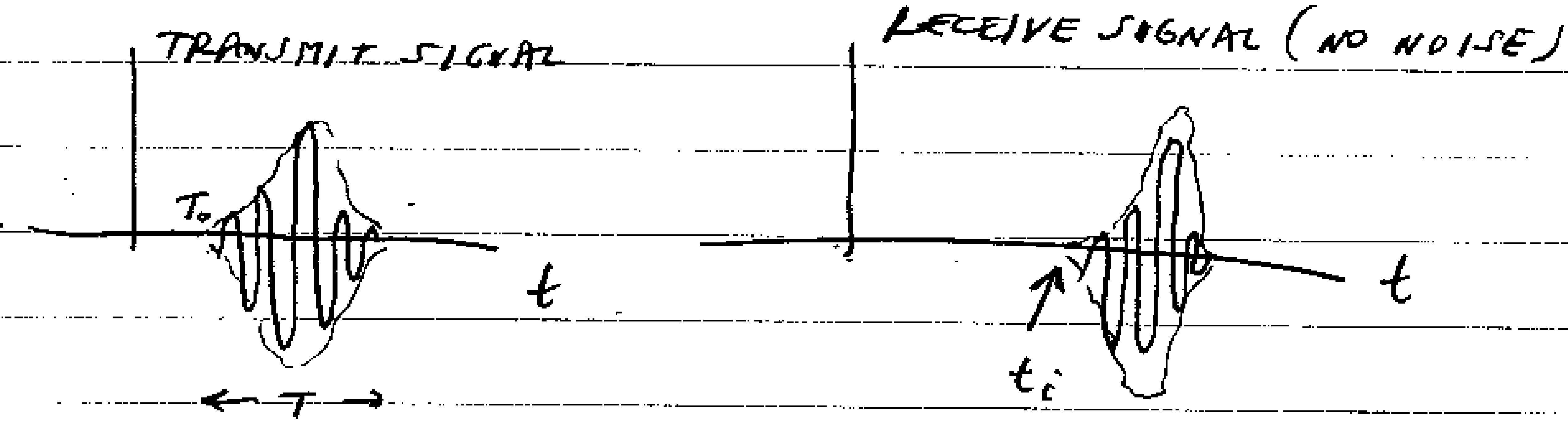
WE KNOW THAT SOURCE IS NEAR NOMINAL POSITION (FROM LAST ESTIMATE MAYBE).

MEASUREMENTS ARE ; t_i $i = 0, 1, \ldots, N-1$ OR ARRIVAL TIMES OF SIGNAL. FOR SIGNAL EMITTED AT $t = T_0$ WE MEASURE

$$t_i = T_0 + \frac{R_i}{c} + \underset{\uparrow \text{ NOISE}}{\epsilon_i}$$

ASSUME ϵ_i's ARE ZERO MEAN, UNCORRELATED WITH VARIANCE σ^2. IN PRACTICE, WE DO NOT KNOW PDF OF ϵ_i's - DEPENDS ON HOW t_i's OBTAINED



HOW COULD WE FIND t_i?

TO ESTIMATE NEW POSITION NEED $\theta = \begin{bmatrix} \delta x_s \\ \delta y_s \end{bmatrix}$
MUST RELATE MEASURED DATA TO θ.

$$R_i = \sqrt{(x_s - x_i)^2 + (y_s - y_i)^2}$$

LOCATION OF i^{th} ANTENNA

$$t_i = T_0 + \frac{\sqrt{(x_s - x_i)^2 + (y_s - y_i)^2}}{c} + \epsilon_i$$

NONLINEAR ☹

FOR $\delta x_s, \delta y_s$ SMALL WE CAN LINEARIZE ABOUT NOMINAL POSITION OR

$$R_i \simeq R_{n_i} + \frac{x_n - x_i}{R_{n_i}} \delta x_s + \frac{y_n - y_i}{R_{n_i}} \delta y_s$$

(TAYLOR EXPANSION OF $R_i(x_s, y_s)$ ABOUT (x_n, y_n))

BUT $\dfrac{x_n - x_i}{R_{n_i}} = \cos \alpha_i$

$\dfrac{y_n - y_i}{R_{n_i}} = \sin \alpha_i$

$$t_i = \underbrace{T_0}_{\text{UNKNOWN}} + \underbrace{\frac{R_{n_i}}{c}}_{\text{KNOWN}} + \underbrace{\frac{\cos\alpha_i}{c}}_{\text{KNOWN}} \delta x_s + \underbrace{\frac{\sin\alpha_i}{c}}_{\text{KNOWN}} \delta y_s + \epsilon_i$$

