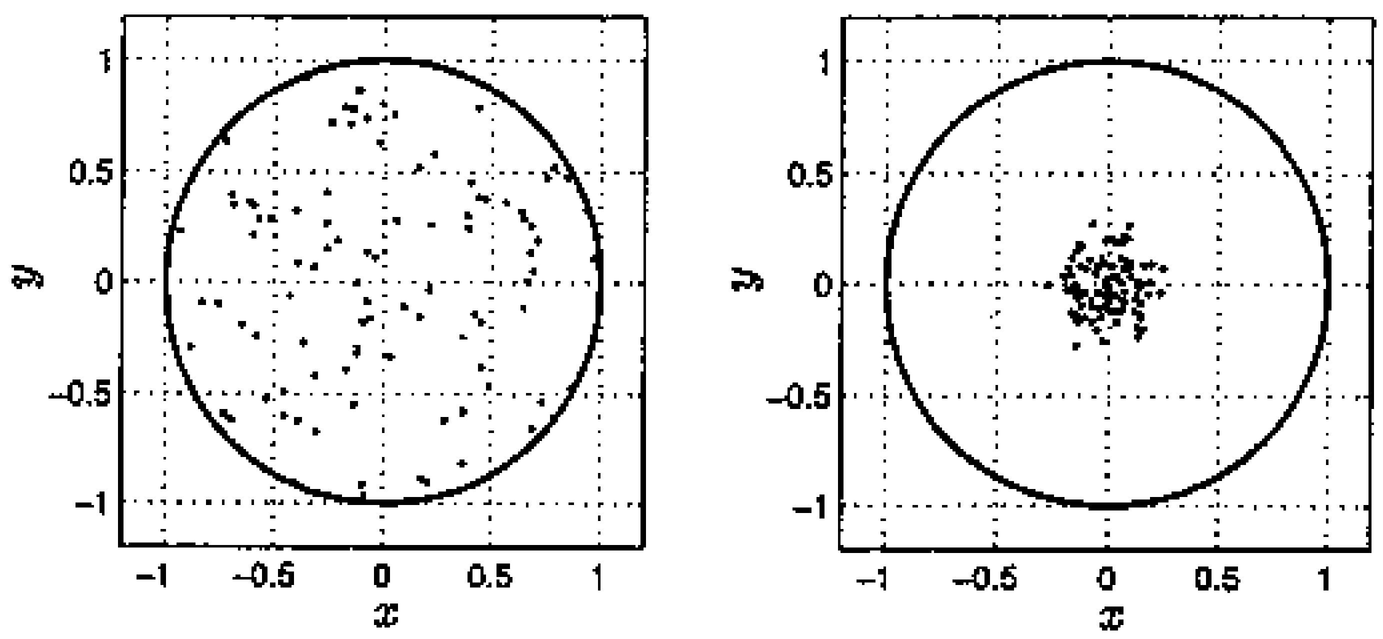
NOW WISH TO FIND PROBABILITIES OF REGIONS IN (x, y) PLANE FOR EXAMPLE,

$$P[\text{BULLSEYE}] = P\left[\sqrt{x^2+y^2} \le \frac{1}{4}\right]$$



(a) Novice (b) Champion

FOR NOVICE ASSUME DART EQUALLY LIKELY TO LAND ANYWHERE

Figure 12.2: Typical outcomes for novice and champion dart player.

FOR NOVICE WE HAVE

$$P[\text{BULLSEYE}] = \frac{\text{AREA OF BULLSEYE}}{\text{TOTAL AREA}}$$

$$= \frac{\pi(1/4)^2}{\pi(1)^2} = \frac{1}{16}$$

HOW DO WE FIND $P[\text{BULLSEYE}]$ FOR CHAMPION?

FOR NOVICE

$$P[\text{BULLSEYE}] = \pi(1/4)^2 \times \frac{1}{\pi}$$

$$= \underbrace{\text{AREA OF BULLSEYE}}_{\text{AREA OF EVENT}} \times \underbrace{\frac{1}{\pi}}_{\text{HEIGHT}}$$

$$= \text{VOLUME}$$

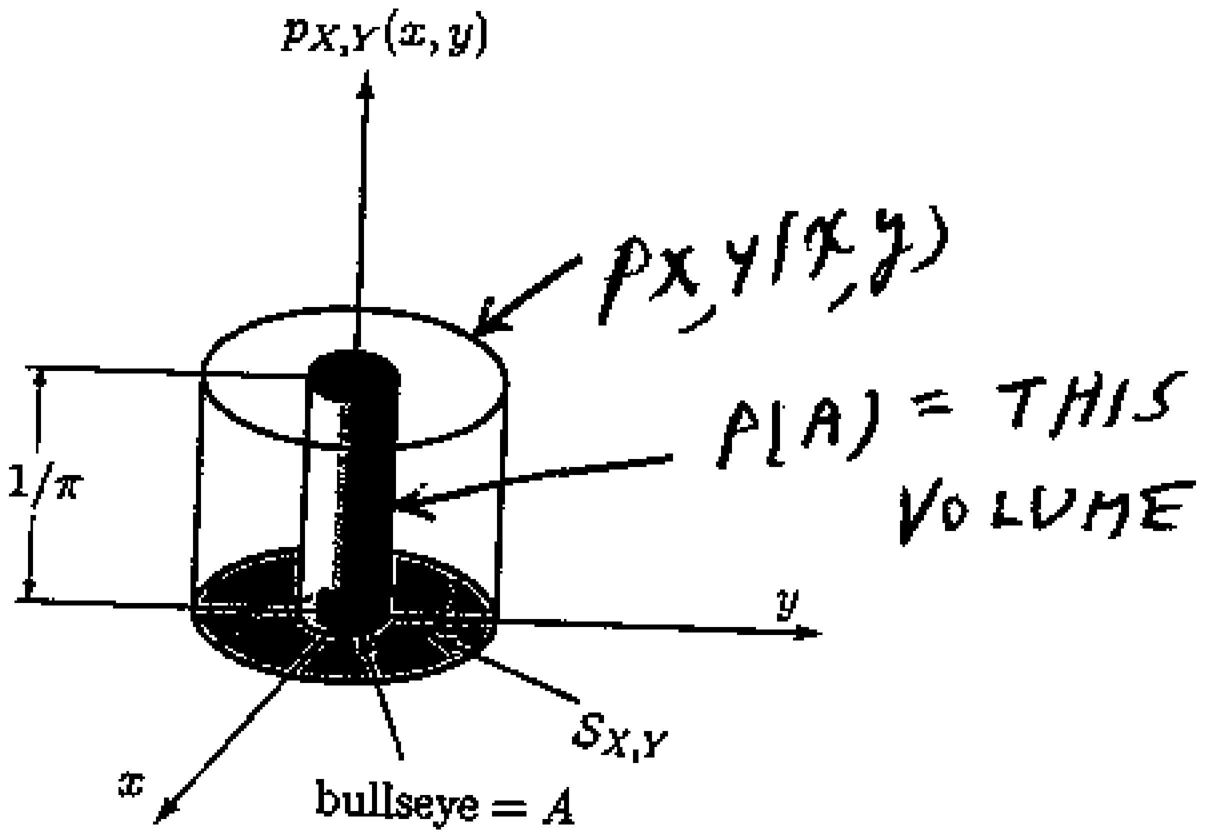


Figure 12.3: Geometric interpretation of bullseye probability calculation for novice dart thrower.

