AND $\dfrac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{bmatrix}$ = JACOBIAN

MATRIX OF TRANSFORMATION FROM (w,z) TO (x,y)

EXAMPLE : LINEAR TRANSFORMATION

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\underline{G}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underline{G}^{-1} \begin{bmatrix} w \\ z \end{bmatrix} \qquad \text{LET } \underline{G}^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}$$

$$x = g^{-1}(w,z) = g^{11}w + g^{12}z$$
$$y = h^{-1}(w,z) = g^{21}w + g^{22}z$$

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix} \partial x/\partial w & \partial x/\partial z \\ \partial y/\partial w & \partial y/\partial z \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \underline{G}^{-1}$$

$$p_{W,Z}(w,z) = p_{X,Y}\left(\underline{G}^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) \left|\mathrm{DET}(\underline{G}^{-1})\right|$$

SAME AS BEFORE

EXAMPLE : $X \sim N(0,1)$, $Y \sim N(0,1)$ IND.

$$\left.\begin{aligned} W &= X \\ Z &= Y/X \end{aligned}\right\} \text{ NONLINEAR TRANSFORMATION}$$

NOTE: IF WE ULTIMATELY INTEGRATE OUT W, THEN HAVE p_Z $\Rightarrow$ AUXILIARY R.V. METHOD

TO FIND $S_{W,Z}$, $\quad W = X \Rightarrow -\infty < w < \infty$

$\quad Z = Y/X \Rightarrow -\infty < z < \infty$

SINCE $-\infty < x < \infty$, $-\infty < y < \infty$.

1) FIND INVERSE TRANSFORMATION

$$x = w$$
$$y = xz = wz$$

$$\frac{\partial(x, y)}{\partial(w, z)} = \begin{bmatrix} \partial x/\partial w & \partial x/\partial z \\ \partial y/\partial w & \partial y/\partial z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & w \end{bmatrix}$$

$$\left| \det \frac{\partial(x, y)}{\partial(w, z)} \right| = |w| \qquad (\text{W CAN BE NEGATIVE})$$

$$p_{W,Z}(w, z) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \Big|_{\substack{x=w \\ y=wz}} |w|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+w^2z^2)} |w|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}w^2(1+z^2)} |w| \qquad \begin{matrix} -\infty < w < \infty \\ -\infty < z < \infty \end{matrix}$$

IF WE WISH TO FIND PDF OF Y/X, THEN

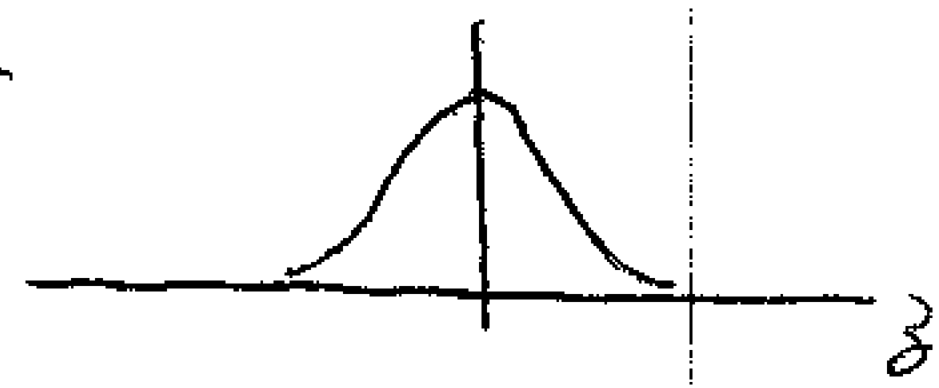
$$p_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}w^2(1+z^2)} |w| \, dw$$

$$= 2 \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}w^2(1+z^2)} |w| \, dw \qquad \text{WHY?}$$

$$= \frac{1}{\pi} \int_0^{\infty} w \, e^{-\frac{1}{2}(1+z^2)w^2} \, dw$$

$$= \frac{1}{\pi} \left. \frac{e^{-\frac{1}{2}(1+z^2)w^2}}{-(1+z^2)} \right|_0^\infty = \frac{1}{\pi(1+z^2)}$$

THIS IS A CAUCHY PDF



EXAMPLE: SEE EX 12.12

IF $X \sim N(0, \sigma^2)$, $Y \sim N(0, \sigma^2)$ AND X, Y ARE IND, THEN

POLAR COORD. $R = \sqrt{X^2+Y^2}$ $R \geq 0$

$\Theta = \arctan(Y/X)$ $0 \leq \Theta < 2\pi$

$$\Rightarrow \quad p_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{1}{2} r^2/\sigma^2} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

THIS IS A RAYLEIGH PDF AND $\Theta \sim U(0, 2\pi)$ AND R, Θ ARE IND.

IMPORTANT IN PRACTICE!

EXPECTED VALUES

WE FIND IT CONVENIENT TO USE VECTOR NOTATION, HENCE $\begin{bmatrix} X \\ Y \end{bmatrix}$. EXPECTED VALUE OF RANDOM VECTOR DEFINED AS

$$E\left[\begin{bmatrix} X \\ Y \end{bmatrix}\right] = \begin{bmatrix} E(X) \\ E(Y) \end{bmatrix} = \text{VECTOR OF EXPECTED VALUES}$$

WHAT IS $E[Z]$ IF $Z = g(x, y)$?

$$E[Z] = \int_{-\infty}^{\infty} z \, p_Z(z) \, dz$$

MUST FIND

ALTERNATIVELY $E[Z] = E[g(x, y)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) \underbrace{p_{X,Y}(x, y)}_{\text{AVERAGING PDF}} \, dx \, dy$$

USE NEW NOTATION

$$E_{X,Y}[\;] = \iint (\;) \, p_{X,Y}(x, y) \, dx \, dy$$

$$E_X[\;] = \int (\;) \, p_X(x) \, dx$$

SHOW THAT $E_{X,Y}[X] = E_X[X]$.