LET $\tau_i = t_i - \dfrac{R_{n_i}}{c}$ (NEW MEASUREMENTS)

$$\tau_i = T_0 + \frac{\cos\alpha_i}{c} \delta x_s + \frac{\sin\alpha_i}{c} \delta y_s + \epsilon_i$$

TO GET RID OF T_0 USE TIME DIFFERENCE OF ARRIVAL MEASUREMENTS (TDOA'S) OR

$$\zeta_1 = \tau_1 - \tau_0$$
$$\zeta_2 = \tau_2 - \tau_1$$
$$\vdots$$
$$\zeta_{N-1} = \tau_{N-1} - \tau_{N-2}$$

FINALLY, OUR MODEL IS

$$\zeta_i = \frac{1}{c}(\cos\alpha_i - \cos\alpha_{i-1})\,\delta x_s + \frac{1}{c}(\sin\alpha_i - \sin\alpha_{i-1})\,\delta y_s + \epsilon_i - \epsilon_{i-1}$$

FOR $i = 1, 2, \ldots, N-1$

NOW WE HAVE OUR LINEAR MODEL WITH

$$\underline{\theta} = [\delta x_s \;\; \delta y_s]^T$$

$$\underline{H} = \frac{1}{c}\underbrace{\begin{bmatrix} \cos\alpha_1 - \cos\alpha_0 & \sin\alpha_1 - \sin\alpha_0 \\ \vdots & \vdots \\ \cos\alpha_{N-1} - \cos\alpha_{N-2} & \sin\alpha_{N-1} - \sin\alpha_{N-2} \end{bmatrix}}_{(N-1)\times 2}$$

$$\underline{w} = \begin{bmatrix} \epsilon_1 - \epsilon_0 \\ \vdots \\ \epsilon_{N-1} - \epsilon_{N-2} \end{bmatrix}$$

NOTE THAT THE NOISE IS NON COLORED DUE TO THE DIFFERENCING.

TO FIND COVARIANCE

$$\underline{W} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \vdots \\ \epsilon_{N-1} \end{bmatrix}}_{\underline{\epsilon}}$$

$$\underline{C} = E\left(\underline{A}\underline{\epsilon}(\underline{A}\underline{\epsilon})^T \right) = \underline{A}\,\sigma^2 \underline{I}\,\underline{A}^T = \sigma^2 \underline{A}\underline{A}^T$$

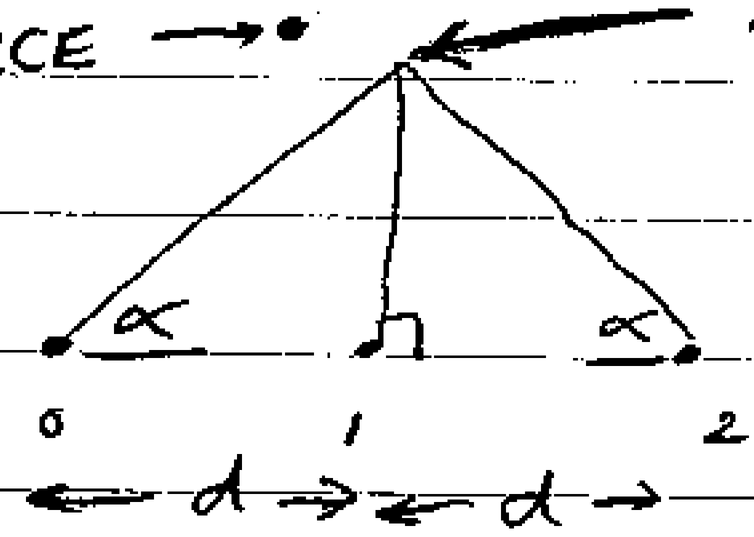
$$\therefore \quad \hat{\underline{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1} \underline{H}^T \underline{C}^{-1} \underline{\xi}$$

$$= \left(\underline{H}^T (\underline{A}\underline{A}^T)^{-1} \underline{H} \right)^{-1} \underline{H}^T (\underline{A}\underline{A}^T)^{-1} \underline{\xi}$$

$$VAR(\hat{\theta}_i) = \sigma^2 \left((\underline{H}^T(\underline{A}\underline{A}^T)^{-1}\underline{H})^{-1} \right)_{ii}$$

$$\underline{C}_{\hat{\theta}} = \sigma^2 \left[\underline{H}^T (\underline{A}\underline{A}^T)^{-1} \underline{H} \right]^{-1}$$