$$p_{X,Y}(x,y) = \frac{1}{\pi} \quad \text{FOR } x^2+y^2 \leq 1$$
$$= 0 \quad \text{OTHERWISE}$$

$$\Rightarrow P(\text{BULLSEYE}) = P(A) = \iint_A p_{X,Y}(x,y)\,dx\,dy$$

SINCE

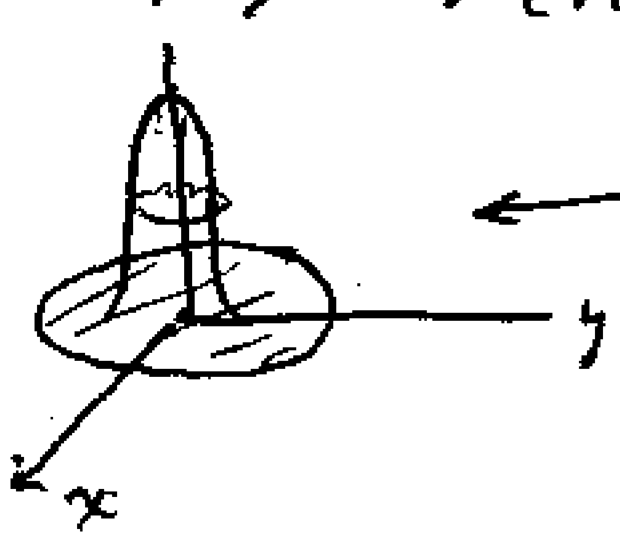
$$P(A) = \iint_{\{(x,y):\, x^2+y^2 \leq (\frac{1}{4})^2\}} \frac{1}{\pi}\,dx\,dy$$

$$= \frac{1}{\pi}\iint_{\{(x,y):\, x^2+y^2 \leq (\frac{1}{4})^2\}} dx\,dy$$

$$= \frac{1}{\pi} \times \text{AREA OF } A = \frac{1}{\pi} \times \pi\left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

HENCE IN GENERAL A = REGION IN (x,y) PLANE AND $P(A)$ IS A VOLUME, i.e., THE VOLUME UNDER $p_{X,Y}$. FOR THE CHAMPION WE HAVE

$$P(\text{BULLSEYE}) = P(A) = \iint_A \underbrace{p_{X,Y}(x,y)}\,dx\,dy$$



$p_{X,Y}$ IS THE <u>JOINT PDF</u>

EXAMPLE : $p_{X,Y}(x,y) = 4(1-|2x-1|)(1-|2y-1|) \quad 0 \le x \le 1,\ 0 \le y \le 1$

$0 \quad$ OTHERWISE

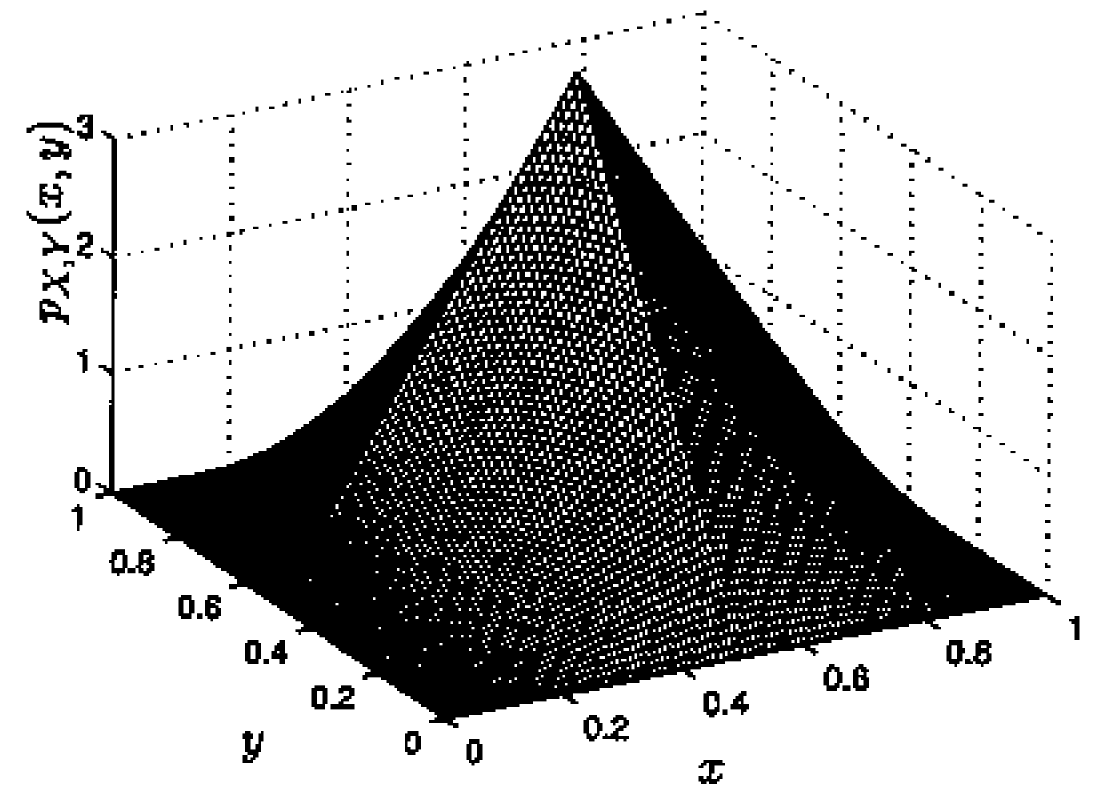


Figure 12.4: Three-dimensional plot of joint PDF.

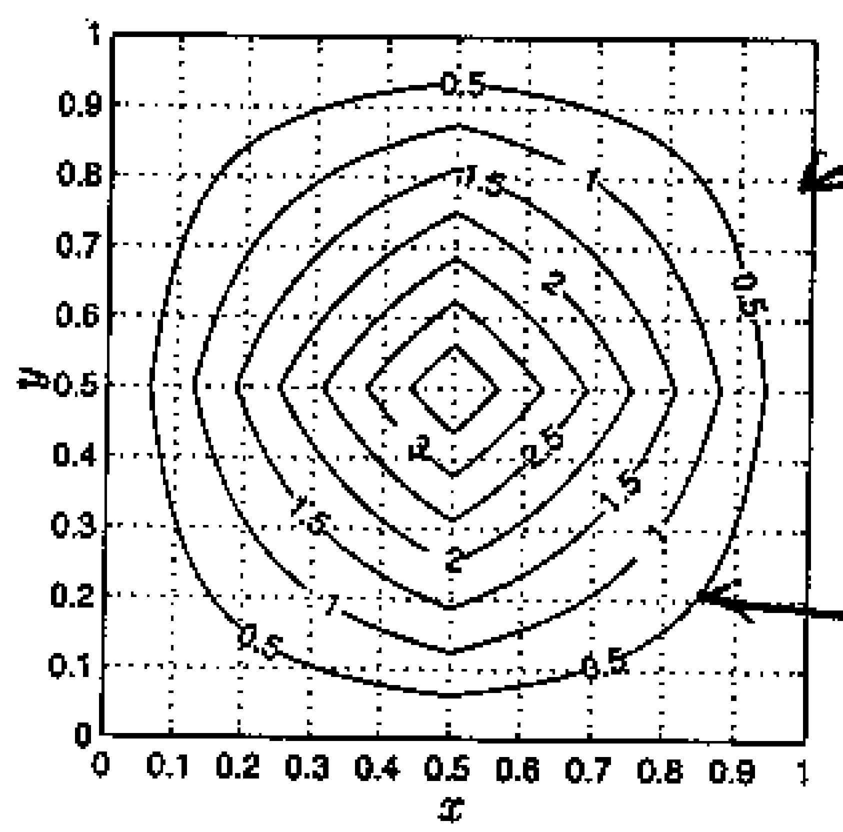


Figure 12.5: Contour plot of joint PDF.

