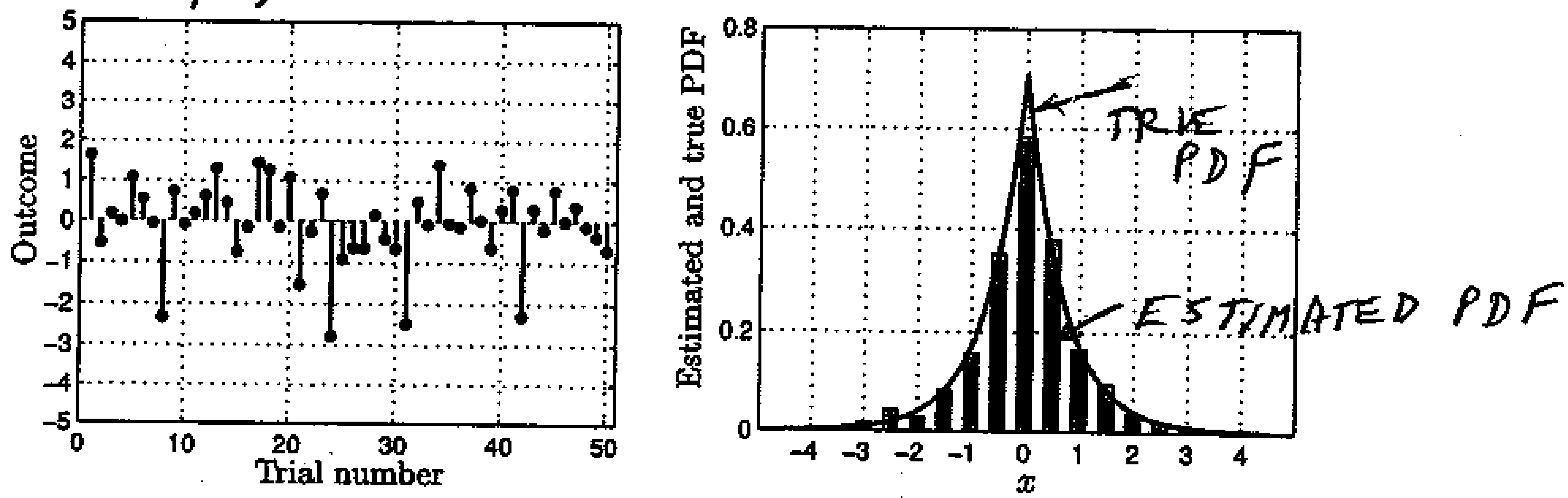
$$x = \sqrt{\sigma^2/2}\ \ln(2u) \qquad 0 < u < \tfrac{1}{2}$$

$$\sqrt{\sigma^2/2}\ \ln\left(\frac{1}{2(1-u)}\right) \qquad \tfrac{1}{2} < u < 1$$

NEGATIVE VALUES OF x

POSITIVE VALUES OF x



(a) First 50 outcomes

(b) True PDF and estimated PDF based on 1000 outcomes

Figure 10.30: Computer generation of Laplacian random variable outcomes using inverse probability integral transformation.

ESTIMATING THE PDF

RECALL $\quad p_X(x_0) = \dfrac{P[x_0 - \Delta x/2 \le X \le x_0 + \Delta x/2]}{\Delta x}$

BUT $P[x_0 - \frac{\Delta x}{2} \le X \le x_0 + \Delta x/2) \approx$

$$\frac{\text{NUMBER OF OUTCOMES IN } [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]}{\underbrace{\text{TOTAL NUMBER}}_{M}}$$

$$\Rightarrow \hat{p}_X(x_0) = \frac{\text{NUMBER OF OUTCOMES IN } [x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}]}{M \Delta x}$$

IN PREVIOUS EXAMPLE $\Delta x = 0.5$ AND **BINS** ARE $[-4.25, -3.75)$, $[-3.75, -3.25)$, ..., $[3.75, 4.25]$

$\hat{p}_X(4)$ $\quad$ $\hat{p}_X(-4)$ $\quad$ $\hat{p}_X(-3.5)$

SEE MATLAB CODE ON PG. 328 AND EXAMPLE 2.1

CHAPTER 11 - EXPECTED VALUES

FOR A DISCRETE R.V. $E[X] = \sum_i x_i \, p_X[x_i]$
(SEE SECTION 6.3)

EXAMPLE: FAIR DIE TOSS

$$S_X = \{1, 2, 3, 4, 5, 6\}$$

$$E[X] = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6) = 3\tfrac{1}{2}$$

FOR A CONT. R.V. WE DEFINE $E(X)$ AS

$$E(X) = \int_{-\infty}^{\infty} x \, p_X(x) \, dx$$

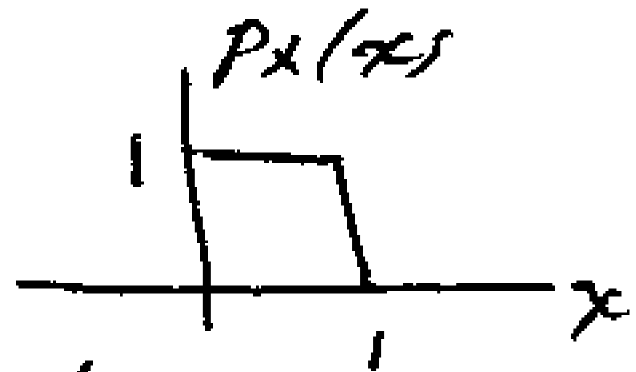
CAN BE THOUGHT OF AS LIMIT OF

$$\sum_{i=1}^{M} x_i \underbrace{p_X(x_i)\Delta x}_{} \qquad \text{AS} \quad \Delta x \to 0$$

$$\simeq \underbrace{P\left[x_i - \tfrac{\Delta x}{2} \le X \le x_i + \Delta x/2\right]}_{\simeq\, p_X[x_i]}$$