SOURCE → • ← NOMINAL POSITION



$N = 3$ (MINIMUM NUMBER)

$$\underline{H} = \frac{1}{c} \begin{bmatrix} -\cos\alpha & 1-\sin\alpha \\ -\cos\alpha & -(1-\sin\alpha) \end{bmatrix}$$

$\alpha_0 = \alpha$
$\alpha_1 = 90°$
$\alpha_2 = 180 - \alpha$

$$\underline{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{C}_{\hat{\theta}} = \sigma^2 c^2 \begin{bmatrix} \frac{1}{2\cos^2\alpha} & 0 \\ 0 & \frac{3/2}{(1-\sin\alpha)^2} \end{bmatrix}$$

FOR SMALL VARIANCES α SHOULD BE CLOSE TO ZERO. $\Rightarrow$ INCREASE d OR BASELINE OF ARRAY SHOULD BE LARGE.

ALSO, BEST ACCURACY FOR SHORT RANGES.

CHAPTER 7 - MAXIMUM LIKELIHOOD ESTIMATION

THIS IS THE "TURN THE CRANK" PROCEDURE. USED IN ALMOST ALL PRACTICAL SYSTEMS. APPROXIMATELY MVU ESTIMATOR (ACTUALLY $\approx$ EFFICIENT) FOR LARGE DATA RECORDS.

<u>EXAMPLE</u>: DC LEVEL IN WGN - MODIFIED

$$x[n] = \underbrace{A}_{\text{UNKNOWN}} + \underbrace{w[n]}_{\text{WGN WITH VARIANCE } \underline{\underline{A}}\ (A>0)}$$

APPROACHES:

1) CRLB
2) RBLS

1) $$p(\underline{x};A) = \frac{1}{(2\pi A)^{N/2}} e^{-\frac{1}{2A}\sum_{n=0}^{N-1}(x[n]-A)^2}$$

$$\frac{\partial \ln p}{\partial A} = -\frac{N}{2A} + \frac{1}{A}\sum_n (x[n]-A) + \frac{1}{2A^2}\sum_n (x[n]-A)^2$$

$$\overset{?}{=} I(A)(\hat{A}-A)$$

DOESN'T APPEAR AS IF CRLB ATTAINED.

CRLB IS $\mathrm{VAR}(\hat{A}) \geq \dfrac{A^2}{N(A+\frac{1}{2})}$

2) TRY TO FACTOR PDF

$$\frac{1}{A}\sum_n (x[n]-A)^2 = \frac{1}{A}\sum_n x^2[n] - 2N\bar{x} + NA$$

$$p(\underline{x};A) = \underbrace{\frac{1}{(2\pi A)^{N/2}} e^{-\frac{1}{2}\left[\frac{1}{A}\sum_n x^2[n] + NA\right)}}_{g\left(\sum_n x^2[n],\ A\right)} \underbrace{e^{N\bar{x}}}_{h(\underline{x})}$$

$\Rightarrow$ BY NEYMAN-FISHER, $\sum_{n=0}^{N-1} x^2[n]$ IS SS FOR A.

ASSUME SS IS COMPLETE AND TRY TO FIND FUNCTION TO MAKE IT UNBIASED.

$$E\left[h\left(\sum_n x^2[n]\right)\right] = A \qquad A > 0$$

BUT $E\left(\sum_n x^2[n]\right) = N E(x^2[n])$

$$= N\left(\mathrm{VAR}(x[n]) + E^2(x[n])\right)$$

$$= N(A+A^2)$$

HOW DO WE FIND h ?

ALTERNATIVELY, SINCE $x[0]$ IS UNBIASED ESTIMATOR OF A TRY

$$\hat{A} = E\left(x[0] \,\Big|\, \sum_{n=0}^{N-1} x^2[n]\right)$$

TOO DIFFICULT MATHEMATICALLY.