LIKE TOPOGRAPHICAL MAP OF MOUNTAIN (OR PYRAMID)

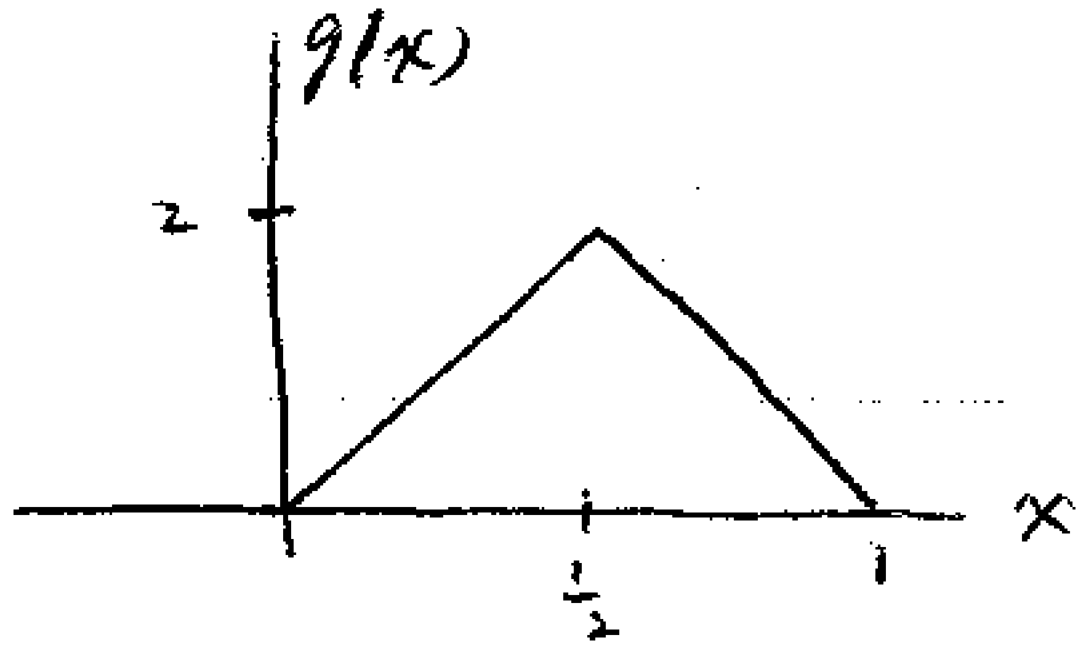
CONTOUR OF CONSTANT PDF ALL ALONG CONTOUR $p_{X,Y}(x,y) = 0.5$

WHAT IS $\int_0^1 \int_0^1 p_{X,Y}(x,y)\,dx\,dy$?

$$P[S_{X,Y}] = \int_0^1 \int_0^1 4(1-|2x-1|)(1-|2y-1|)\,dx\,dy$$

$$= \int_0^1 2(1-|2x-1|)\,dx \int_0^1 2(1-|2y-1|)\,dy$$

$p_{X,Y}$ SEPARATES INTO $g(x)g(y) \Rightarrow$ <u>GREATLY</u> SIMPLIFIES DOUBLE INTEGRAL



$\int_0^1 g(x)\,dx = 1$ WHY?

$\Rightarrow P[S_{X,Y}] = 1$

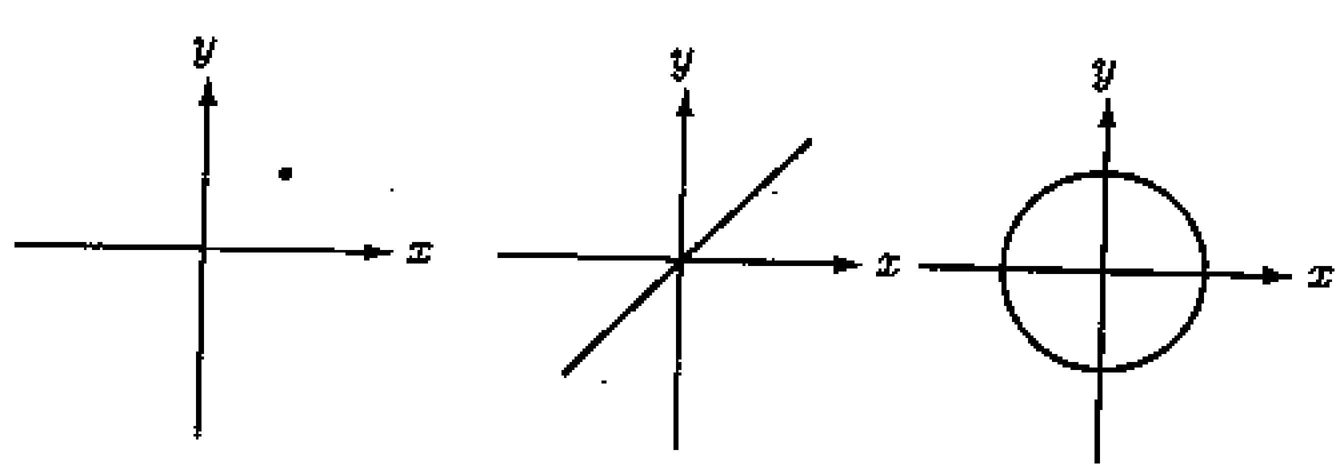
TRY $P[\,0 \le X \le \frac{1}{2},\ \frac{1}{2} \le Y \le 3/4\,]$.

IN GENERAL, IT MAY BE HARD TO EVALUATE THE DOUBLE INTEGRAL UNLESS

1) $p_{X,Y}$ IS SEPARABLE
2) INTEGRATION REGION IS RECTANGULAR

(REVIEW EVALUATION OF ITERATED INTEGRALS)

SINCE IN 3 DIMENSIONS A NONZERO VOLUME REQUIRES A NONZERO AREA BASE, MANY REGIONS IN (x, y) HAVE $P[A] = 0$



(a) Point (b) Line (c) Curve

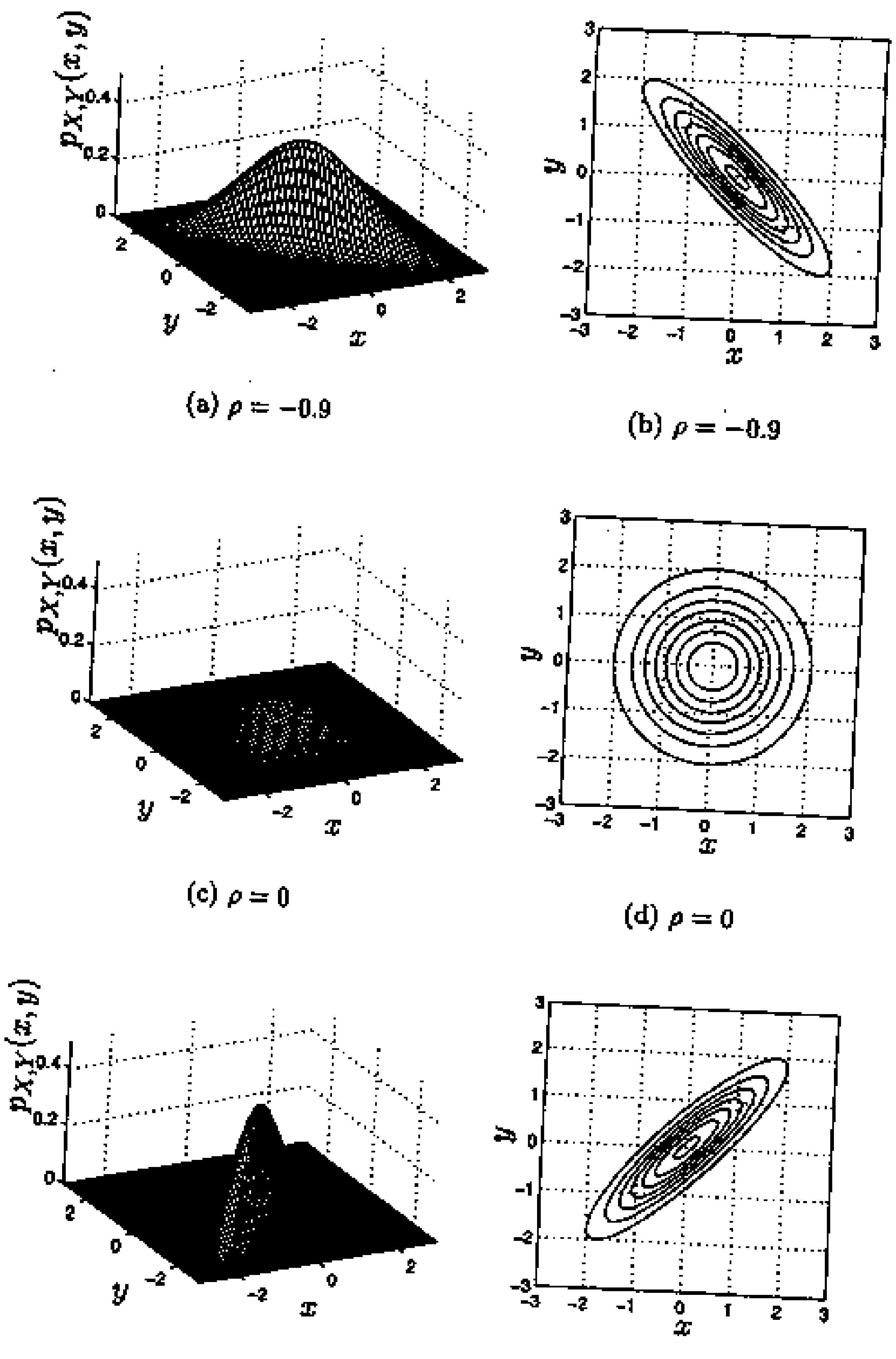
Figure 12.8: Examples of zero probability events for jointly distributed continuous random variables X and Y. All regions in the x-y plane have zero area.