AS BEFORE EXPECTATION IS LINEAR

$$E_{X,Y}[a g(x, y) + b h(x, y)] = a E_{X,Y}[g(x, y)] + b E_{X,Y}[h(x, y)]$$

NOW RECALL HOW THINGS SIMPLIFY FOR IND. RVs. CONSIDER $E_{X,Y}[XY]$ (CALLED A JOINT MOMENT) FOR X, Y IND.

$$E_{X,Y}[\underbrace{XY}_{g(x,y)}] = \iint xy \, p_{X,Y}(x, y) \, dx \, dy$$

$$= \int\int xy\, p_X(x) p_Y(y)\, dx\, dy$$
$$= \int x p_X(x)\, dx \int y p_Y(y)\, dy$$
$$= E_X[X] E_Y[Y]$$

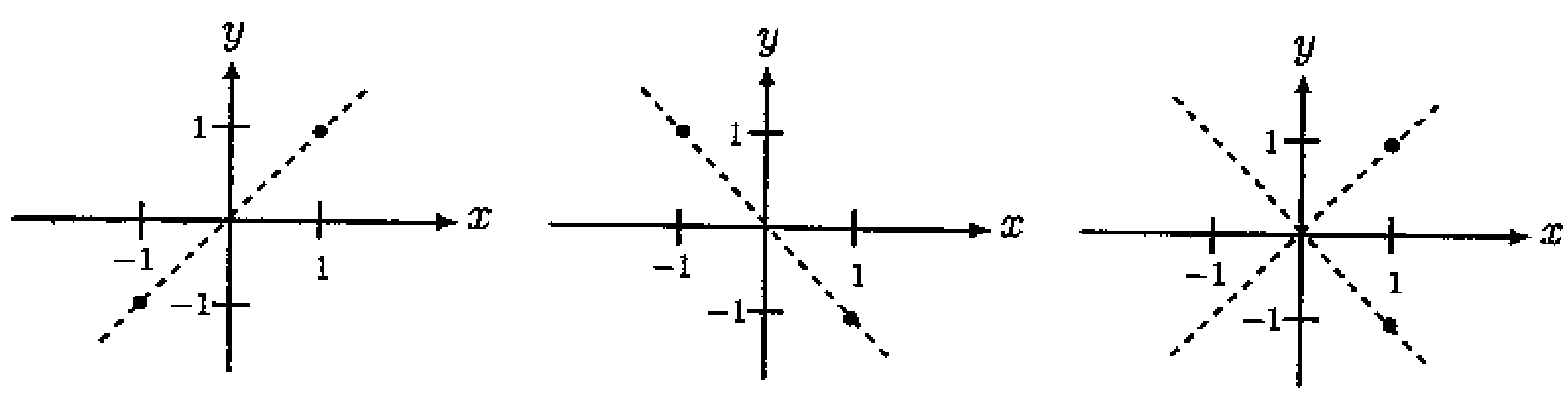
OR $E_{X,Y}[XY] = E_X[X] E_Y[Y]$

THE COVARIANCE

RECALL THAT $\text{VAR}(X) = E[(X - E[X])^2]$
$= E[(X - E[X])(X - E[X])]$

DEFINE THE COVARIANCE AS
$\text{COV}(X, Y) = E_{X,Y}[(X - E_X[X])(Y - E_Y[Y])]$

INDICATES HOW X AND Y VARY WITH RESPECT TO EACH OTHER - ON THE AVERAGE.



(a) $p_{X,Y}[-1,-1] = p_{X,Y}[1,1] = 1/2$ (b) $p_{X,Y}[1,-1] = p_{X,Y}[-1,1] = 1/2$ (c) $p_{X,Y}[1,1] = p_{X,Y}[1,-1] = 1/2$

Figure 7.6: Joint PMFs depicting different relationships between the random variables X and Y.

TO FIND COVARIANCES FIRST FIND $E_X[X]$, $E_Y[Y]$. $p_X[x_i] = \sum_j p_{X,Y}[x_i, y_j]$
$p_Y[y_j] = \sum_i p_{X,Y}[x_i, y_j]$

a)

$X \backslash Y$	-1	1	p_X
-1	$\frac{1}{2}$	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$
p_Y	$\frac{1}{2}$	$\frac{1}{2}$	

$\Rightarrow E_X[X] = 0$

$E_Y(Y) = 0$

SAME AS FOR b).

FOR c) $E_X[X] = 1$

$E_Y[Y] = 0$

$\Rightarrow \quad \mathrm{COV}(X,Y) = E_{X,Y}[XY]$

a) $XY = 1$ ALWAYS $\Rightarrow \mathrm{COV}(X,Y) = 1$

b) $XY = -1$ ALWAYS $\Rightarrow \mathrm{COV}(X,Y) = -1$

c) $XY = 1$ WITH PROB $= \frac{1}{2}$

$XY = -1$ WITH PROB $= \frac{1}{2}$ $\Rightarrow \mathrm{COV}(X,Y) = 0$

IN a) X, Y VARY DIRECTLY $\Rightarrow \mathrm{COV}(X,Y) > 0$

b) X, Y VARY IN OPPOSITE (ONE +, ONE −) DIRECTIONS $\Rightarrow \mathrm{COV}(X,Y) < 0$

c) X, Y VARY EITHER DIRECTLY OR IN OPPOSITE DIRECTIONS $\Rightarrow \mathrm{COV}(X,Y) = 0$

THE COVARIANCE INDICATES HOW OUTCOMES OF X ARE RELATED (LINEARLY) TO OUTCOMES OF Y.

IF X, Y ARE INDEPENDENT, $\mathrm{COV}(X,Y) = 0$.

PROOF: $\mathrm{COV}(X,Y) = E_{X,Y}[(X - E_X[X])(Y - E_Y[Y])]$

$= E_{X,Y}[XY - X E_Y[Y] - Y E_X[X] + E_X[X] E_Y[Y]]$

$E_X\{X\}, E_Y\{Y\}$ ARE CONSTANTS $\Rightarrow$

$$= E_{X,Y}\{XY\} - E_Y\{Y\}\underbrace{E_{X,Y}\{X\}}_{E_X\{X\}} - E_X\{X\}\underbrace{E_{X,Y}\{Y\}}_{E_Y\{Y\}} + E_X\{X\}E_Y\{Y\}$$

$\therefore \mathrm{COV}(X,Y) = E_{X,Y}\{XY\} - E_X\{X\}E_Y\{Y\}$ ALTERNATIVE FORMULA

FOR X,Y IND $\Rightarrow E_{X,Y}\{XY\} = E_X\{X\}E_Y\{Y\}$
$\Rightarrow \mathrm{COV}(X,Y) = 0$

CONVERSE NOT TRUE IN GENERAL
$\mathrm{COV}(X,Y) = 0 \not\Rightarrow X,Y$ IND. (SEE PG. 191)

ALSO REFER TO FIG. 12.9 FOR STANDARD BIVARIATE. THERE $E_X\{X\} = E_Y\{Y\} = 0 \Rightarrow$
$\mathrm{COV}(X,Y) = E_{X,Y}\{XY\}$ - RELATIONSHIP TO ρ?