SEE BOOK FOR DISCRETE TO CONT. DEFINITION EXAMPLE

<u>EXAMPLE</u>: $X \sim U(0,1)$



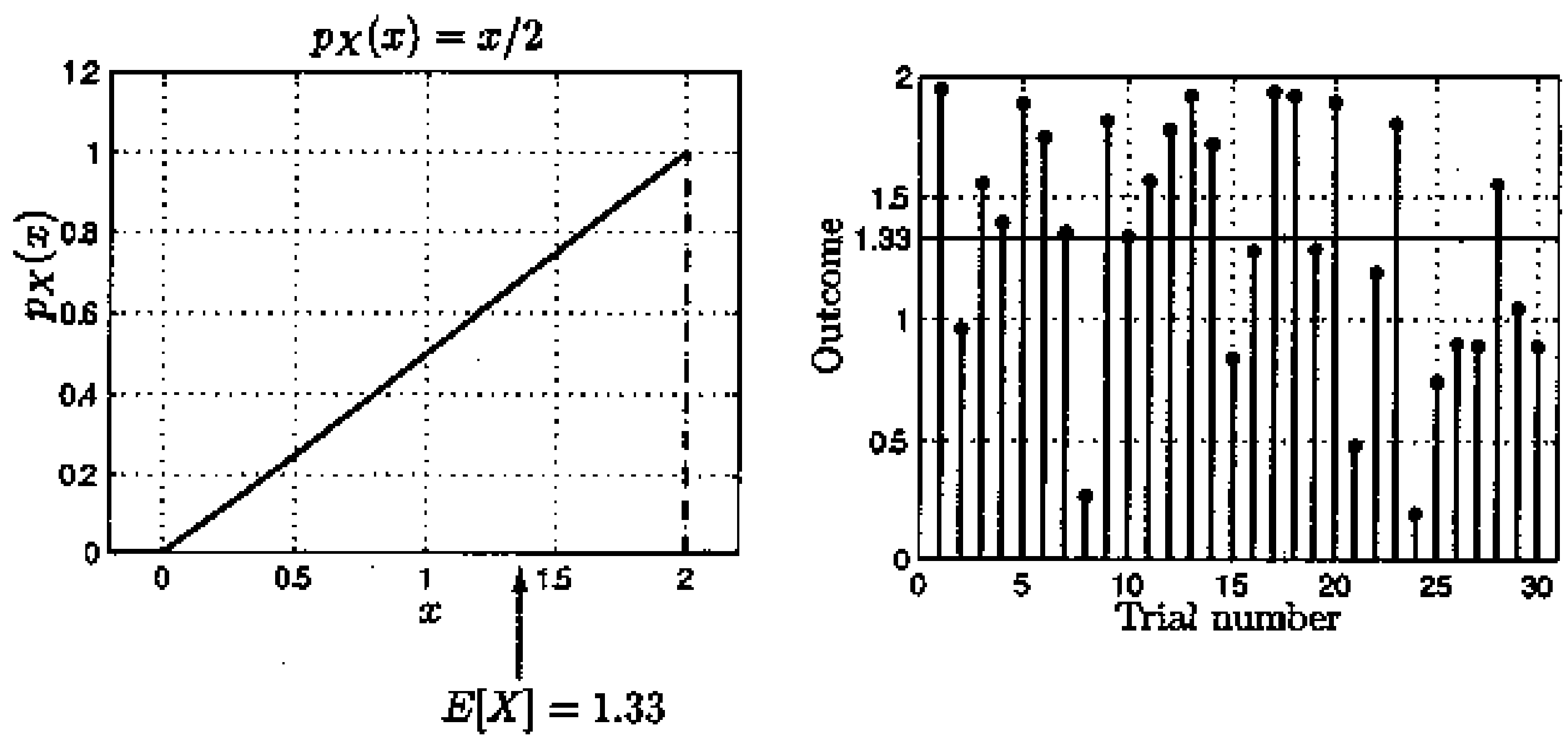
$$E[X] = \int_0^1 x \cdot 1\, dx = \tfrac{1}{2} x^2 \Big|_0^1 = \tfrac{1}{2}$$

<u>EXAMPLE</u>:

$$E[X] = \int_0^2 x \frac{x}{2}\, dx$$

$$= \frac{x^3}{6}\Big|_0^2$$

$$= \frac{8}{6} = 1.33$$



(a) PDF

(b) Typical outcomes and expected value of 1.33

Figure 11.1: Example of nonuniform PDF and its mean.

ANALOGOUS TO CENTER OF MASS

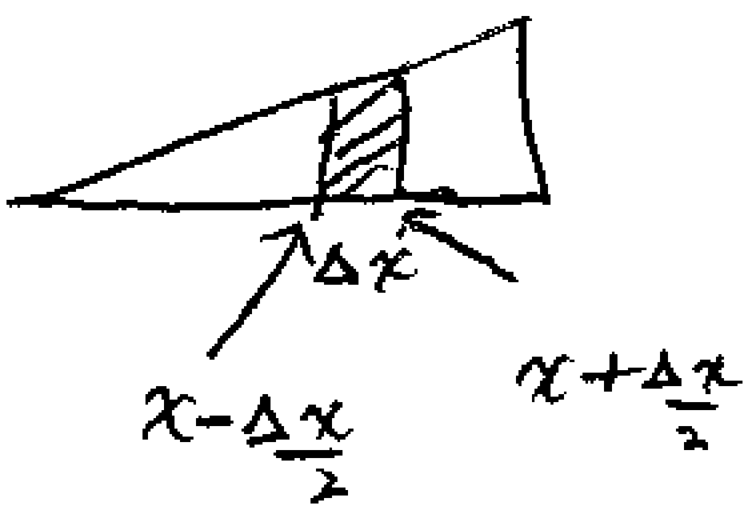
$$CM = \int_0^2 x\, m(x)\, dx$$

↑ MASS DENSITY $= \dfrac{\Delta m}{\Delta x}$ AS $\Delta x \to 0$

TOTAL VOLUME = 1

$m(x) = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$

$\Delta V = 1 \cdot \Delta A$ ← AREA



TOTAL MASS = 1

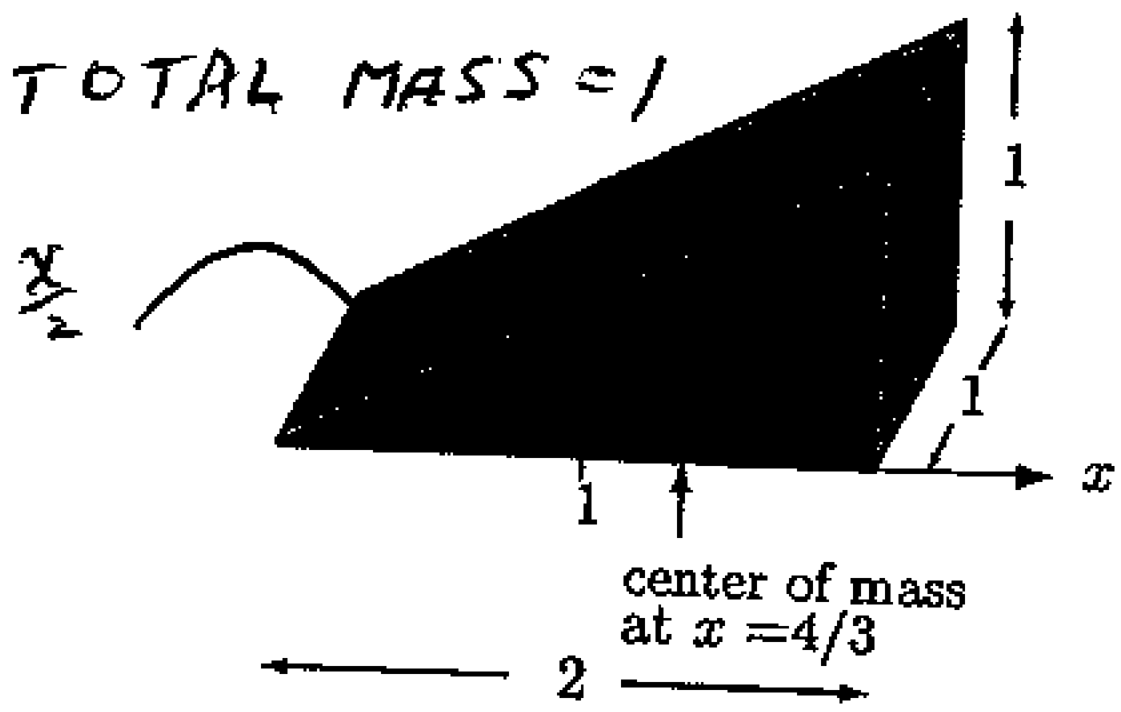


Figure 11.2: Center of mass (CM) analogy to average value.