EXAMPLE: STANDARD BIVARIATE GAUSSIAN (NORMAL) PDF

FIRST OF MANY GAUSSIAN PDFS FOR MULTIPLE R.V.S.

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

WHERE $-1 < \rho < 1$



(a) $\rho = -0.9$ (b) $\rho = -0.9$

(c) $\rho = 0$ (d) $\rho = 0$

(e) $\rho = 0.9$ (f) $\rho = 0.9$

Figure 12.9: Three-dimensional and constant PDF contour plots of standard bivariate Gaussian PDF.

CONTOURS OF CONSTANT PDF ALL THOSE VALUES OF (x, y) FOR WHICH

$p_{X,Y}(x,y) = c$ $\quad$ C A CONSTANT

$$\Rightarrow \quad x^2 - 2\rho x y + y^2 = r^2$$

FOR THESE (x, y) WE HAVE

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} r^2} = c$$

IF $\rho = 0 \Rightarrow$ CIRCLE

$\rho \neq 0 \Rightarrow$ ELLIPSE

ALSO, SINCE $|\rho| < 1$, $x^2 - 2\rho x y + y^2 = r^2 > 0$

<u>PROOF</u>: $\quad x^2 - 2\rho x y + y^2 = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{= Q = \text{QUADRATIC FORM}}$

T = TRANSPOSE

SYMMETRIC

<u>ASIDE</u>: A SYMMETRIC MATRIX <u>A</u> IS DEFINED TO BE <u>POSITIVE DEFINITE</u> IF

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}_{\underline{A}} \begin{bmatrix} x \\ y \end{bmatrix} > 0 \quad \text{FOR ALL} \quad \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

HENCE, IF A MATRIX IS POSITIVE DEFINITE, ITS ASSOCIATED QUADRATIC FORM (USUALLY DENOTED BY Q) IS POSITIVE.

TO PROVE THAT A 2×2 MATRIX IS P.D. WE MUST SHOW THAT

$$a > 0$$
$$\det(\underline{A}) > 0$$

NOW FOR $Q = x^2 - 2\rho xy + y^2 > 0$, $\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$ MUST BE P.D. BUT

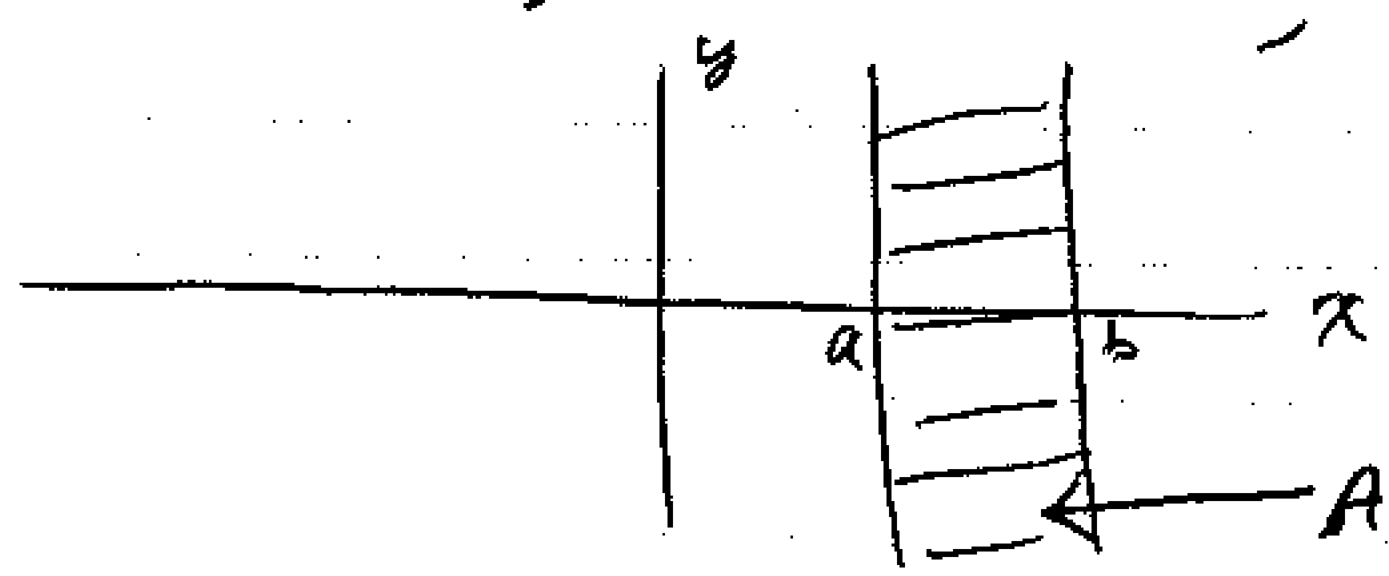
$$a = 1 > 0$$
$$\det\left(\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\right) = 1 - \rho^2 > 0 \quad \text{SINCE } |\rho| < 1$$

MARGINAL PDFs

FOR A JOINTLY DISTRIBUTED RANDOM VARIABLE, THE MARGINAL PDF IS THAT PDF WHEN INTEGRATED YIELDS $P[a \le X \le b]$ FOR ANY Y OUTCOME. (SEE SECTION 7.4 FOR DISCRETE R.V.s)

ON $S_{X,Y}$ WE DESIRE $P[A]$, WHERE

$$A = \{(x, y) : a \le x \le b, \ -\infty < y < \infty\}$$



$$P[a \le X \le b] = P[a \le X \le b, -\infty < Y < \infty]$$

$$= \iint_A p_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_a^b p_{X,Y}(x,y)\,dx\,dy$$