<u>EXAMPLE</u>: STANDARD BIVARIATE GAUSSIAN
SINCE $E_X\{X\} = E_Y\{Y\} = 0$
$\mathrm{COV}(X,Y) = E_{X,Y}\{XY\}$

$$E_{X,Y}\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}\, dx\, dy$$

BY COMPLETING SQUARE CAN REDUCE TO 1-D INTEGRALS (SEE BOOK)

$\Rightarrow E_{X,Y}\{XY\} = \rho \Rightarrow \mathrm{COV}(X,Y) = \rho$

ACTUALLY, FOR STANDARD BIVARIATE GAUSSIAN

$$\rho = \rho_{X,Y} = \frac{COV(X,Y)}{\sqrt{VAR(X)\,VAR(Y)}}$$

$= $ CORRELATION COEFFICIENT

SINCE $VAR(X) = VAR(Y) = 1$ (RECALL $X \sim N(0,1)$, $Y \sim N(0,1)$)

$\rho_{X,Y}$ TELLS US HOW WELL WE CAN PREDICT Y FROM X.

ASIDE - LINEAR PREDICTION
(SEE SECTION 7.9)

ASSUME WE OBSERVE $X = x$ AND WE WISH TO "PREDICT" OR ESTIMATE OUTCOME OF Y (SIMILAR TO PREDICTING WEIGHT FROM HEIGHT). USE LINEAR PREDICTOR

$$\hat{y} = a x$$

(ASSUMES $E_X(X) = E_Y(Y) = 0$ OR ELSE NEED $\hat{y} = ax + \underset{\text{CONSTANT}}{b}$)

TO FIND BEST a
WE MINIMIZE MEAN SQUARE ERROR

$$mse(a) = E_{X,Y}[(Y - \hat{Y})^2]$$

AVERAGES ERROR^2 OVER ALL OUTCOMES OF (x, y).

$$mse = E_{x,y}\{(y-ax)^2\}$$

$$\frac{d\,mse}{da} = E_{x,y}\left(\frac{d}{da}(y-ax)^2\right) \qquad \text{WHY?}$$

$$= E_{x,y}[2(y-ax)(-x)] = 0$$

$$\text{OR} \quad E_{x,y}[(y-ax)x] = 0 \qquad (*)$$

$$E_{x,y}[xy] = a\,E_{x,y}[x^2]$$

$$a_{OPT} = \frac{E_{x,y}[xy]}{E_{x,y}[x^2]} \leftarrow = E_x[x^2]$$

$$= \frac{COV(x,y)}{VAR(x)} \qquad \text{WHY?}$$

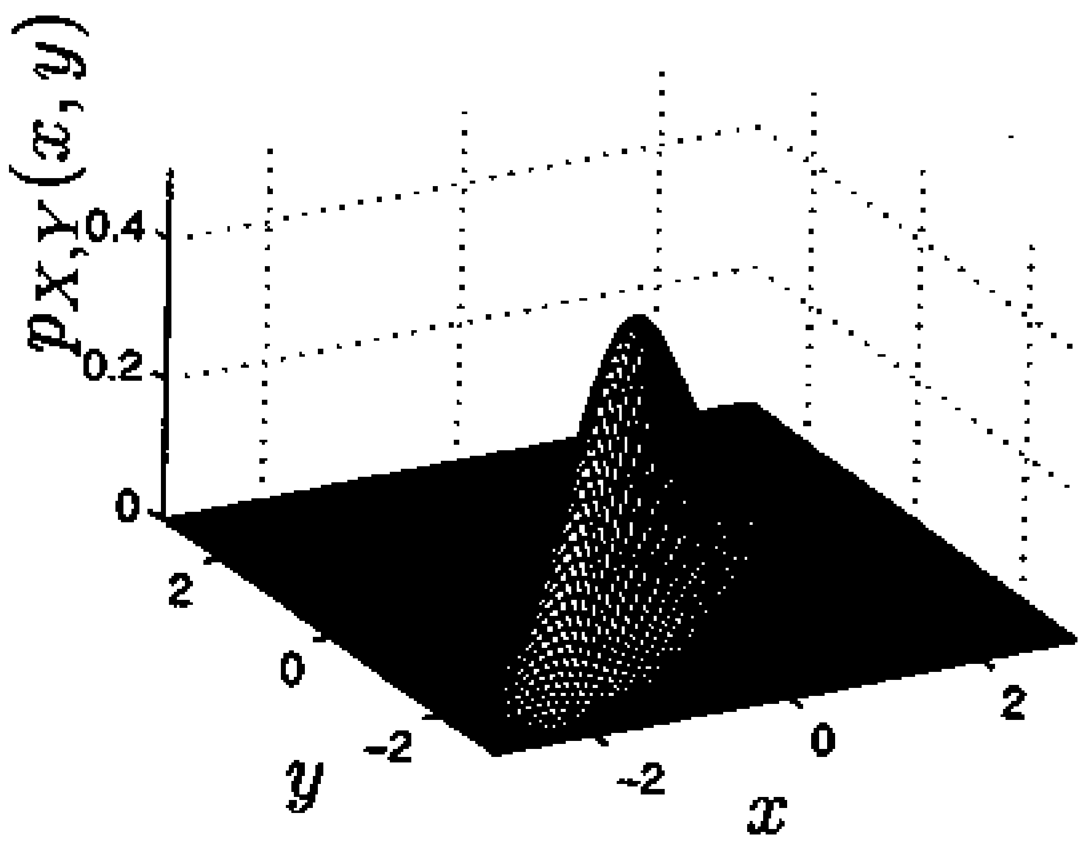
$$\therefore \quad \hat{y} = \frac{COV(x,y)}{VAR(x)}\,x \qquad \text{OUTCOME OF } X$$

