}$$

$A \sim$ RAYLEIGH

$$E(A) = \sqrt{\tfrac{\pi}{2} N \sigma^2}$$

$$\Rightarrow \quad N = \frac{2}{\pi \sigma^2} E^2(A)$$

SINCE WE GET ONLY ONE (U, V) MEASUREMENT PER PULSE TRANSMISSION (RECALL FOR RAYLEIGH FADING SINUSOID ENVELOPE IS NEARLY CONSTANT), SEND MULTIPLE PULSES.

RECEIVE $X_m(t)$, $m = 1, 2, \ldots, M$

$\Rightarrow$ MEASURE ENVELOPES AS

$$\hat{A}_m = \sqrt{U_m^2 + V_m^2}$$

FINALLY, $$\hat{N} = \frac{2}{\pi \sigma^2} \left(\frac{1}{M} \sum_{m=1}^{M} \hat{A}_m \right)^2$$