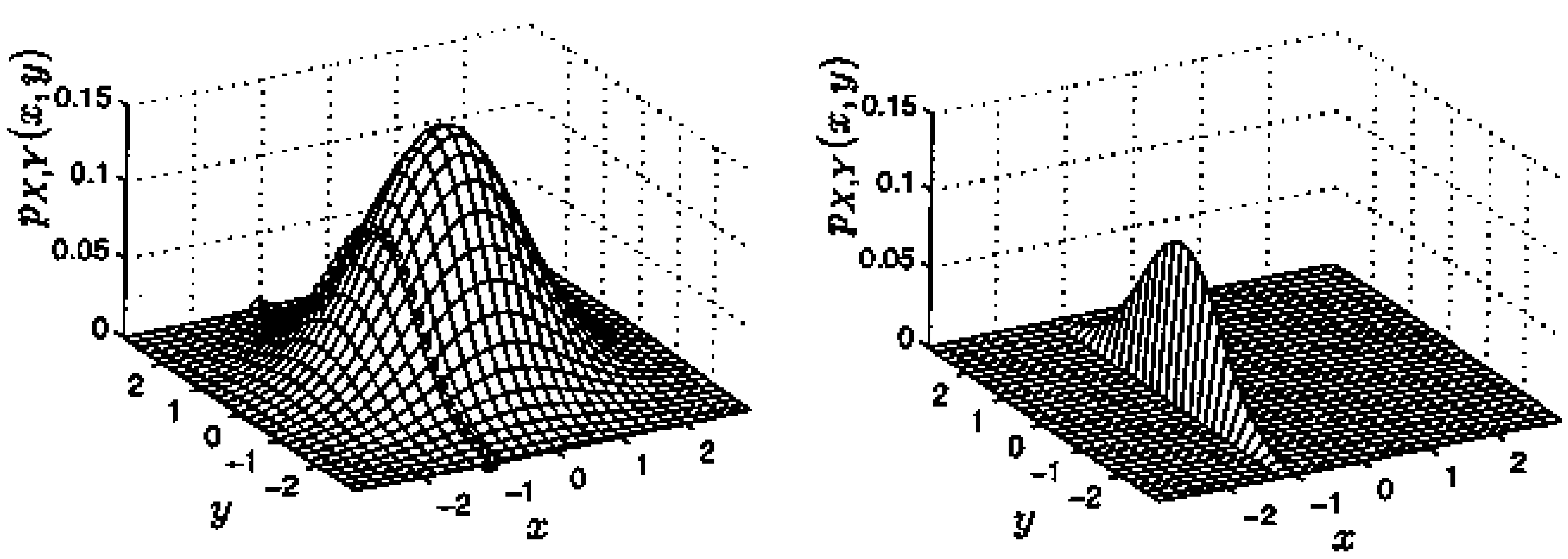
$$= \int_a^b \underbrace{\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy}_{p_X(x)}\,dx$$

HENCE, THE MARGINAL PDF IS GIVEN BY

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

WE "INTEGRATE OUT" y. ANALOGOUS TO THE CASE OF DISCRETE RVS WHERE WE SUM OVER A ROW OR COLUMN OF PMF,

$$p_X[x_i] = \sum_{j=1}^{\infty} p_{X,Y}[x_i, y_j]$$



(a) Curve is $p_{X,Y}(-1,y)$ (b) Area under curve is $p_X(-1)$

Figure 12.10: Obtaining the marginal PDF of X from the joint PDF of (X,Y).

EXAMPLE: STANDARD BIVARIATE GAUSSIAN

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho x y + y^2)} dy$$

TRICK TO HARD INTEGRATIONS IS TO PRODUCE A KNOWN INTEGRAL, IN THIS CASE A GAUSSIAN.

$$\begin{aligned} Q &= y^2 - 2\rho x y + x^2 \\ &= y^2 - 2\rho x y + \rho^2 x^2 - \rho^2 x^2 + x^2 \quad \text{COMPLETE SQUARE IN } y \\ &= (y-\rho x)^2 + (1-\rho^2)x^2 \end{aligned}$$

$$p_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2} e^{-\frac{1}{2(1-\rho^2)}(1-\rho^2)x^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} dy}_{1}$$

$$\underset{>0}{\sigma^2 = 1-\rho^2}, \quad \mu = \rho x$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \qquad X \sim N(0,1)$$

LIKEWISE $\quad p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx$

PRODUCES $Y \sim N(0,1)$

NOTE THAT $p_{X,Y} \Rightarrow p_X$ AND p_Y <u>BUT</u>

$p_X, p_Y \not\Rightarrow p_{X,Y}$

EXAMPLE: SEE EXAMPLES 7.3, 7.5

	$j=0$	$j=1$	$p_X[i]$
$i=0$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$i=1$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$p_Y[j]$	$\frac{1}{2}$	$\frac{1}{2}$	

	$j=0$	$j=1$	$p_X[i]$
$i=0$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
$i=1$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
$p_Y[j]$	$\frac{1}{2}$	$\frac{1}{2}$	

DISCRETE RVS

$p_{X,Y}[i,j]$ JOINT PMFS

CLEARLY, $p_{X,Y} \Rightarrow p_X, p_Y$ BUT CANNOT INFER $p_{X,Y}$ FROM ONLY p_X AND p_Y

WHY? TO SPECIFY p_X, IF $S_X = \{0,1\}$, NEED ONE PROBABILITY, LIKEWISE FOR p_Y
$\Rightarrow P[X=0], P[Y=0]$ SUFFICIENT
HOW MANY NUMBERS REQUIRED TO SPECIFY $p_{X,Y}$?

INDEPENDENCE

RECALL THAT TWO EVENTS DEFINED ON A SAMPLE SPACE S ARE DEFINED TO BE INDEPENDENT IF

$$P[A \cap B] = P[A]P[B]$$

SINCE THEN $P[A|B] = \dfrac{P[A \cap B]}{P[B]}$

$$= \frac{P(A)P(B)}{P(B)} = P(A)$$

KNOWLEDGE OF EVENT B OCCURING DOES NOT CHANGE PROBABILITY OF A.

TWO CONT. RVS ARE DEFINED TO BE INDEPENDENT IF FOR ALL EVENTS A, B

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

EQUIVALENT TO $P(Y \in B \mid X \in A) = P(Y \in B)$

IT CAN BE SHOWN THAT X, Y ARE INDEPENDENT IF AND ONLY IF $p_{X,Y}$ FACTORS OR

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y) \qquad \text{FOR ALL } x, y$$

(SEE PG. 179 FOR DISCRETE R.V. PROOF).

EXAMPLE: EXPONENTIAL R.V

$$p_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0,\ y \geq 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

REWRITE AS $p_{X,Y}(x,y) = e^{-(x+y)} u(x) u(y)$

WHERE $u(x)$ IS UNIT STEP FUNCTION

$u(x)\,u(y) = 0$ UNLESS BOTH $u(x) = 1$ AND $u(y) = 1 \Rightarrow x \geq 0$ and $y \geq 0$

$$p_{X,Y}(x,y) = \underbrace{e^{-x}u(x)}_{p_X(x)}\ \underbrace{e^{-y}u(y)}_{p_Y(y)}$$

$\underset{\sim}{X} \sim EXP(1) \qquad Y \sim EXP(1)$

NOTE: CANNOT CLAIM INDEPENDENCE UNLESS EACH FACTOR IS A VALID PDF (INTEGRATES TO 1)

<u>EXAMPLE</u>:

$$p_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)} & x \geq 0,\ y \geq 0 \ \underline{\text{AND}}\ y < x \\ 0 & \text{OTHERWISE} \end{cases}$$

NOT FACTORABLE BECAUSE OF REGION OVER WHICH $p_{X,Y}$ IS NONZERO.

<u>EXAMPLE</u>: STANDARD BIVARIATE GAUSSIAN WITH $\rho = 0$

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}$$

$$= \frac{1}{2\pi}\, e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}}_{p_X(x)}\ \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}}_{p_Y(y)}$$

AS BEFORE $X \sim N(0,1)$, $Y \sim N(0,1)$ BUT FOR $\rho = 0$, X, Y ARE INDEPENDENT. SEE FIG. 12.9

RECALL THAT FACTORABILITY OF $p_{X,Y}$ ALLOWS EASIER INTEGRATION. WILL HAVE FACTORABILITY IF WE CAN ASSUME X AND Y ARE INDEPENDENT.