ESTIMATE OF y ← OUTCOME OF y

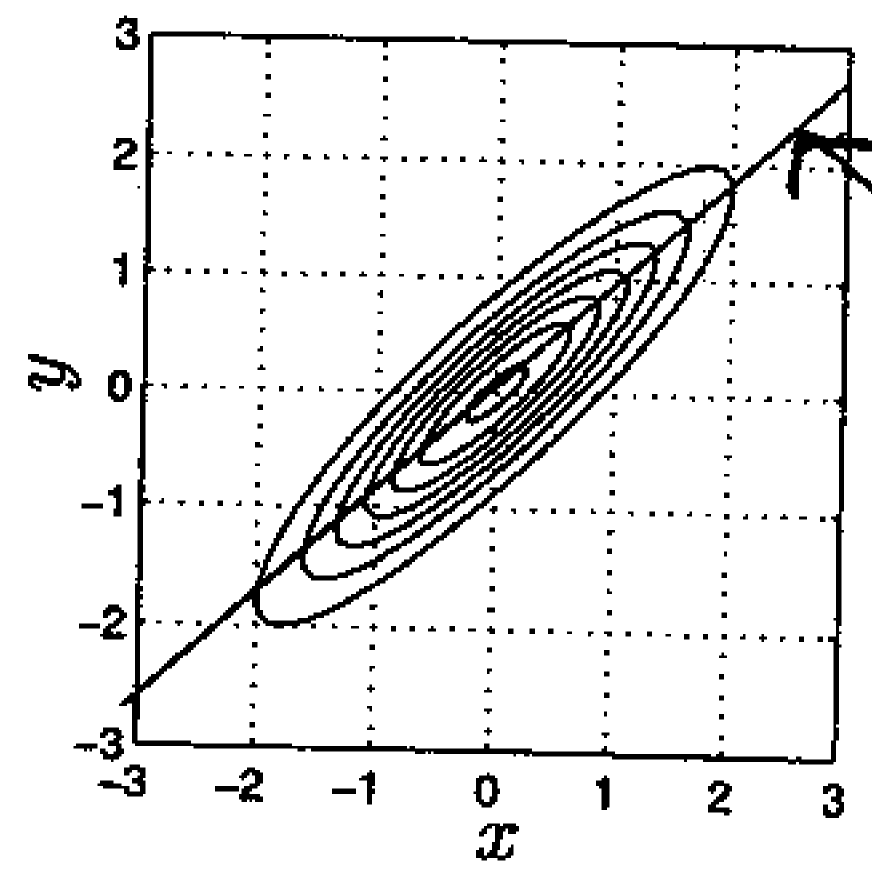
FOR STANDARD BIVARIATE GAUSSIAN $COV(x,y) = \rho$, $VAR(x) = 1 \Rightarrow$

$$\hat{y} = \rho x$$

CAN YOU NOW MAKE SENSE OF FIGURE 12.9?



(e) $\rho = 0.9$



$y = \rho x$

(f) $\rho = 0.9$

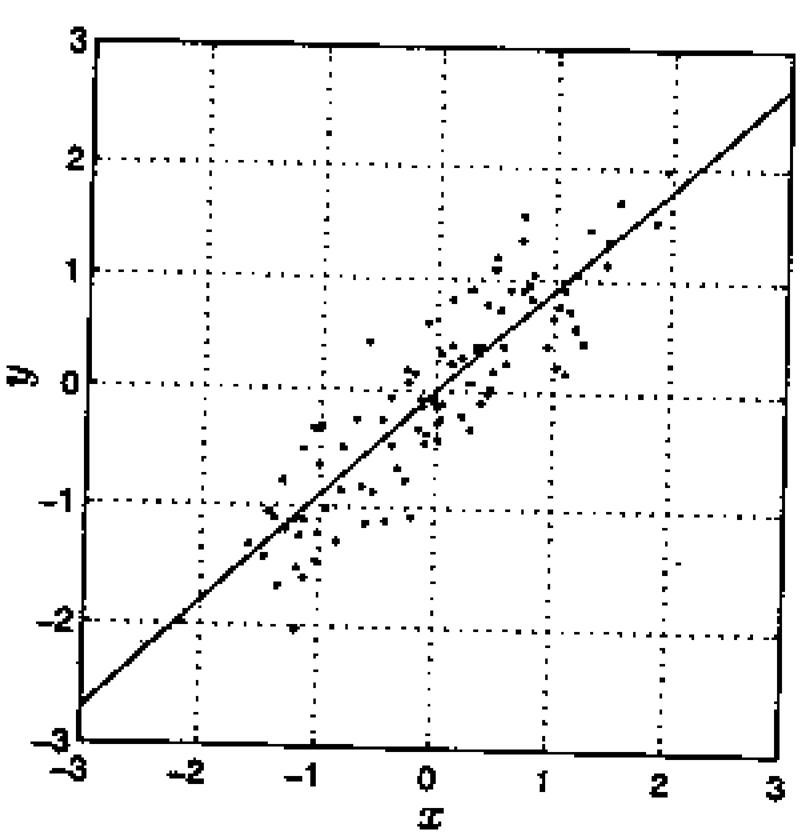
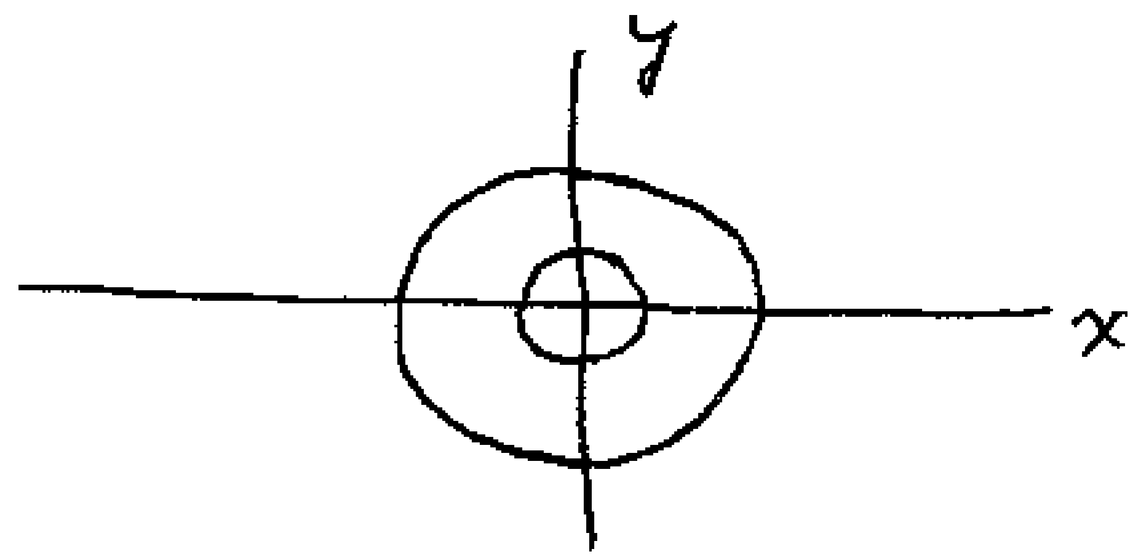


Figure 12.20: 100 outcomes of bivariate Gaussian random vector with zero means and covariance matrix given by (12.40). The best prediction of Y when $X = x$ is observed is given by the line.