WE PROPOSE THE FOLLOWING: FIND $\hat{A}$ SO THAT $\hat{A}$ IS <u>ASYMPTOTICALLY</u> OPTIMAL OR AS $N \to \infty$

$$E(\hat{A}) \to A$$
$$\text{VAR}(\hat{A}) \to \text{CRLB}$$

$\Rightarrow$ ASYMPTOTICALLY EFFICIENT $\Rightarrow$ ASYMPTOTICALLY MVU

CONSIDER $\hat{A} = -\frac{1}{2} + \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] + 1/4}$

BIASED SINCE

$$E(\hat{A}) = E\left(-\frac{1}{2} + \sqrt{\frac{1}{N}\sum x^2[n] + 1/4}\right)$$
$$\neq -\frac{1}{2} + \sqrt{E\left(\frac{1}{N}\sum x^2[n]\right) + 1/4}$$
$$= -\frac{1}{2} + \sqrt{E(x^2[n]) + 1/4}$$

$$= -\tfrac{1}{2} + \sqrt{A + A^2 + 1/4} = -\tfrac{1}{2} + (A + 1/2) = A$$

REASONABLE IN THAT AS $N \to \infty$

$$\frac{1}{N} \sum x^2(n) \to E(x^2(n)) = A + A^2$$

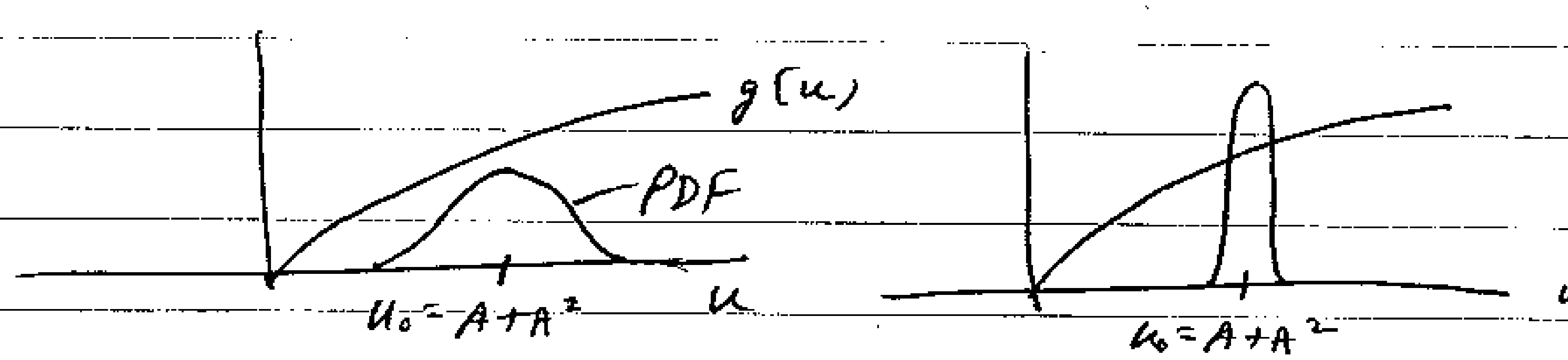
$$\Rightarrow \hat{A} \to A$$ BY LAW OF LARGE NUMBERS

$\hat{A}$ IS SAID TO BE CONSISTENT (AT LEAST FOR LARGE DATA RECORDS ESTIMATOR YIELDS TRUE VALUE)

TO FIND MEAN AND VARIANCE APPROXIMATELY (AS $N \to \infty$) WE CAN LINEARIZE ESTIMATOR.

$$\hat{A} = g(u) \qquad u = 1/N \sum_{n=0}^{N-1} x^2(n)$$

$$g(u) = -\tfrac{1}{2} + \sqrt{u + 1/4}$$



SMALL N　　　　LARGE N

LINEARIZING $g(u)$ ABOUT $u = u_0 = E(x^2(n))$

$$g(u) \approx g(u_0) + \frac{dg(u_0)}{du_0}(u-u_0)$$

$$\hat{A} = A + \frac{\frac{1}{2}}{A+\frac{1}{2}}\left[\frac{1}{N}\sum_n x^2[n] - (A+A^2)\right]$$

VALID AS $N \to \infty$.