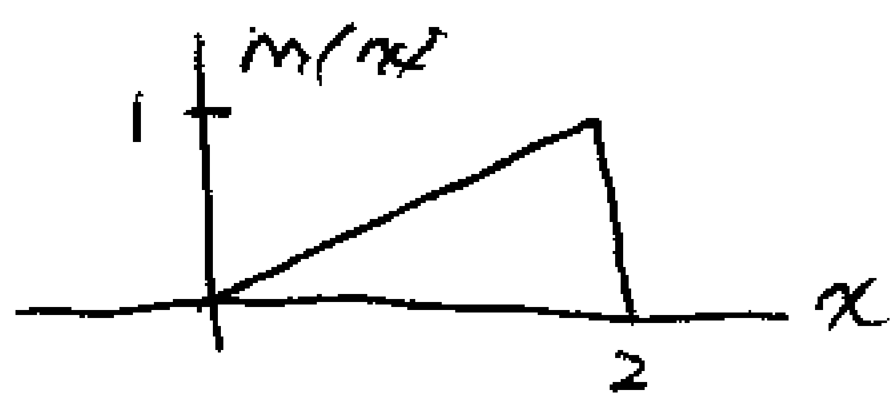
$$\Delta A = \frac{1}{2} \Delta x \left(\frac{x - \frac{\Delta x}{2}}{2} + \frac{x + \frac{\Delta x}{2}}{2} \right) = \frac{1}{2} x \, \Delta x$$

$$\Delta V = \frac{1}{2} x \, \Delta x$$

BUT $D = M/V = 1 \Rightarrow \Delta M = \Delta V$

$$\frac{\Delta M}{\Delta x} = \frac{\Delta V}{\Delta x} = \frac{1}{2} x \quad \text{AND AS } \Delta x \to 0$$

$$\frac{\Delta M}{\Delta x} \to \frac{1}{2} x = m(x)$$



$$CM = \int_0^2 x \, m(x) \, dx = E(x)$$

SAYS WE CAN BALANCE CHEESE AT $x = 4/3$ OR $\int_0^2 \underbrace{(x - CM)}_{\text{MOMENT ARM}} \underbrace{m(x)}_{\text{MASS}} dx = 0$ ↑ TORQUE

NOTE FROM DEFINITION OF $E(x)$

$$\int_{-\infty}^{\infty} (x - E[x]) \, p_X(x) \, dx = 0$$

IF $p_X(x)$ SYMMETRIC ABOUT $x = a \Rightarrow$
$E[X] = a$. CONVERSE TRUE?

NOT ALL PDFS HAVE EXPECTED VALUES.
TRY SIMULATING $p_X(x) = \begin{cases} \frac{1}{2x^{3/2}} & x \geq 1 \\ 0 & x < 1 \end{cases}$

AND AVERAGING VALUES. FOR $E(X)$
TO EXIST REQUIRE $\int_{-\infty}^{\infty} |x| p_X(x)\,dx < \infty$

EXPECTED VALUES FOR IMPORTANT PDFS

1) $X \sim U(a, b)$ $\quad E[X] = \frac{1}{2}(a+b)$

2) $X \sim \text{Exp}(\lambda)$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \left[-x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}\right]\Big|_0^\infty = \frac{1}{\lambda}$$

(SEE TABLE OF INTEGRALS, SERIES, AND PRODUCTS BY GRADSHTEYN AND RYZHIK, ACADEMIC PRESS, 1994)

3) $X \sim N(\mu, \sigma^2)$

PDF IS SYMMETRIC ABOUT $a = \mu$
$\Rightarrow E(X) = \mu$ ALSO CALLED <u>MEAN</u> OR <u>AVERAGE</u> <u>VALUE</u>

A DIRECT COMPUTATION YIELDS

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$+ \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

LET $u = x-\mu$ IN FIRST INTEGRAL

$$= \int_{-\infty}^{\infty} \underbrace{u}_{\text{ODD}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u^2}}_{\text{EVEN}} du = 0$$

SECOND INTEGRAL $= \mu$ WHY?

4) LAPLACIAN

$$p_X(x) = \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{2/\sigma^2}\,|x|} \quad -\infty < x < \infty$$

$\Rightarrow E(X) = 0$ WHY?

SEE OTHERS IN BOOK

EXPECTED VALUE OF $Y = g(X)$

ASSUME X AND Y ARE CONT. RVs

BY DEFINITION $E(Y) = \int_{-\infty}^{\infty} y\, p_Y(y)\, dy$

REQUIRES US TO FIND $p_Y(y)$. TO AVOID THIS, USE

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, p_X(x)\, dx$$

VERY USEFUL! SEE APPENDIX IIA FOR PROOF.

(IF $g(X) = X \Rightarrow$ DEFINITION OF $E(X)$).

EXAMPLE: $g(X) = aX + b$

$$E[g(X)] = \int_{-\infty}^{\infty} (ax+b)\, p_X(x)\, dx$$
$$= a \underbrace{\int_{-\infty}^{\infty} x\, p_X(x)\, dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} p_X(x)\, dx}_{=1}$$

$$= a E(X) + b$$

IN GENERAL, $E[a_1 g_1(X) + a_2 g_2(X)] = a_1 E[g_1(X)] + a_2 E[g_2(X)] \Rightarrow$ EXPECTATION OPERATION IS LINEAR.

EXAMPLE: $Y = X^2$ $\quad X \sim N(0,1)$

$E(Y) =$ AVERAGE POWER ACROSS 1 OHM RESISTOR (X = VOLTAGE)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

USE INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \quad u = x \qquad dv = x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$
$$du = dx \qquad v = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

ALSO, $E(X^2) = 2\int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

WHY?

$$E(X^2) = 2\left[-x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\Big|_0^\infty - \int_0^\infty -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx\right]$$

$$= 2\left(0 + \tfrac{1}{2}\right) = 1$$

NOTE: LIMIT OF $x e^{-\frac{1}{2}x^2}$ AS $x \to \infty$ IS ZERO (L HOSPITAL'S RULE) AND $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{2}$ WHY?