WHAT IS BEST PREDICTION OF Y IF



CIRCULAR PDF CONTOURS ?

WHAT VALUE OF ρ GIVES BEST PREDICTION?

$$mse_{MIN} = E_{x,y}[(y - \hat{y}_{OPT})^2]$$

$$= E_{x,y}\left[\left(y - \frac{COV(x,y)}{VAR(x)}x\right)^2\right]$$

$$= E_{x,y}\left[y^2 - 2yx\frac{COV(x,y)}{VAR(x)} + \frac{COV^2(x,y)}{VAR^2(x)}x^2\right]$$

$$= E_{x,y}[y^2] - 2\frac{COV(x,y)}{VAR(x)}E_{x,y}[xy] + \frac{COV^2(x,y)}{VAR^2(x)}E[x^2]$$

SINCE $E_x[x] = E_y[y] = 0$

$$mse_{MIN} = \underbrace{E_y[y^2]}_{VAR(y)} - 2\frac{COV^2(x,y)}{VAR(x)} + \frac{COV^2(x,y)}{VAR^2(x)}VAR(x)$$

$$= VAR(y) - \frac{COV^2(x,y)}{VAR(x)}$$

$$= VAR(y)\left[1 - \frac{COV^2(x,y)}{VAR(x)VAR(y)}\right]$$

$$\therefore\ mse_{MIN} = VAR(y)(1 - \rho_{x,y}^2)$$

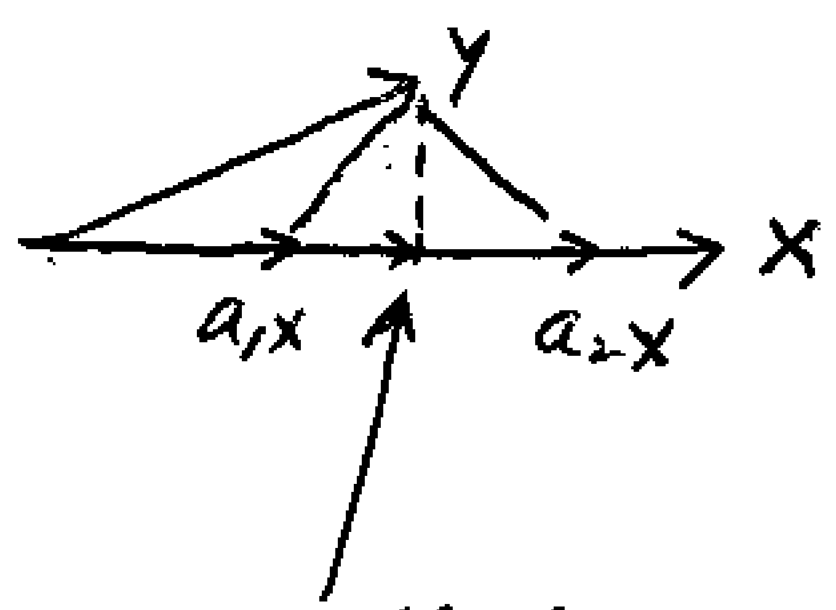
FOR STANDARD BIVARIATE GAUSSIAN, $\rho_{xy} = \rho$, $VAR(y) = 1$

$$\Rightarrow\ mse_{MIN} = 1 - \rho^2$$

$$= 0 \quad \text{FOR } \rho = \pm 1$$

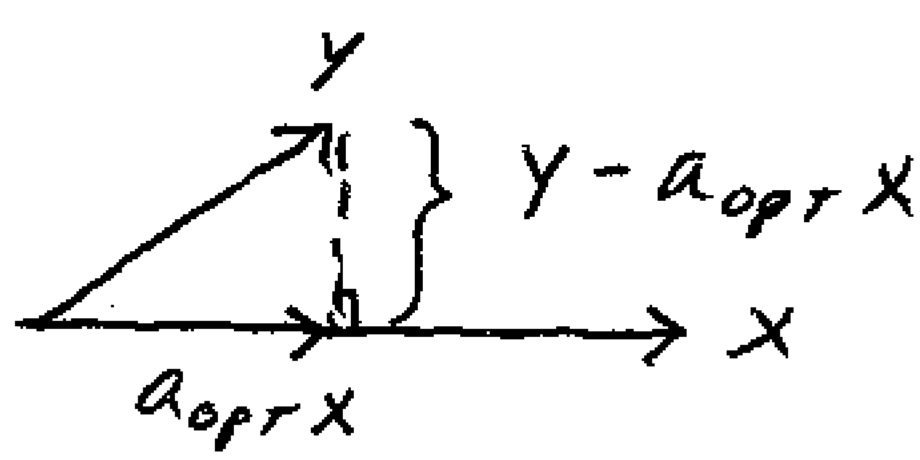
LASTLY, NOTE THAT TO FIND a_{OPT},
FROM (*) $E_{X,Y}[(Y-aX)X] = 0$
THINK OF X AND Y AS EUCLIDEAN VECTORS

X, Y CANNOT BE ALIGNED OR ELSE
$Y = 2X$ FOR EXAMPLE. HENCE,



WANT TO MINIMIZE
$(Y-aX)^2$ = DISTANCE^2
(ON AVERAGE)

MINIMIZES DISTANCE



$(Y - a_{OPT}X) \perp X$

TWO "VECTORS" ARE SAID TO BE $\perp$ IF
$X \cdot Y = 0$, HERE $X \cdot Y \sim E_{X,Y}[XY]$

$\Rightarrow (Y - a_{OPT}X) \perp X$ MEANS

$$E_{X,Y}[(Y - a_{OPT}X)X] = 0$$

OR ERROR $\perp$ DATA
$Y - a_{OPT}X$ $\quad$ X

CALLED THE <u>ORTHOGONALITY PRINCIPLE</u>

HOLDS EVEN FOR $\hat{Y} = a_1 X_1 + a_2 X_2 + \ldots + a_N X_N$
(WILL BE USEFUL LATER)

BACK TO COVARIANCES

CONSIDER STANDARD BIVARIATE GAUSSIAN

$$p_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)}$$

$$\text{IF } \rho = 0, \quad p_{XY}(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}_{p_X(x)} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}}_{p_Y(y)}$$

X, Y ARE IND OR
$COV(X,Y) = 0 \Rightarrow X, Y$ ARE IND.