TRANSFORMATIONS

RECALL THAT FOR A SINGLE RV. X, WE DETERMINED PDF OF $Y = g(X)$. SIMILARLY, NOW WANT TO DETERMINE PDF OF $Z = g(X,Y)$. FOR EXAMPLE, IF $X \sim N(0,1)$, $Y \sim N(0,1)$ AND INDEPENDENT, WHAT IS PDF OF DISTANCE FROM $(x,y) = (0,0)$ OR $Z = \sqrt{X^2+Y^2}$?

TWO APPROACHES:

1) CDF APPROACH, FIND $F_Z(z)$ AND
$$p_Z(z) = \frac{dF_Z(z)}{dz}$$
2) EXTENSION OF $p_Y(y) = p_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$ FORMULA.

WE ILLUSTRATE CDF APPROACH FIRST.

EXAMPLE: $U_1 \sim U(0,1)$ $\quad U_2 \sim U(0,1)$

AND U_1, U_2 INDEPENDENT

CAN YOU GUESS RESULT FOR $X = U_1 + U_2$?

		U_2				
		0.00	0.25	0.50	0.75	1.00
	0.00	0.00	0.25	0.50	0.75	1.00
	0.25	0.25	0.50	0.75	1.00	1.25
U_1	0.50	0.50	0.75	1.00	1.25	1.50
	0.75	0.75	1.00	1.25	1.50	1.75
	1.00	1.00	1.25	1.50	1.75	2.00

Table 2.1: Possible values for $X = U_1 + U_2$ for intuition-building experiment.

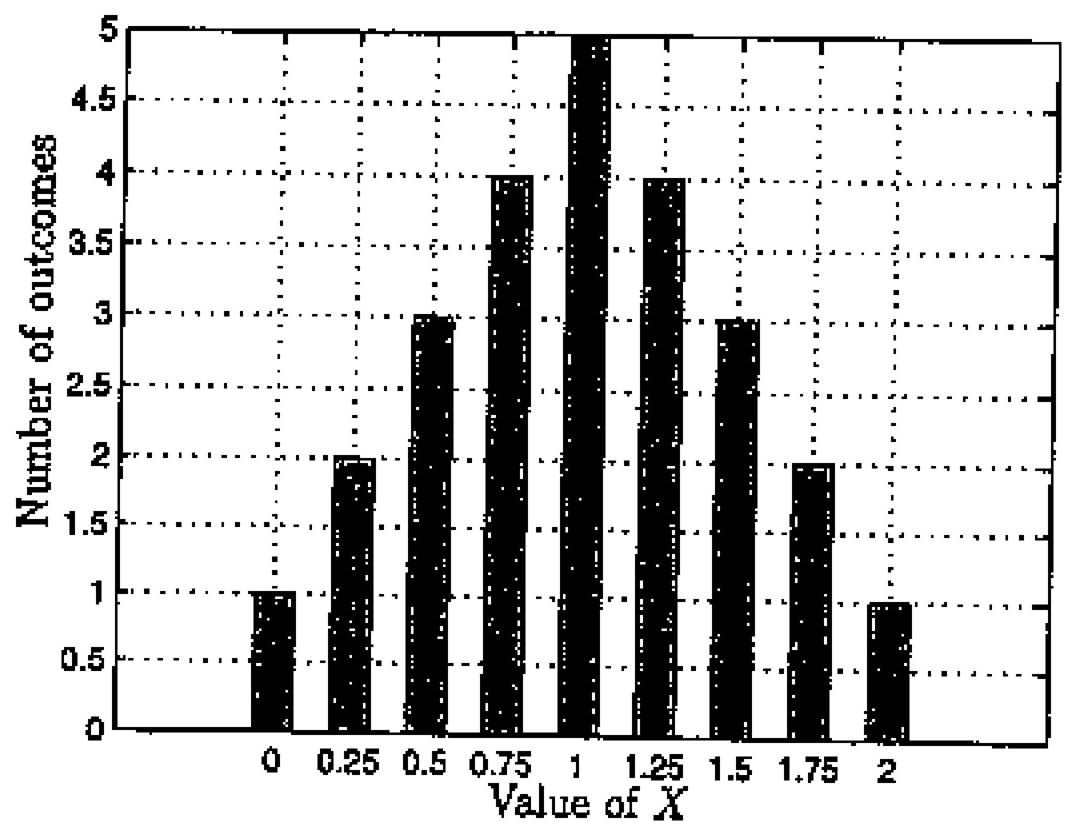


Figure 2.3: Histogram for X for intuition-building experiment.

TRY ALSO MATLAB $\quad$ `x = rand(1000,1)+rand(1000,1)`

`hist(x,50)`

GENERAL APPROACH IS

$$F_Z(z) = P(Z \le z)$$

$$= P(g(X,Y) \le z)$$

$$= \iint_{\{(x,y):\, g(x,y) \le z\}} p_{X,Y}(x,y)\, dx\, dy$$

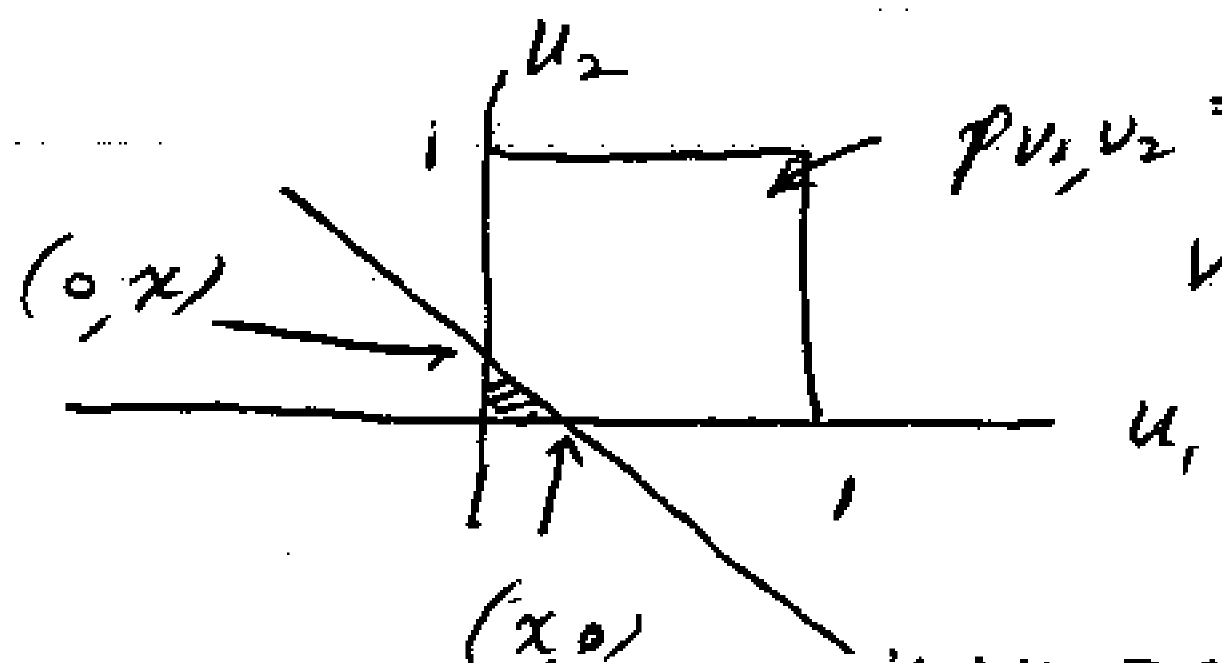
HERE WE HAVE $X = U_1 + U_2$

$$F_X(x) = \iint_{\{(u_1,u_2)\,:\,u_1+u_2 \le x\}} p_{U_1,U_2}(u_1, u_2)\, du_1\, du_2$$