VARIANCE AND MOMENTS

VARIANCE MEASURES VARIABILITY OF R.V. OUTCOMES

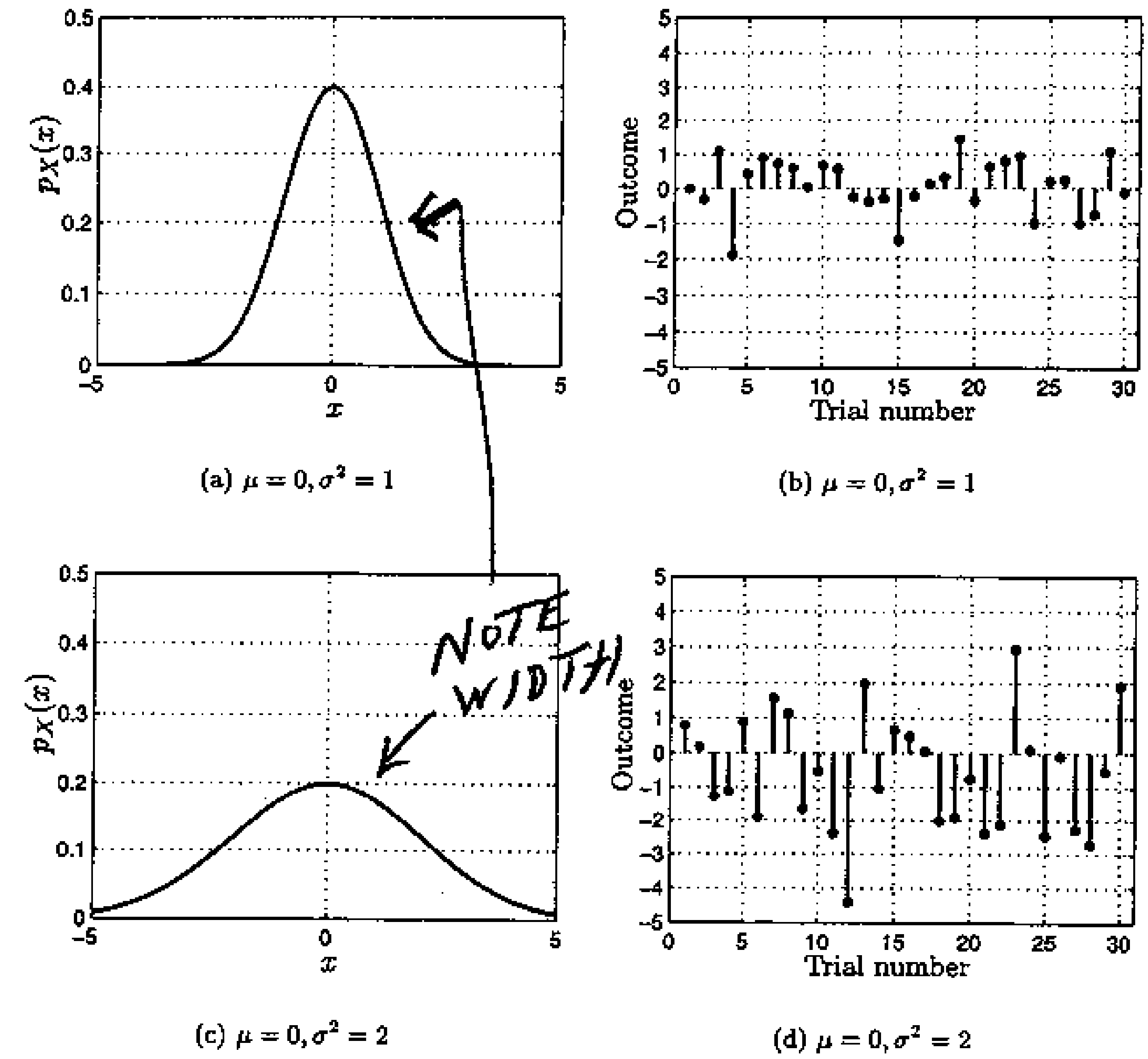


Figure 10.9: Examples of Gaussian PDF with different σ^2's.

APPEARS AS IF THE WIDER THE PDF THE MORE VARIABILITY. TO MEASURE WIDTH DEFINE VARIANCE

$$\mathrm{VAR}(X) = \int_{-\infty}^{\infty} (x - \underbrace{E[X]}_{\text{MEAN}})^2 \, \underbrace{p_X(x)}_{\text{AVERAGING PDF}} \, dx$$

= AVERAGE SQUARED DEVIATION FROM MEAN

EXAMPLE : $X \sim N(\mu, \sigma^2)$

$$\mathrm{VAR}(X) = \int_{-\infty}^{\infty} (x - \underbrace{E(X)}_{\mu})^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

LET $u = \frac{x-\mu}{\sigma}$ $\qquad \sigma = \sqrt{\sigma^2} > 0$

$$= \int_{-\infty}^{\infty} \sigma^2 u^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma \, du$$

$$= \sigma^2 \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \sigma^2 E[X^2]$$

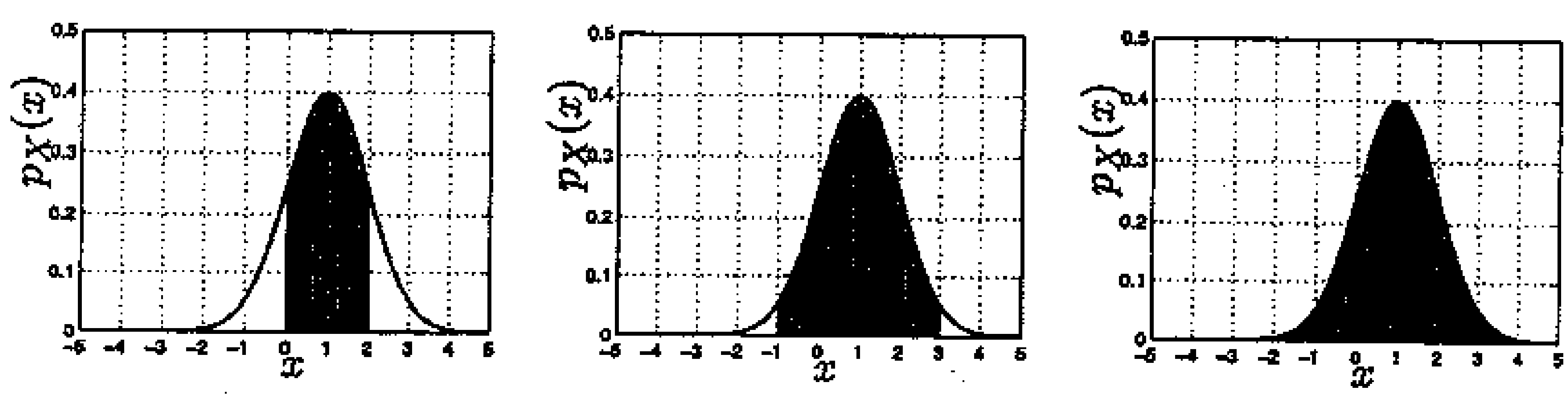
FOR $X \sim N(0,1)$

$$= \sigma^2$$

HENCE, $X \sim N(\underset{\text{MEAN}}{\mu}, \underset{\text{VARIANCE}}{\sigma^2})$

$\sigma = \sqrt{\sigma^2}$ CALLED THE <u>STANDARD DEVIATION</u>

NOTE THAT A $N(\mu, \sigma^2)$ RV WILL DEVIATE FROM MEAN ABOUT 99.8% OF TIME
$\mu - 3\sigma \le X \le \mu + 3\sigma$ (USEFUL FOR QUICK ASSESSMENT OF RANGE OF OUTCOMES).



(a) 68.2% for 1 standard deviation

(b) 95.5% for 2 standard deviations

(c) 99.8% for 3 standard deviations

Figure 11.5: Percentage of outcomes of $\mathcal{N}(1,1)$ random variable that are within $k = 1, 2,$ and 3 standard deviations from the mean. Shaded regions denote area within interval $\mu - k\sigma \le x \le \mu + k\sigma$.