$E(\hat{A}) = A \Rightarrow$ ASYMPTOTICALLY UNBIASED

$$\mathrm{VAR}(\hat{A}) = \left(\frac{\frac{1}{2}}{A+\frac{1}{2}}\right)^2 \mathrm{VAR}\left(\frac{1}{N}\sum_n x^2[n]\right)$$

$$= \frac{1/4}{N(A+1/2)^2}\mathrm{VAR}(x^2[n])$$

BUT $\mathrm{VAR}(x^2[n]) = E(x^4[n]) - E^2(x^2[n])$

$$= E\left((A+w[n])^4\right) - \underbrace{E^2(x^2[n])}_{(A+A^2)^2}$$

$$E((A+w[n])^4) = E\left(\sum_{k=0}^{4}\binom{4}{k}A^k w[n]^{4-k}\right)$$

$$= E(w^4[n]) + \binom{4}{2}A^2E(w^2[n]) + A^4$$

(ODD ORDER MOMENTS = 0)

$$= 3A^2 + 6A^3 + A^4$$

$$\mathrm{VAR}(x^2[n]) = 3A^2 + 6A^3 + A^4 - A^2 - 2A^3 - A^4$$
$$= 4A^3 + 2A^2$$

$$\mathrm{VAR}(\hat{A}) = \frac{1/4}{N(A+\frac{1}{2})^2} \, 4A^2(A+\tfrac{1}{2})$$

$$= \frac{A^2}{N(A+\frac{1}{2})} = \mathrm{CRLB} \; !$$

$\therefore$ $\hat{A}$ IS <u>ASYMPTOTICALLY</u> EFFICIENT

FOR FINITE DATA RECORDS CANNOT SAY MUCH - IN PRACTICE, WORKS WELL.

ALSO, AS $N \to \infty$

$$\hat{A} = A + \frac{\frac{1}{2}}{A+\frac{1}{2}}\left[\underbrace{\frac{1}{N}\sum_n x^2[n]}_{\substack{\text{BY CENTRAL LIMIT}\\ \text{THEOREM} \approx \text{GAUSSIAN}}} - (A+A^2)\right]$$

$\hat{A}$ = LINEAR FUNCTION OF $\frac{1}{N}\sum_n x^2[n]$

$\Rightarrow$ $\hat{A} \approx$ GAUSSIAN

$\hat{A}$ IS MAXIMUM LIKELIHOOD ESTIMATOR (MLE).