BUT $p_{U_1,U_2} = p_{U_1} p_{U_2}$ INDEPENDENCE

$= 1 \qquad 0 < u_1 < 1,\ 0 < u_2 < 1$

$0 \qquad$ OTHERWISE



$p_{U_1,U_2} = 1$ ON SQUARE

WANT AREA FOR WHICH $u_1 + u_2 \le x$

VALID FOR $0 < x < 1$

AREA $= \frac{1}{2}x^2 \qquad 0 \le x < 1$

$= 0 \qquad x < 0$

SEE BOOK – FOR $1 \le x < 2$

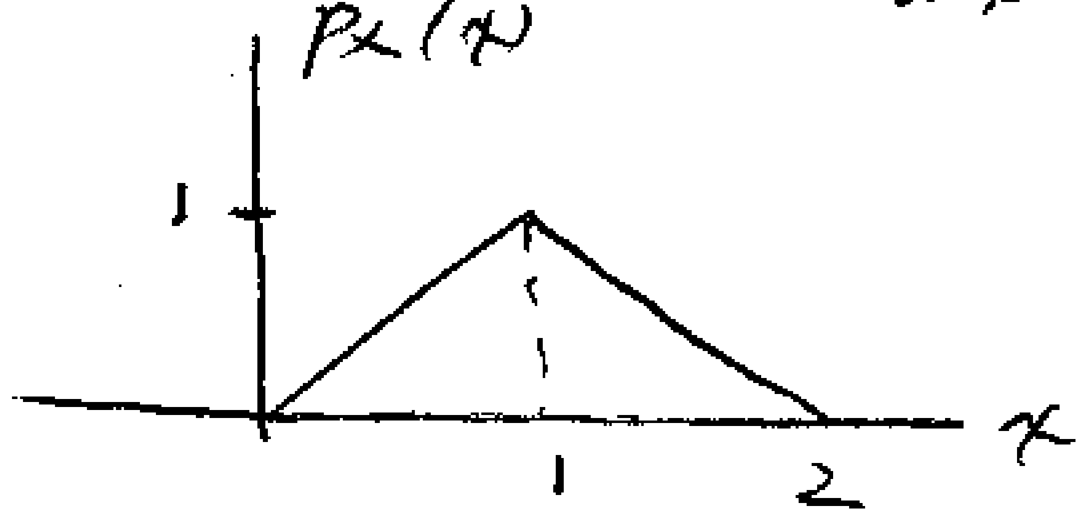
AREA $= 1 - \frac{1}{2}(2-x)^2$

FOR $x > 2$

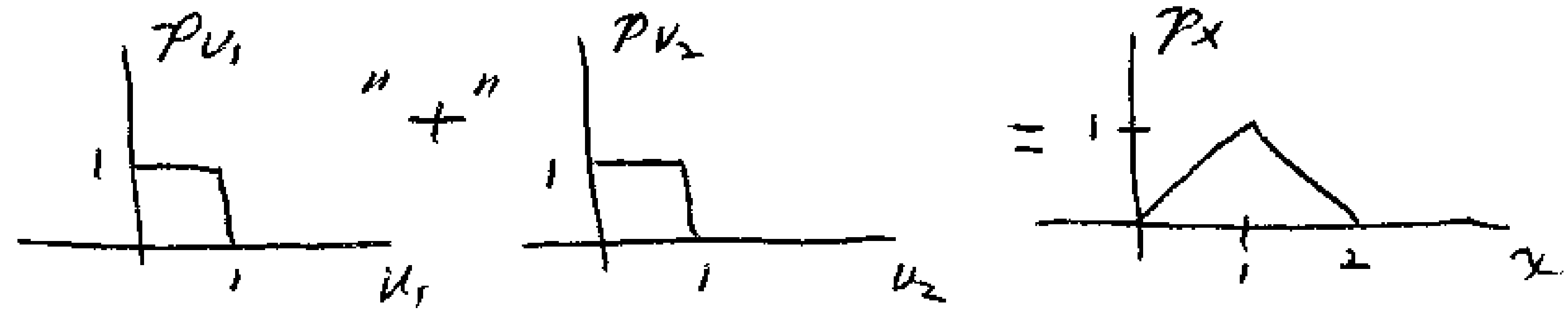
AREA $= 1$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \le x < 1 \\ 1 - \frac{1}{2}(2-x)^2 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

$$p_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases}$$



NOTE:



LOOK FAMILIAR?

NOW CONSIDER MORE GENERALLY $Z = X + Y$, X, Y ARE IND.

$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

NOW FIX z AND DETERMINE ALL VALUES OF (x, y) FOR WHICH $x + y \le z$. THIS IS THE INTEGRATION REGION.

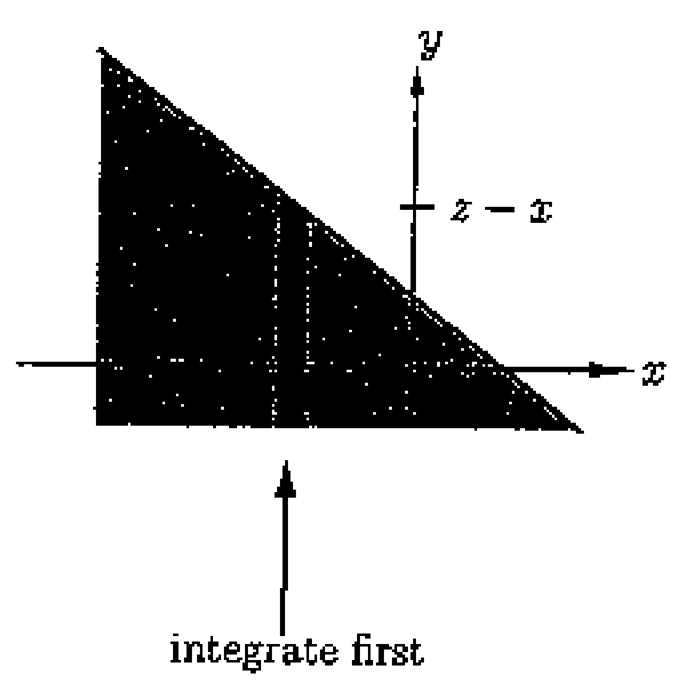


Figure 12.16: Iterated integral evaluation – shaded region is $y \le z - x$. Integrate first in y direction for a fixed x and then integrate over $-\infty < x < \infty$.

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} p_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} p_X(x)\,p_Y(y)\,dy\,dx \qquad \text{(IND.)}$$

$$= \int_{-\infty}^{\infty} p_X(x) \underbrace{\int_{-\infty}^{z-x} p_Y(y)\,dy}_{F_Y(z-x)}\,dx$$

$$\frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} p_X(x) \underbrace{\frac{dF_Y(z-x)}{dz}}_{p_Y(z-x)}\,dx \qquad \text{CHAIN RULE}$$

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)\,p_Y(z-x)\,dx$$

CONVOLUTION INTEGRAL

$$p_Z = p_X \star p_Y$$

<u>ADDING TWO INDEPENDENT R.V.S ⇒</u>
<u>WE CONVOLVE THEIR PDFS</u>

HOW COULD WE GET PDF OF $U_1 + U_2 + U_3$, WHERE $U_1 + U_2$ IS IND. OF U_3 AND U_1, U_2 ARE IND?

FOR THE CONVOLUTION PHOBES, HOW CAN YOU AVOID THIS OPERATION?