VERIFY THIS $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = ?$
FOR $X \sim N(\mu, \sigma^2)$

PROPERTIES:

1) $VAR(C) = 0$ $\quad$ C = CONSTANT
2) $VAR(X + C) = VAR(X)$
3) $VAR(CX) = C^2 \, VAR(X)$

NOTE: $VAR(g_1(X) + g_2(X)) \neq VAR(g_1(X)) + VAR(g_2(X))$

NOT LINEAR

ALSO $VAR(X) = E(X^2) - E^2(X)$

$E(X^2)$ CALLED SECOND MOMENT

$E(X)$ ALSO CALLED FIRST MOMENT
$E(X^r) = r^{th}$ MOMENT

(IF $E(X^s) < \infty \Rightarrow E(X^r) < \infty$ FOR $r < s$
IF $E(X^r) = \infty \Rightarrow E(X^s) = \infty$)

IF RV HAS $E(X) = \infty$
$\Rightarrow E(X^2) = \infty \Rightarrow VAR(X)$ NOT DEFINED

EXAMPLE : $X \sim EXP(\lambda)$

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx \qquad n = 1, 2, 3, \ldots$$

USE INTEGRATION BY PARTS TO FIND $E[X^n]$ AS FUNCTION OF $E[X^{n-1}]$ (STANDARD TRICK)

LET $u = x^n$, $dv = \lambda e^{-\lambda x} dx$
$du = n x^{n-1} dx \qquad v = -e^{-\lambda x}$

$$E[X^n] = \underbrace{-x^n e^{-\lambda x}\Big|_0^\infty}_{=0} - \int_0^\infty -e^{-\lambda x} n x^{n-1} dx$$

$$= n \int_0^\infty x^{n-1} e^{-\lambda x} dx = \frac{n}{\lambda} \underbrace{\int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx}_{E[X^{n-1}]}$$

SINCE $E[X^1] = 1/\lambda$ (EASY TO VERIFY)

$$E[X^2] = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$E[X^3] = \frac{3}{\lambda}\frac{2}{\lambda^2} = \frac{3 \cdot 2}{\lambda^3}$$

$$\vdots$$

$$E[X^n] = \frac{n!}{\lambda^n}$$

CHARACTERISTIC FUNCTIONS

USEFUL TO FIND MOMENTS, AND LATER WILL ALLOW US TO FIND PDF OF

SUM OF R.V.s

DEFINED AS $\phi_X(\omega) = E[e^{j\omega X}]$

RECALL $E[g(X)] = \int_{-\infty}^{\infty} g(x)\, p_X(x)\, dx$

$$\phi_X(\omega) = E[\cos \omega X + j \sin \omega X]$$

$$= E[\underbrace{\cos \omega X}_{g_1(X)}] + j\, E[\underbrace{\sin \omega X}_{g_2(X)}]$$

$$= \int_{-\infty}^{\infty} \cos \omega x\, p_X(x)\, dx + j \int_{-\infty}^{\infty} \sin \omega x\, p_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{j\omega x}\, p_X(x)\, dx$$

OR $$= \int_{-\infty}^{\infty} p_X(x)\, e^{j\omega x}\, dx$$

CONT-"TIME" FOURIER TRANSFORM

RECALL $s(t) \leftrightarrow S(\omega)$

NOW $p_X(x) \leftrightarrow \phi_X(\omega)$

ONLY DIFFERENCE IS USE OF $+j$ OR $e^{+j\omega x}$ INSTEAD OF $e^{-j\omega x}$.

ALSO, USING FOURIER TRANSFORM THEORY

$$p_X(x) = \int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, \frac{d\omega}{2\pi}$$

JUST AN <u>INVERSE</u> FOURIER TRANSFORM

TO FIND MOMENTS USING $\phi_X(\omega)$:

$$E[x^n] = \frac{1}{j^n} \left. \frac{d^n \phi_x(\omega)}{d\omega^n} \right|_{\omega=0}$$

ALWAYS EASIER TO DIFFERENTIATE THAN INTEGRATE! (ONCE $\phi_X(\omega)$ KNOWN)

<u>EXAMPLE</u>: $X \sim EXP(\lambda)$

TO FIND $\phi_X(\omega)$:

$$\phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} p_X(x) e^{j\omega x} dx$$

$$= \int_0^{\infty} \lambda e^{-\lambda x} e^{j\omega x} dx = \int_0^{\infty} \lambda e^{-(\lambda - j\omega)x} dx$$

$$= \lambda \left. \frac{e^{-(\lambda - j\omega)x}}{-(\lambda - j\omega)} \right|_0^{\infty} = -\frac{\lambda}{\lambda - j\omega} [e^{-(\lambda - j\omega)\infty} - 1]$$

BUT $\lim_{x \to \infty} e^{-(\lambda - j\omega)x} = 0$ FOR $\lambda > 0$

WHY?

$\phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$ (OR COULD LOOK UP IN TABLES)

TO FIND $E[X^n]$ USE FORMULA.

$$\frac{d\,\phi_x(\omega)}{d\omega} = \frac{d}{d\omega}\lambda(\lambda - j\omega)^{-1} = \lambda(-1)(\lambda - j\omega)^{-2}(-j)$$

$$\frac{d^2\phi_x(\omega)}{d\omega^2} = \lambda(-1)(-2)(\lambda - j\omega)^{-3}(-j)^2$$

$$\vdots$$

$$\frac{d^n\phi_x(\omega)}{d\omega^n} = \lambda(-1)(-2)\ldots(-n)(\lambda - j\omega)^{-(n+1)}(-j)^n$$

$$= \lambda j^n n!\,(\lambda - j\omega)^{-(n+1)}$$

AT $\omega = 0$

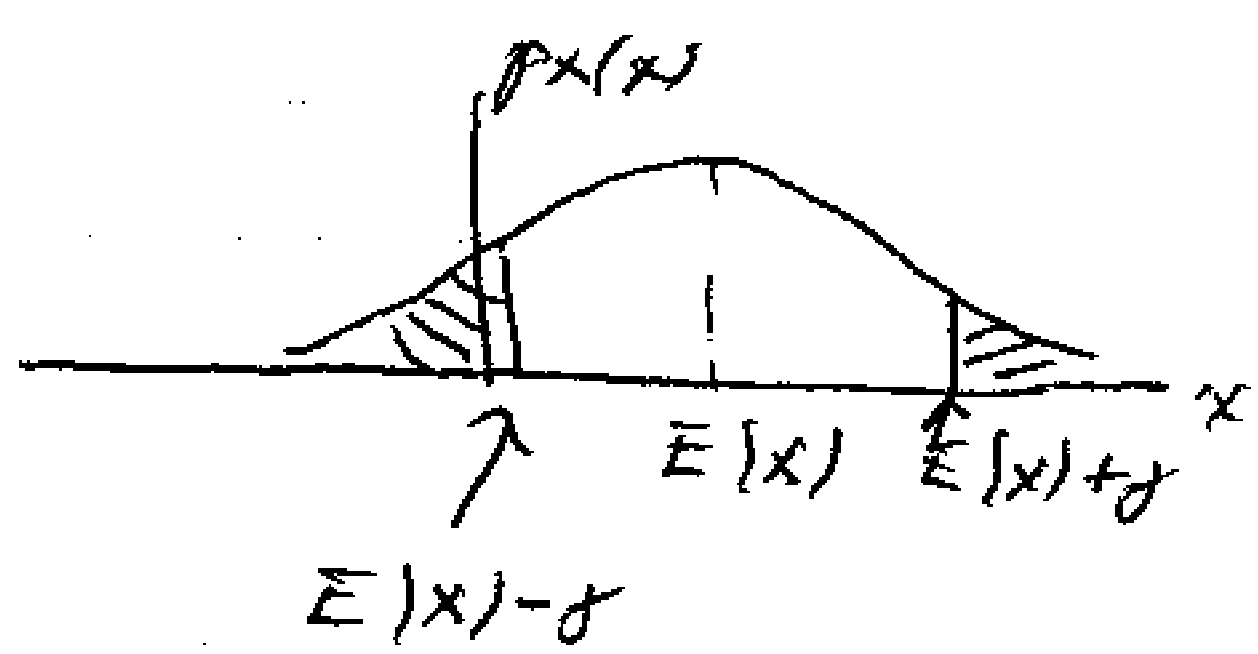
$$= \lambda j^n n!\,\lambda^{-(n+1)} = j^n \frac{n!}{\lambda^n}$$