<u>ONLY HOLDS</u> FOR THIS <u>SPECIAL CASE</u>
(HOW ABOUT CONVERSE, X, Y IND $\Rightarrow COV(X,Y) = 0$?
DOES THIS HOLD?)

COVARIANCES ALSO ARE NECESSARY TO DETERMINE $VAR(X+Y)$ SINCE

$$VAR(X+Y) = E_{X,Y}\left[\left((X+Y) - E_{X,Y}[X+Y]\right)^2\right]$$

$$= E_{X,Y}\left[\left((X - E_X[X]) + (Y - E_Y[Y])\right)^2\right]$$

NOTE CROSS-TERM!

$$\mathrm{VAR}(x+y) = E_{x,y}\left[(x-E_x[x])^2 + 2(x-E_x[x])(y-E_y[y]) + (y-E_y[y])^2\right]$$

$$= \mathrm{VAR}(x) + 2\,\mathrm{COV}(x,y) + \mathrm{VAR}(y)$$
$$= \mathrm{VAR}(x) + \mathrm{VAR}(y) + 2\,\mathrm{COV}(x,y)$$

IS VARIANCE INCREASED BY ADDING RANDOM VARIABLES? CAN $\mathrm{VAR}(x+y) = 0$?

COVARIANCE MATRIX

CONVENIENT TO PUT ALL VARIANCES, COVARIANCES INTO MATRIX AS

$$\underline{C} = \begin{bmatrix} \mathrm{VAR}(x) & \mathrm{COV}(x,y) \\ \mathrm{COV}(y,x) & \mathrm{VAR}(y) \end{bmatrix} \quad 2\times 2$$

PROPERTIES: 1) SYMMETRIC, $\underline{C}^T = \underline{C}$ WHY?
2) POSITIVE DEFINITE (PD)

PROOF:
RECALL: FOR $\underline{C}$ TO BE PD
$(\underline{C})_{11} > 0$ AND $\mathrm{DET}(\underline{C}) > 0$

$$\mathrm{DET}(\underline{C}) = \mathrm{VAR}(x)\,\mathrm{VAR}(y) - \mathrm{COV}^2(x,y)$$
$$= \mathrm{VAR}(x)\,\mathrm{VAR}(y)\left(1 - \frac{\mathrm{COV}^2(x,y)}{\mathrm{VAR}(x)\,\mathrm{VAR}(y)}\right)$$

$= VAR(X)\,VAR(Y)\,(1-\rho_{XY}^2) > 0$ FOR $|\rho_{XY}| < 1$

SINCE $|\rho_{XY}| \le 1$ (SEE PG. 197, PROPERTY 7.7)

---

3) $VAR(X+Y)$ = SUM OF ALL ELEMENTS IS $\underline{C}$

WHERE IS $\underline{C}$ USED? RECALL (PG 80 OF NOTES)

IF $X, Y \sim$ STANDARD BIVARIATE GAUSSIAN

AND $W = \sigma_W X$, $Z = \sigma_Z Y \Rightarrow$

$$p_{W,Z}(w,z) = \frac{1}{2\pi(\det^{1/2}(\underline{C}))} e^{-\frac{1}{2}\begin{bmatrix} w \\ z\end{bmatrix}^T \underline{C}^{-1}\begin{bmatrix} w \\ z\end{bmatrix}}$$

$$\underline{C} = \begin{bmatrix} \sigma_W^2 & \rho\sigma_W\sigma_Z \\ \rho\sigma_W\sigma_Z & \sigma_Z^2 \end{bmatrix}$$

$$= \begin{bmatrix} VAR(W) & COV(W,Z) \\ COV(Z,W) & VAR(Z) \end{bmatrix} \qquad \begin{array}{l} \sigma_W^2 \overset{?}{=} VAR(W) \\ \rho\sigma_W\sigma_Z \overset{?}{=} COV(W,Z) \end{array}$$

$$VAR(W) = E[W^2] - \underbrace{E^2[W]}_{\substack{=0 \\ \text{WHY?}}} = E[W^2] = E[(\sigma_W X)^2]$$

$$= \sigma_W^2 E[X^2] = \sigma_W^2 VAR(X)$$

$$= \sigma_W^2$$

JUSTIFY $\rho\sigma_W\sigma_Z = COV(W,Z)$?

FROM EXAMPLE 12.10, IF $X, Y \sim$ STANDARD BIVARIATE GAUSSIAN

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} \sigma_W & 0 \\ 0 & \sigma_Z \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \mu_W \\ \mu_Z \end{bmatrix}$$

↑ NOW HAVE NONZERO MEANS

$$p_{W,Z}(w,z) = \frac{1}{2\pi \, \mathrm{DET}^{1/2}(\underline{C})} e^{-\frac{1}{2}\begin{bmatrix} w-\mu_W \\ z-\mu_Z \end{bmatrix}^T \underline{C}^{-1} \begin{bmatrix} w-\mu_W \\ z-\mu_Z \end{bmatrix}}$$

MOST GENERAL GAUSSIAN PDF FOR TWO RVS - CALLED THE MULTIVARIATE GAUSSIAN PDF. CAN SHOW THAT MARGINALS ARE $W \sim N(\mu_W, \sigma_W^2)$

$Z \sim N(\mu_Z, \sigma_Z^2)$

W, Z ARE NOT INDEPENDENT UNLESS $\mathrm{COV}(W,Z) = 0$ OR $\underline{C}$ IS DIAGONAL MATRIX.

NOW REVERT BACK TO X AND Y.

$$p_{X,Y}(x,y) = \frac{1}{2\pi \, \mathrm{DET}^{1/2}(\underline{C})} e^{-\frac{1}{2}\begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \underline{C}^{-1} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}}$$

$-\infty < x < \infty$

$-\infty < y < \infty$

ONE OF MOST IMPORTANT PROPERTIES OF THIS PDF IS THAT IF

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underline{G} \begin{bmatrix} X \\ Y \end{bmatrix} \qquad \underline{G} \text{ INVERTIBLE}$$

THEN W, Z ARE STILL MULTIVARIATE GAUSSIAN BUT WITH DIFFERENT MEANS AND COVARIANCE MATRIX.