APPROACH 2 : INTRODUCE AUXILIARY RV OR LET

$$W = X \qquad W = \text{AUXILIARY RV}$$
$$Z = g(X,Y)$$

FIND JOINT PDF OF (W,Z) AND "INTEGRATE OUT W"

NEED TO DETERMINE HOW TO FIND $p_{W,Z}$ FOR

$$W = g(X,Y) \qquad \text{CHANGED NOTATION SLIGHTLY!}$$
$$Z = h(X,Y)$$

$\left(\begin{array}{l} W = X \\ Z = g(X,Y) \end{array} \right.$ IS SPECIAL CASE)

TO SIMPLIFY DISCUSSION ASSUME

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\underline{G}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

LINEAR TRANSFORMATION

$\underline{G}$ IS NONSINGULAR OR $\underline{G}^{-1}$ EXISTS

RECALL THAT IF $Y = 2X$, WHERE $X \sim U(1,2)$

WE HAD $p_Y(y) = p_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

$$= p_X(y/2)\, \tfrac{1}{2}$$

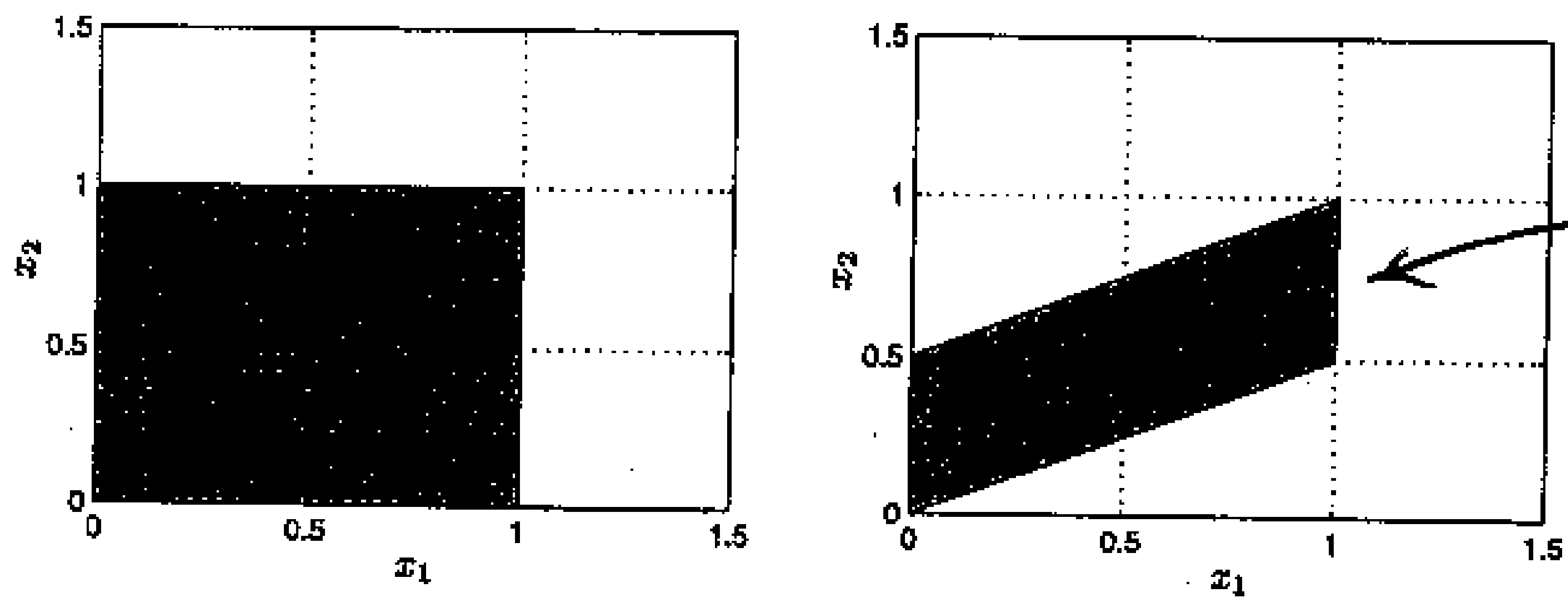
↑ FACTOR NEEDED TO ACCOUNT FOR STRETCHING OF DOMAIN VIA $y = 2x$

IN 2-D THINGS ARE MORE COMPLICATED - GET STRETCHING/COMPRESSION AND/OR ROTATION

<u>EXAMPLE</u>:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ (u_1+u_2)/2 \end{bmatrix} \qquad u_1 \sim U(0,1) \quad u_2 \sim U(0,1)$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



AREA $= \frac{1}{2}$

LINEAR MAPPING CHANGES RECTANGLES TO PARALLELOGRAMS

(a) No dependency

(b) Dependency

Figure 2.12: Relationships between random variables.

CAN SHOW THAT AREA IS MULTIPLIED BY $|DET(\underline{G})| = |1 \cdot \frac{1}{2} - 0 \cdot \frac{1}{2}| = \frac{1}{2}$ $\Rightarrow$ TRANSFORMED PDF MUST BE SCALED BY 2 OR $1/|DET(\underline{G})| = |DET(\underline{G}^{-1})|$. (IN 1-D FOR

$y = 2x \Rightarrow [y] = \underset{\uparrow \underline{G}}{[2]}(x_1)$

SCALING IS $|DET(\underline{G}^{-1})| = \frac{1}{2}$)

RESULT FOR LINEAR TRANSFORMATION:

$$p_{\underline{W}Z}(\underline{w}, z) = p_{XY}\left(\underline{G}^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) |DET(\underline{G}^{-1})|$$

EXAMPLE : $W = \sigma_W X$, $Z = \sigma_Z Y$ $\quad$ $X, Y \sim$ STANDARD BIVARIATE GAUSSIAN

$\sigma_W > 0, \sigma_Z > 0$

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_W & 0 \\ 0 & \sigma_Z \end{bmatrix}}_{\underline{G}} \begin{bmatrix} X \\ Y \end{bmatrix} \qquad \underline{G}^{-1} = \begin{bmatrix} 1/\sigma_W & 0 \\ 0 & 1/\sigma_Z \end{bmatrix}$$

$$\mathrm{DET}(\underline{G}^{-1}) = \frac{1}{\sigma_W \sigma_Z} \qquad \underline{G}^{-1}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} w/\sigma_W \\ z/\sigma_Z \end{bmatrix}$$

$$p_{WZ}(w, z) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[(w/\sigma_W)^2 - 2\rho \frac{w}{\sigma_W}\frac{z}{\sigma_Z} + \left(\frac{z}{\sigma_Z}\right)^2\right]} \cdot \frac{1}{\sigma_W \sigma_Z}$$

$$= \frac{1}{2\pi\sqrt{(1-\rho^2)\sigma_W^2\sigma_Z^2}} e^{-\frac{1}{2(1-\rho^2)}\left[(w/\sigma_W)^2 - 2\rho\frac{w}{\sigma_W}\frac{z}{\sigma_Z} + \left(\frac{z}{\sigma_Z}\right)^2\right]}$$

$$= \frac{1}{2\pi \, \mathrm{DET}^{1/2}(\underline{C})} e^{-\frac{1}{2}\begin{bmatrix} w \\ z \end{bmatrix}^T \underline{C}^{-1}\begin{bmatrix} w \\ z \end{bmatrix}} \qquad \begin{matrix} -\infty < w < \infty \\ -\infty < z < \infty \end{matrix}$$

WHERE $\underline{C} = \begin{bmatrix} \sigma_W^2 & \rho\sigma_W\sigma_Z \\ \rho\sigma_W\sigma_Z & \sigma_Z^2 \end{bmatrix}$ (WILL SHOW LATER THAT $\underline{C}$ IS A COVARIANCE MATRIX)

MORE GENERALLY IF $W = g(x, y)$, $Z = h(x, y)$

WHERE g, h ARE NONLINEAR, THEN

$$p_{W,Z}(w, z) = p_{X,Y}(g^{-1}(w, z), h^{-1}(w, z)) \left| \mathrm{DET} \frac{\partial(x, y)}{\partial(w, z)} \right|$$

WHERE $x = g^{-1}(w, z)$, $y = h^{-1}(w, z)$ $\Big\}$ ASSUMES A SINGLE SOLUTION FOR (x, y) GIVEN (w, z)