$$E[x^n] = \frac{1}{j^n}\frac{d^n\phi_x(\omega)}{d\omega^n}\bigg|_{\omega=0} = \frac{n!}{\lambda^n}$$

CHEBYSHEV INEQUALITY

THE VARIANCE CAN ALSO BE USED TO BOUND A PROBABILITY. CONSIDER FINDING

$$P[\,|X - E[x]| > \gamma]$$



WHAT CAN BE SAID IF WE CAN'T INTEGRATE $p_x(x)$ OR IF WE DON'T KNOW $p_x(x)$?

ASSUME WE KNOW $E[X]$ AND $VAR(X)$ (WE WILL SEE HOW TO ESTIMATE THESE NEXT!). THEN, CHEBYSHEV'S INEQUALITY PROVIDES A BOUND B SO THAT

$$P(|X - E[X]| > \gamma) \leq B$$

PROBABILITY OF X DEVIATING FROM MEAN BY MORE THAN γ IS LESS THAN OR EQUAL TO B.

$$B = \frac{VAR(X)}{\gamma^2}$$

<u>EXAMPLE</u>: $X \sim N(0,1)$

$E[X] = 0$, $\gamma = 3$, $VAR(X) = 1$

$$P(|X - 0| > 3) \leq \frac{1}{3^2} = \frac{1}{9} = 0.11$$

ACTUALLY, $P(|X| > 3) = 2P(X > 3) = 2Q(3)$

$= 0.0027$

BOUND HOLDS BUT NOT VERY "TIGHT" FOR THIS EXAMPLE.

<u>EXAMPLE</u>: FOR A LAPLACIAN PDF WITH $\sigma^2 = 1 = VAR(X)$

$$P(|X| > 3) \leq \frac{1}{3^2} = 0.11$$

SAME BOUND FOR ALL PDFS WITH $VAR(X) = 1$ (DON'T NEED TO KNOW PDF)

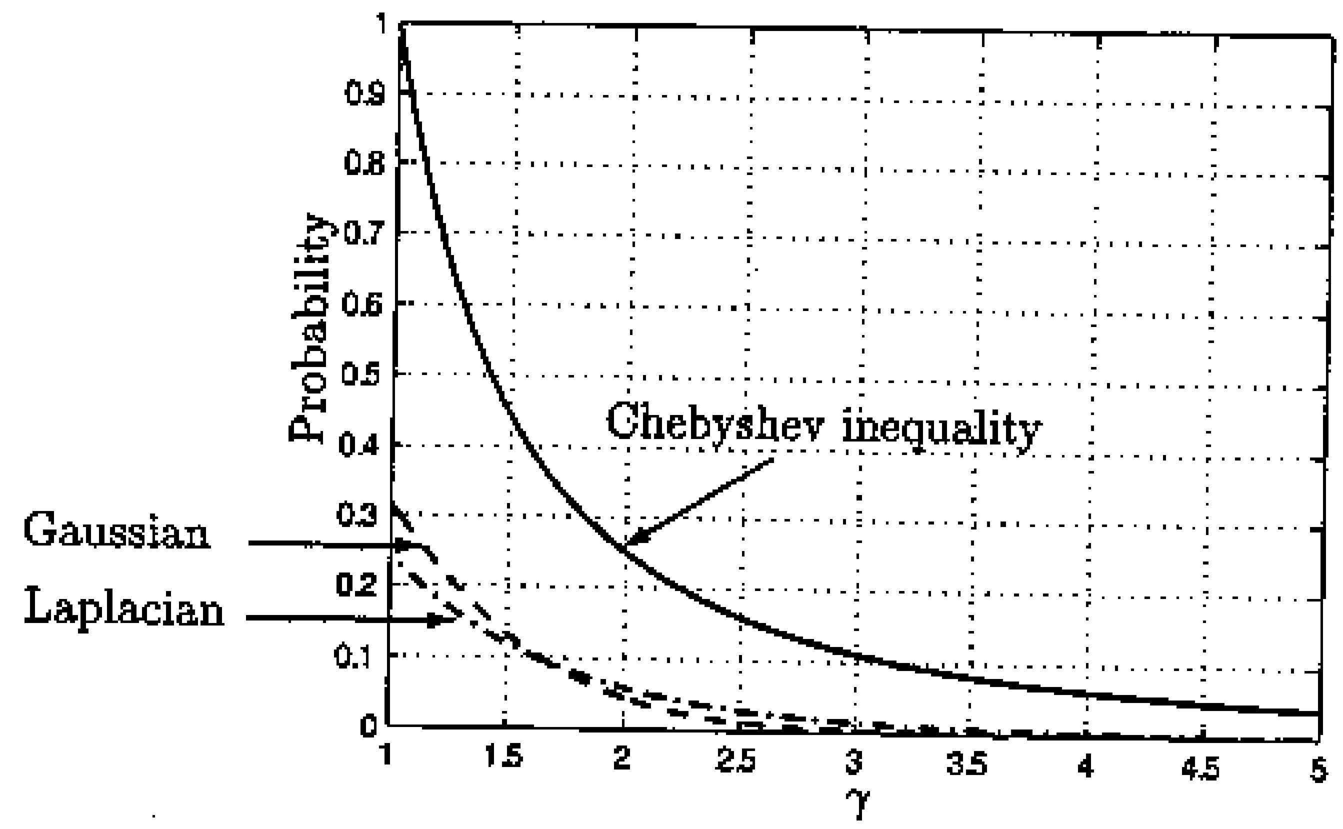


Figure 11.6: Probabilities $P[|X| > \gamma]$ for Gaussian and Laplacian random variables with zero mean and unity variance compared to Chebyshev inequality.

$P[|X|>3] = 0.0027$ FOR GAUSSIAN

$P[|X|>3] = 0.0144$ FOR LAPLACIAN

MOST USEFUL FOR THEORETICAL WORK - CAN PROVE THAT AS $VAR(X) \to 0$, $P[|X-E[X]| > \gamma] \to 0$ FOR ANY $\gamma > 0$.