USE $p_{W,Z}(w,z) = p_{X,Y}\left(\underline{G}^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) \left|\mathrm{DET}(\underline{G}^{-1})\right|$

$$\Rightarrow p_{W,Z}(w,z) = \frac{1}{2\pi\, \mathrm{DET}^{1/2}(\underline{G}\,\underline{C}\,\underline{G}^T)}$$

$$\cdot\, e^{-\frac{1}{2}\begin{bmatrix} w-\mu_W \\ z-\mu_Z \end{bmatrix}^T (\underline{G}\,\underline{C}\,\underline{G}^T)^{-1}\begin{bmatrix} w-\mu_W \\ z-\mu_Z \end{bmatrix}}$$

SEE PP 408-410

WHERE $\begin{bmatrix} \mu_W \\ \mu_Z \end{bmatrix} = \underline{G}\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$

(WILL PROVE LATER - SEE SLIDE 107)

NEW MEAN VECTOR IS $\underline{G}\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$ AND NEW COVARIANCE MATRIX IS $\underline{G}\,\underline{C}\,\underline{G}^T$.

ALLOWS US TO MODIFY MEANS AND COVARIANCES VIA LINEAR "FILTERING" BUT RETAIN GAUSSIAN PDF! INVALUABLE IN PRACTICE.

EXAMPLE: LET $\mu_X = \mu_Y = 0 \Rightarrow$
$\mu_W = \mu_Z = 0$

LET $\underline{C} = \begin{bmatrix} 26 & 6 \\ 6 & 26 \end{bmatrix} \Rightarrow X, Y$ ARE CORRELATED

CONSIDER $\underline{G} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (MODAL MATRIX)^T FOR $\underline{C}$ -

$= \underline{V}^T$

SEE PG 794

THERE $\underline{G} \to \underline{V}^T$

$\underset{\underline{G}}{\underline{V}^T}\,\underset{\underline{C}}{\underline{A}}\,\underset{\underline{G}^T}{\underline{V}} = \underline{\Lambda} \leftarrow$ DIAGONAL

$$\Rightarrow \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\underline{G}\,\underline{C}\,\underline{G}^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} 26 & 6 \\ 6 & 26 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\begin{bmatrix} \frac{1}{\sqrt{2}} 20 & \frac{1}{\sqrt{2}} 32 \\ \frac{1}{\sqrt{2}}(-20) & \frac{1}{\sqrt{2}} 32 \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{1}{2}(20) + \frac{1}{2}(20) & \frac{1}{2}(32) - \frac{1}{2}(32) \\ \frac{1}{2}(20) - \frac{1}{2}(20) & \frac{1}{2}(32) + \frac{1}{2}(32) \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 \\ 0 & 32 \end{bmatrix} \quad \text{DIAGONAL !}$$

$\Rightarrow$ W, Z ARE NOW INDEPENDENT (SINCE THEY ARE MULTIVARIATE GAUSSIAN <u>AND</u> UNCORRELATED)

NOTE: EIGENVALUES OF $\underline{C}$ ARE $\lambda_1 = 20$ AND $\lambda_2 = 32$

ALSO CAN SHOW THAT $\underline{G}$ IS A ROTATION MATRIX OR

$$\underline{G} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{FOR } \theta = \pi/4$$

SEE PG. 264

ACTS TO ROTATE VECTOR θ RADIANS

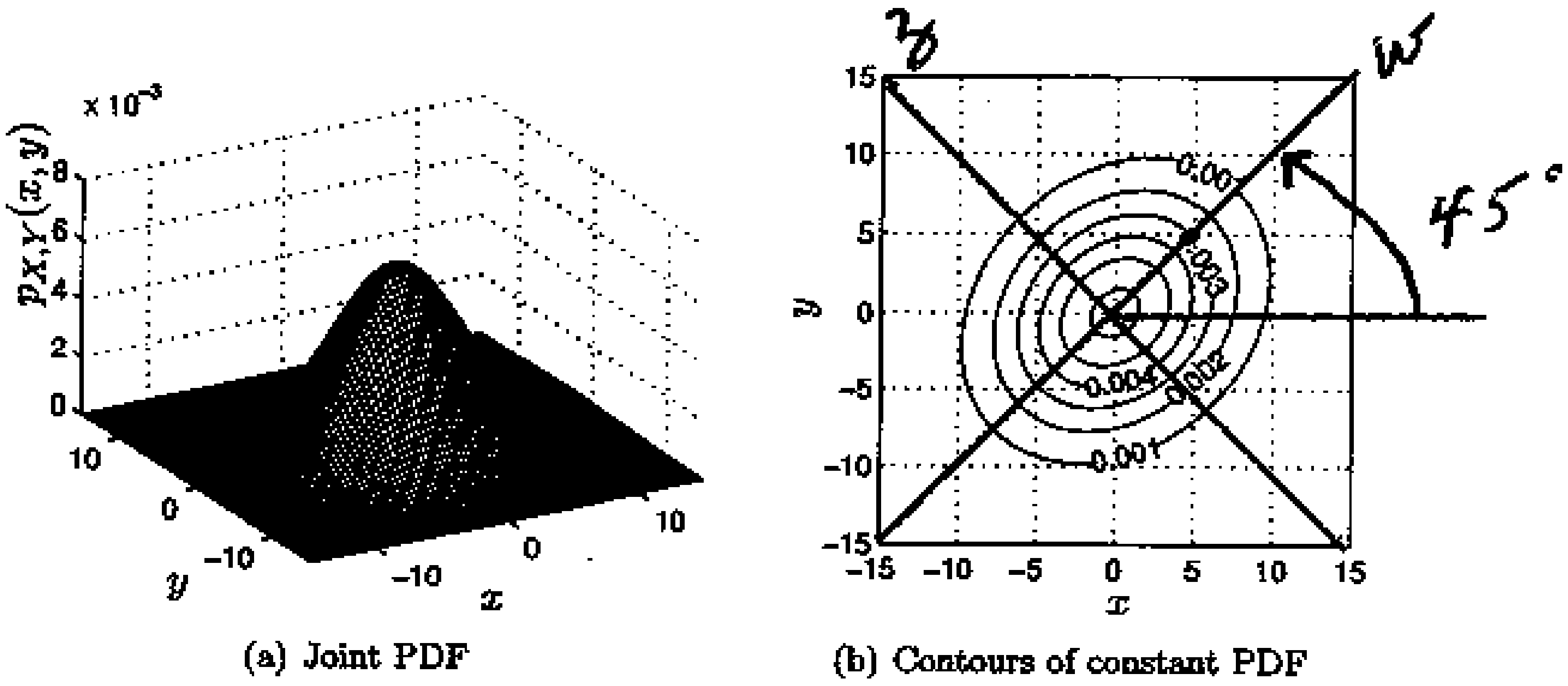


Figure 12.18: Example of joint PDF for correlated Gaussian random variables.