PROOF:

$$VAR(X) = \int_{-\infty}^{\infty} (x-E[X])^2 p_X(x)\,dx$$

$$= \int_{\{x:|x-E[X]|>\gamma\}} + \int_{\{x:|x-E[X]|\le\gamma\}}$$

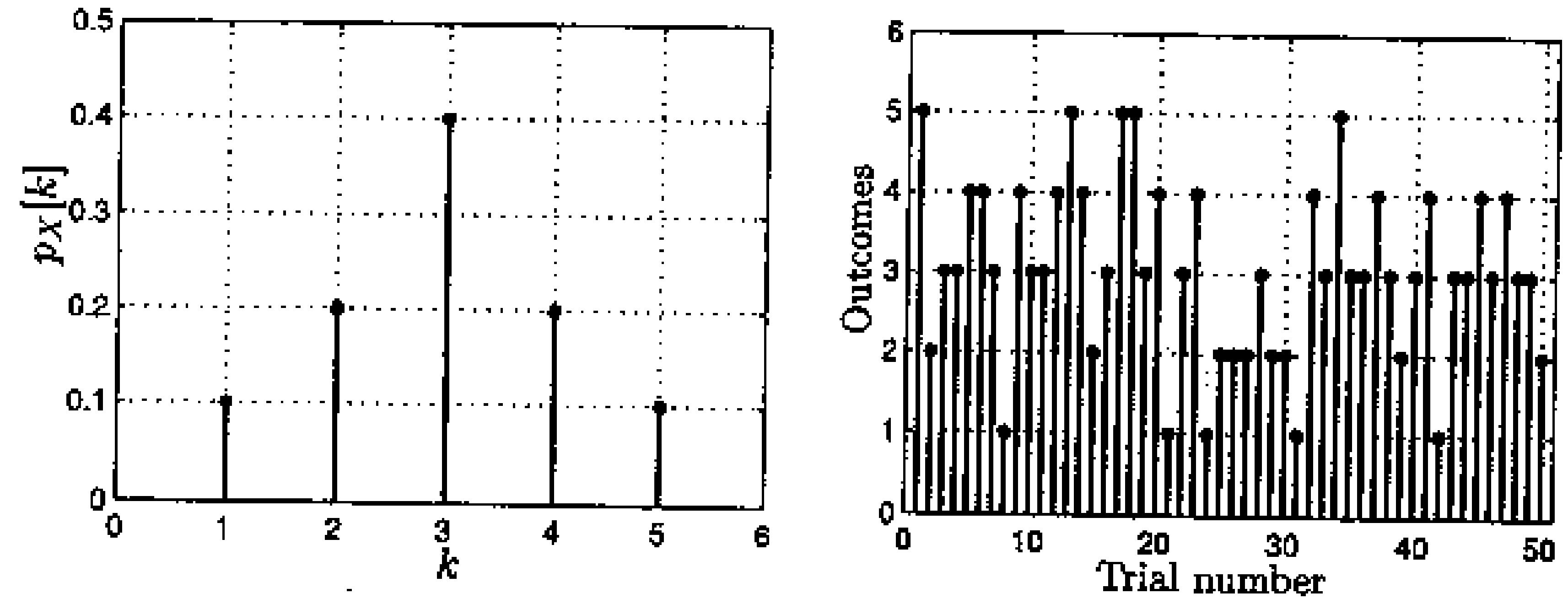
$$\ge \int_{\{x:|x-E[X]|>\gamma\}} (x-E[X])^2 p_X(x)\,dx$$

$$\ge \int_{\{x:|x-E[X]|>\gamma\}} \gamma^2 p_X(x)\,dx$$

$$= \gamma^2 P[|X-E[X]| > \gamma]$$

ESTIMATING MEAN AND VARIANCE

ASSUME X IS A DISCRETE RANDOM VARIABLE



(a) PMF

(b) Simulated data

Figure 6.6: PMF and computer generated data used to illustrate estimation of mean and variance.

$$E(X) = \sum_{k=1}^{5} k \, p_X(k)$$

BUT $p_X(k) = P(X = k) \approx \frac{N_k}{M}$

N_k: NUMBER OF TIMES k OCCURED

M: TOTAL NUMBER OF OUTCOMES

$$E(X) \approx \sum_{k=1}^{5} k \frac{N_k}{M} = \frac{\sum_{k=1}^{5} k N_k}{M}$$

BUT IF $\{1, 1, 5, 3, 2, 3, 4, 1\}$ OCCURED

$N_1 = 3, \quad N_2 = 1, \quad N_3 = 2, \quad N_4 = 1, \quad N_5 = 1$

$$\sum_{k=1}^{5} k N_k = 1(3) + 2(1) + 3(2) + 4(1) + 5(1)$$

$$= 20 = \sum_{i=1}^{8} x_i \qquad (M = 8)$$

$$\Rightarrow \quad E[X] \simeq \frac{\sum_{i=1}^{8} x_i}{8}$$

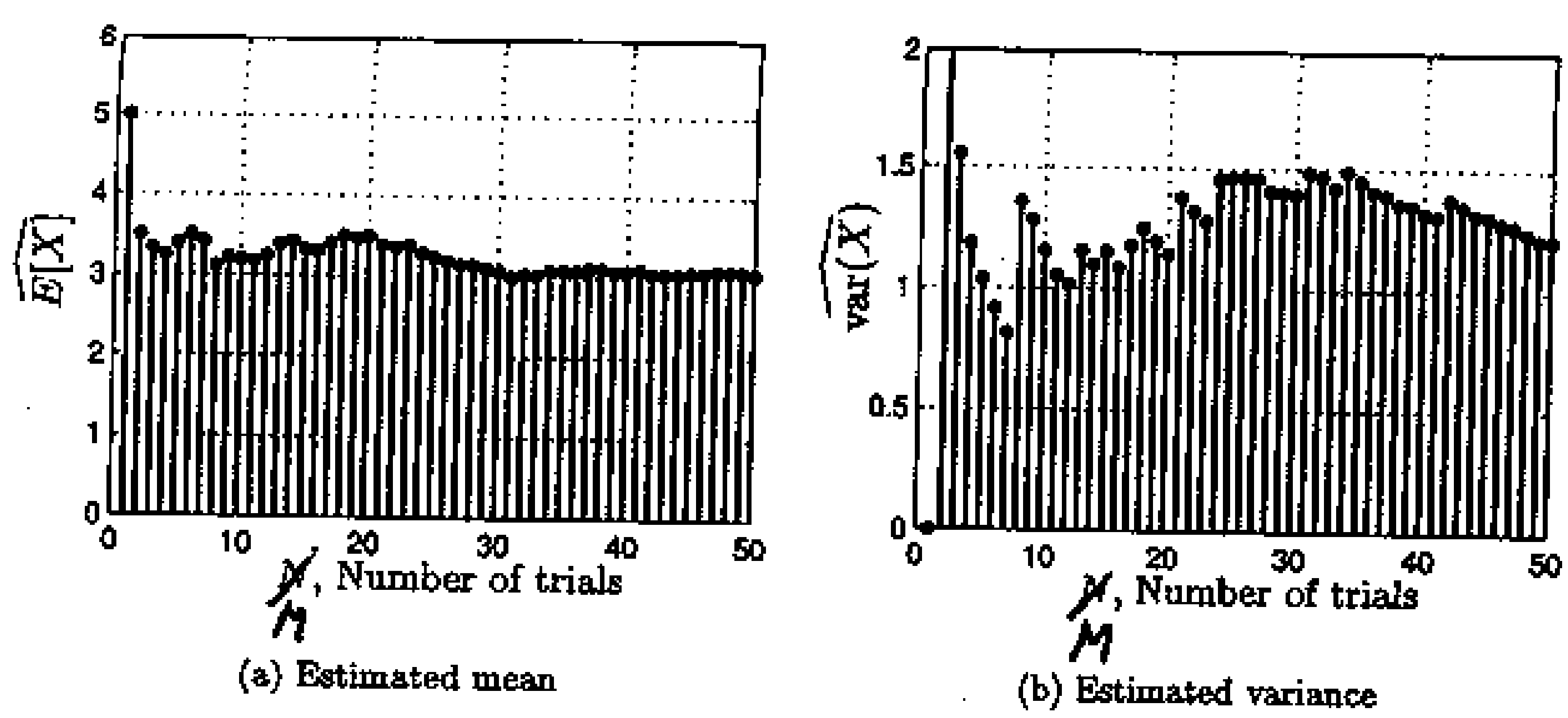
OR IN GENERAL $\hat{E}[X] = \frac{1}{M} \sum_{i=1}^{M} x_i$

$=$ SAMPLE MEAN $\quad \wedge \Rightarrow$ ESTIMATE

IN GENERAL $\hat{E}[g(X)] = \frac{1}{M} \sum_{i=1}^{M} g(x_i)$

SO THAT $\quad VAR(X) = E[X^2] - E^2[X]$

$$\Rightarrow \quad \widehat{VAR}(X) = \frac{1}{M} \sum_{i=1}^{M} x_i^2 - \left(\frac{1}{M} \sum_{i=1}^{M} x_i \right)^2$$



$E[X] = 3$
$VAR(X) = 1.2$

Figure 6.7: Estimated mean and variance for computer data shown in Figure 6.6.

SEE ALSO EXAMPLE 2,3.

FOR A CONT. R.V. $\quad E[X] = \int_{-\infty}^{\infty} x \, p_X(x) \, dx$

$$\simeq \sum_k x_k \, p_X(x_k) \Delta x = \sum_k x_k \, P\left[x_k - \frac{\Delta x}{2} \le X \le x_k + \frac{\Delta x}{2}\right]$$

$$= \sum_k x_k \frac{N_{x_k}}{M} \qquad N_{x_k} = \text{NUMBER OF } X\text{'s FALLING IN } x_k \text{ INTERVAL}$$

$$= \frac{1}{M} \sum_{i=1}^{M} x_i \qquad \text{(AS BEFORE)}$$

WE WILL JUSTIFY THIS MORE RIGOROUSLY LATTER (LAW OF LARGE NUMBERS)

CHAPTER 12 - MULTIPLE CONT. RVS

WE WILL CONSIDER TWO RVS, X AND Y. THEY ARE SAID TO BE JOINTLY DISTRIBUTED IF THE ORIGINAL EXPERIMENTAL SAMPLE SPACE S MAPS INTO TWO NUMBERS

$X(s) = x$ AND $Y(s) = y$ $\quad s \in S$

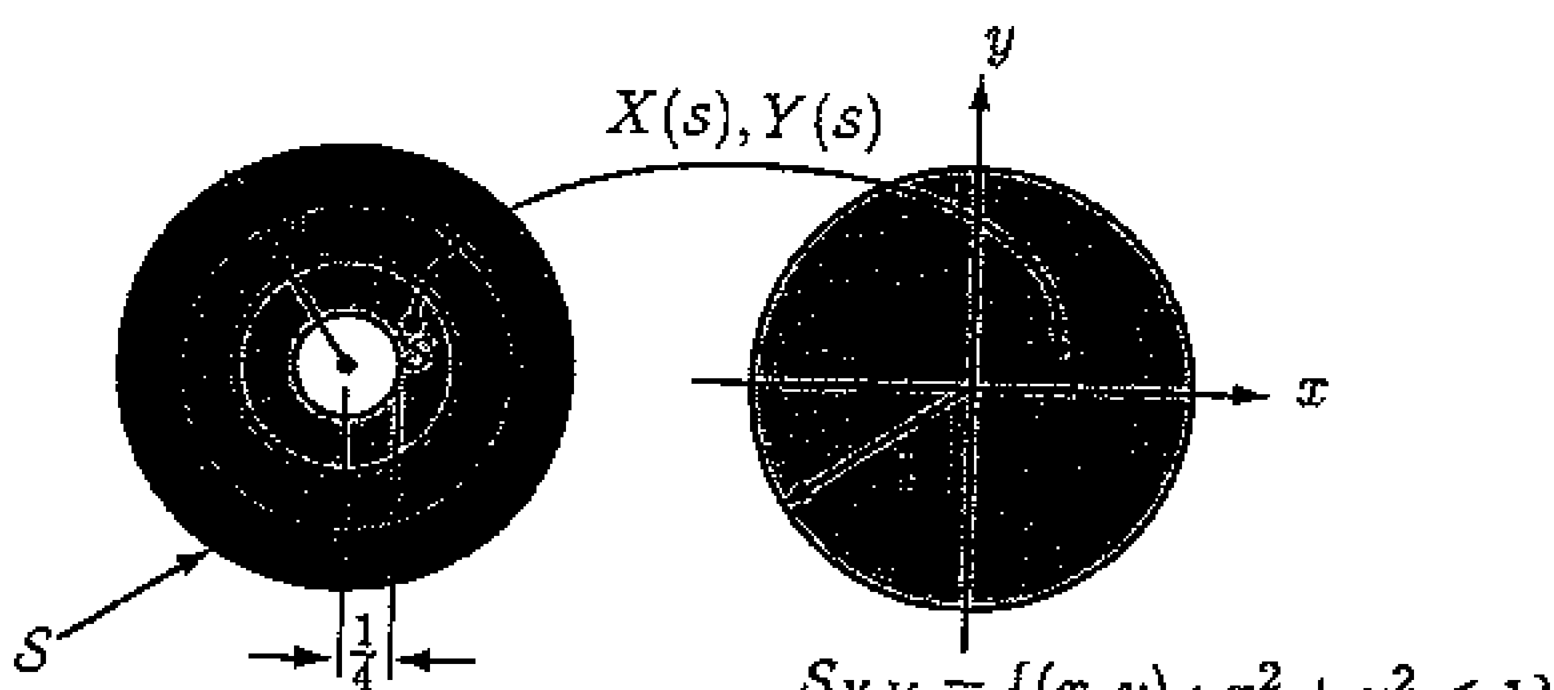


Figure 12.1: Mapping of the outcome of a thrown dart to the plane (example of jointly continuous random variables).

IN GENERAL,
OUTCOMES ARE NOW PAIRS OF NUMBERS (x, y) AND HENCE $S_{X,Y} = \{(x, y) : -\infty < x < \infty, -\infty < y < \infty\}$

(FOR DISCRETE RVS HEIGHT AND WEIGHT OF INDIVIDUAL ARE JOINTLY DISTRIBUTED)