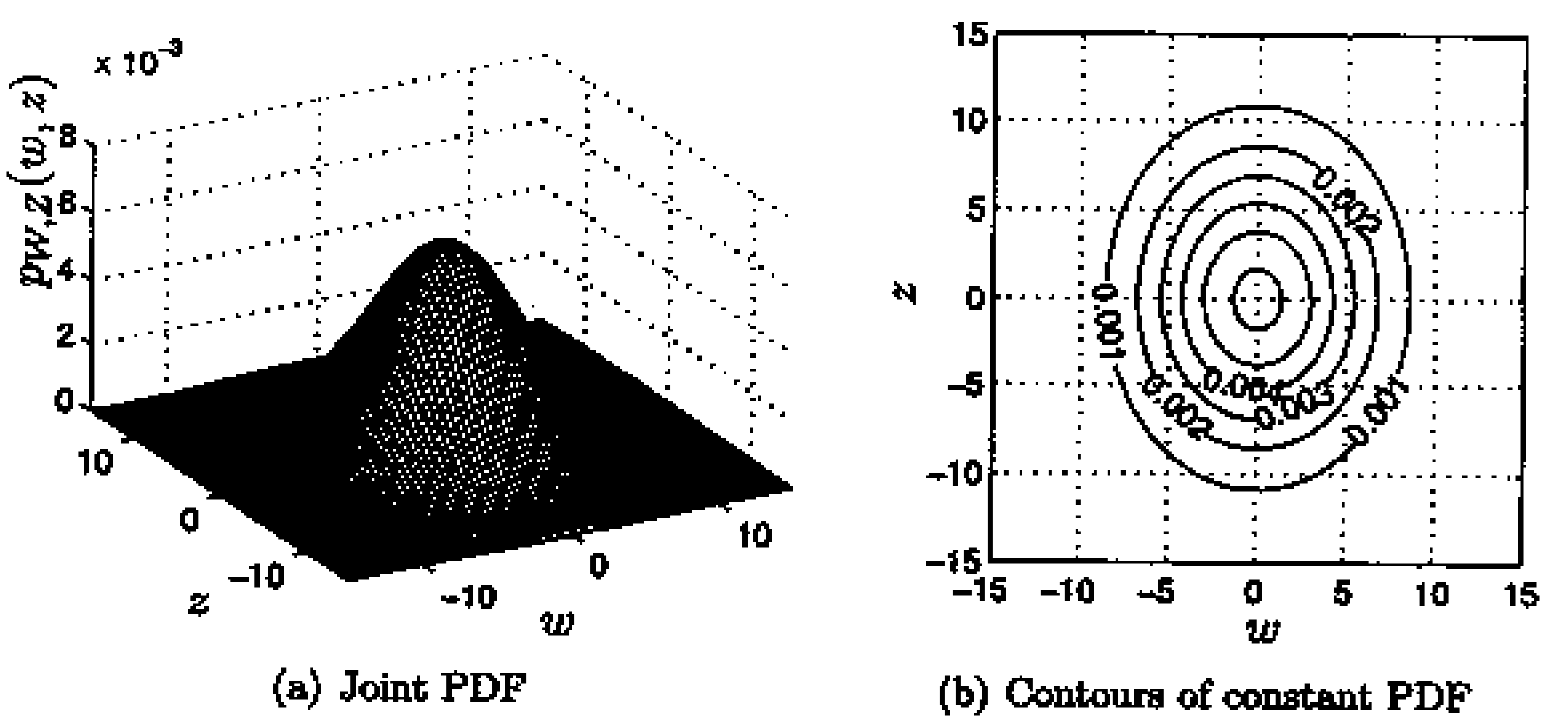


Figure 12.19: Example of joint PDF for transformed correlated Gaussian random variables. The random variables are now uncorrelated and hence independent.

COMPUTER GENERATION OF MULTIVARIATE GAUSSIAN RVs

IN MATLAB $X \sim N(0,1)$ USE randn(1,1)

$Y \sim N(0,1)$ " "

SUPPOSEDLY X, Y ARE INDEPENDENT (MATHWORKS SAYS SO).

FORM $\begin{bmatrix} W \\ Z \end{bmatrix} = \underline{G} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

LET $a = \mu_W$, $b = \mu_Z$

SINCE $\underline{C}_{X,Y} = \begin{bmatrix} VAR(X) & COV(X,Y) \\ COV(Y,X) & VAR(Y) \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I}$$

$$\underbrace{\underline{C}_{W,Z}}_{\text{GIVEN}} = \underline{G}\,\underline{C}_{X,Y}\,\underline{G}^T = \underbrace{\underline{G}\,\underline{G}^T}_{\text{FIND THIS}}$$

INFINITE NUMBER OF $\underline{G}$'S CAN BE FOUND. ASSUME LOWER TRIANGULAR OR

$$\underline{G} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \Rightarrow \underline{G}\,\underline{G}^T = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

$$\underline{C}_{W,Z} = \begin{bmatrix} \sigma_W^2 & \rho\sigma_W\sigma_Z \\ \rho\sigma_W\sigma_Z & \sigma_Z^2 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

$$\Rightarrow \quad a = \sigma_W \qquad b = \rho\sigma_Z \qquad c = \sigma_Z\sqrt{1-\rho^2}$$

OR $\underline{G} = \begin{bmatrix} \sigma_W & 0 \\ \rho\sigma_Z & \sigma_Z\sqrt{1-\rho^2} \end{bmatrix}$

FINALLY, $\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} \sigma_W & 0 \\ \rho\sigma_Z & \sigma_Z\sqrt{1-\rho^2} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{pmatrix} \mu_W \\ \mu_Z \end{pmatrix}$

<u>EXAMPLE</u>: WANT $\mu_W = \mu_Z = 1$

$$\underline{C}_{W,Z} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$$