<u>PROPERTIES</u> : 1) CONSISTENT

2) ASYMPTOTICALLY EFFICIENT
3) " GAUSSIAN

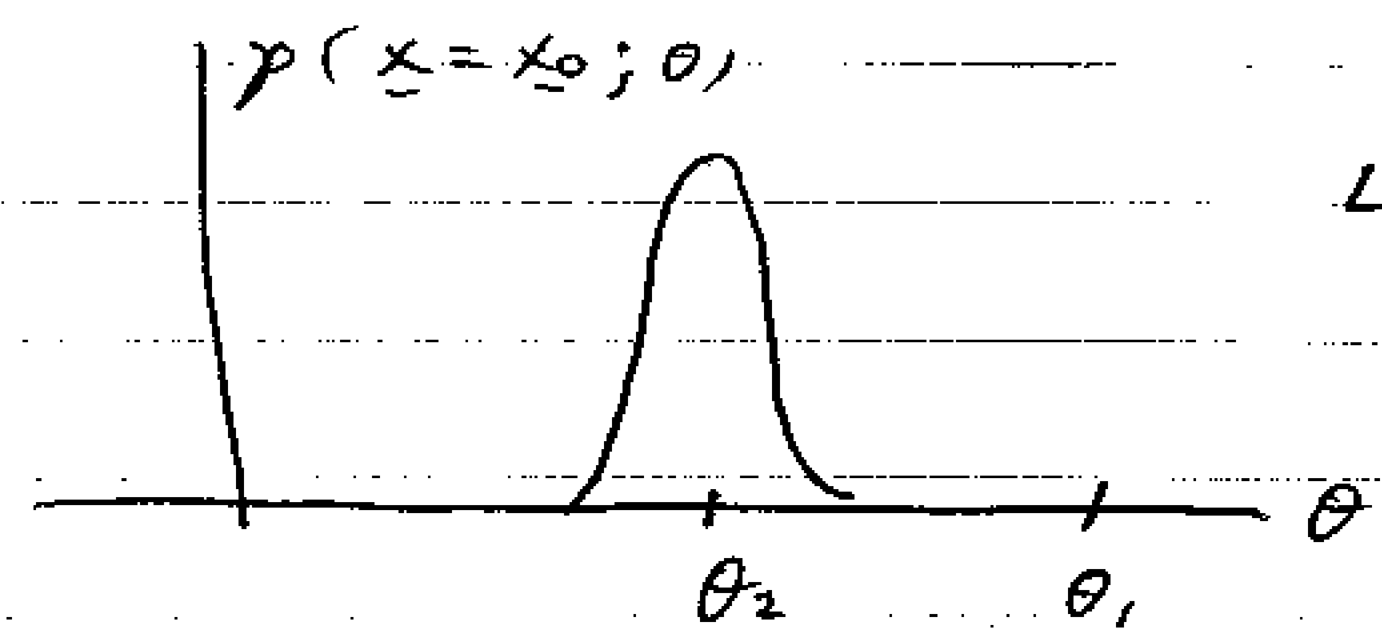
$\Rightarrow$ ASYMPTOTICALLY MVU

$\Rightarrow$ ASYMPTOTICALLY <u>OPTIMAL</u>.

FINDING MLE

DEFINITION: MLE OF θ IS THAT VALUE MAXIMIZING $p(\underline{x};\theta)$ OVER ALLOWABLE RANGE OF θ.

RATIONALE: IF $\underline{x} = \underline{x}_0$ OBSERVED



IF $\theta = \theta_1$, PROBABILITY OF OBSERVING $\underline{x} = \underline{x}_0$ IS SMALL. BUT WE DID OBSERVE $\underline{x} = \underline{x}_0 \Rightarrow$ PROBABILITY MUST HAVE BEEN LARGE $\Rightarrow$ $\theta = \theta_2$ IS MORE REASONABLE. OUR ESTIMATE SHOULD BE $\hat{\theta} = \theta_2 = \arg\max_{\theta} p(\underline{x};\theta)$.

$p(\underline{x};\theta)$ FOR GIVEN θ = PDF OF $\underline{x}$

$p(\underline{x};\theta)$ FOR GIVEN $\underline{x}$ = LIKELIHOOD FUNCTION

EXAMPLE : PREVIOUS EXAMPLE

$$p(\underline{x};A) = \frac{1}{(2\pi A)^{N/2}} e^{-\frac{1}{2A}\sum_n (x[n]-A)^2}$$

MAXIMIZE OVER A

$$\frac{\partial \ln p}{\partial A} = -\frac{N}{2A} + \frac{1}{A}\sum (x[n]-A) + \frac{1}{2A^2}\sum_n (x[n]-A)^2$$

$$= 0$$

$$\Rightarrow \; -NA + 2A\cancel{\sum x[n]} - 2NA^2 + \sum x^2[n]$$
$$- 2A\cancel{\sum x[n]} + NA^2 = 0$$

$$-NA^2 - NA + \sum x^2[n] = 0$$
$$A^2 + A - \frac{1}{N}\sum x^2[n] = 0$$

$$\hat{A} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{N}\sum x^2[n]}$$

SINCE $A > 0$ CHOOSE + SIGN

$$\hat{A} = -\frac{1}{2} + \sqrt{\frac{1}{N}\sum x^2[n] + \frac{1}{4}}$$

SOMETIMES WE ARE LUCKY AND MLE YIELDS AN EFFICIENT ESTIMATOR FOR FINITE DATA RECORDS.

EXAMPLE : DC LEVEL IN WGN

$$p(\underline{x};A) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\left(\sum_n (x[n]-A)^2\right)}$$

$$\frac{\partial \ln p}{\partial A} = \frac{1}{\sigma^2}\sum_n (x[n]-A) = 0$$

$$\Rightarrow \hat{A} = \frac{1}{N}\sum_n x[n]$$

KNOWN TO BE EFFICIENT.

TRUE IN GENERAL, IF EFFICIENT ESTIMATOR EXIST, MLE WILL BE IT.

MLE PROPERTIES

AS $N \to \infty$, $\hat{\theta} \sim N(\theta, I^{-1}(\theta))$

UNBIASED CRLB

GAUSSIAN

OR $\hat{\theta} \overset{a}{\sim} N(\theta, I^{-1}(\theta))$ $a \equiv$ ASYMPTOTICALLY

EXAMPLE : DC LEVEL IN WGN – MODIFIED

HOW LARGE MUST N BE FOR ASYMPTOTIC RESULTS TO APPLY?

USE MONTECARLO COMPUTER SIMULATION

ESTIMATES

$$\widehat{E(\hat{A})} = \frac{1}{M} \sum_{i=1}^{M} \hat{A}_i \qquad M = 1000 \text{ REALIZATIONS}$$

$$\widehat{\text{VAR}(\hat{A})} = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{A}_i - \widehat{E(\hat{A})}\right)^2$$

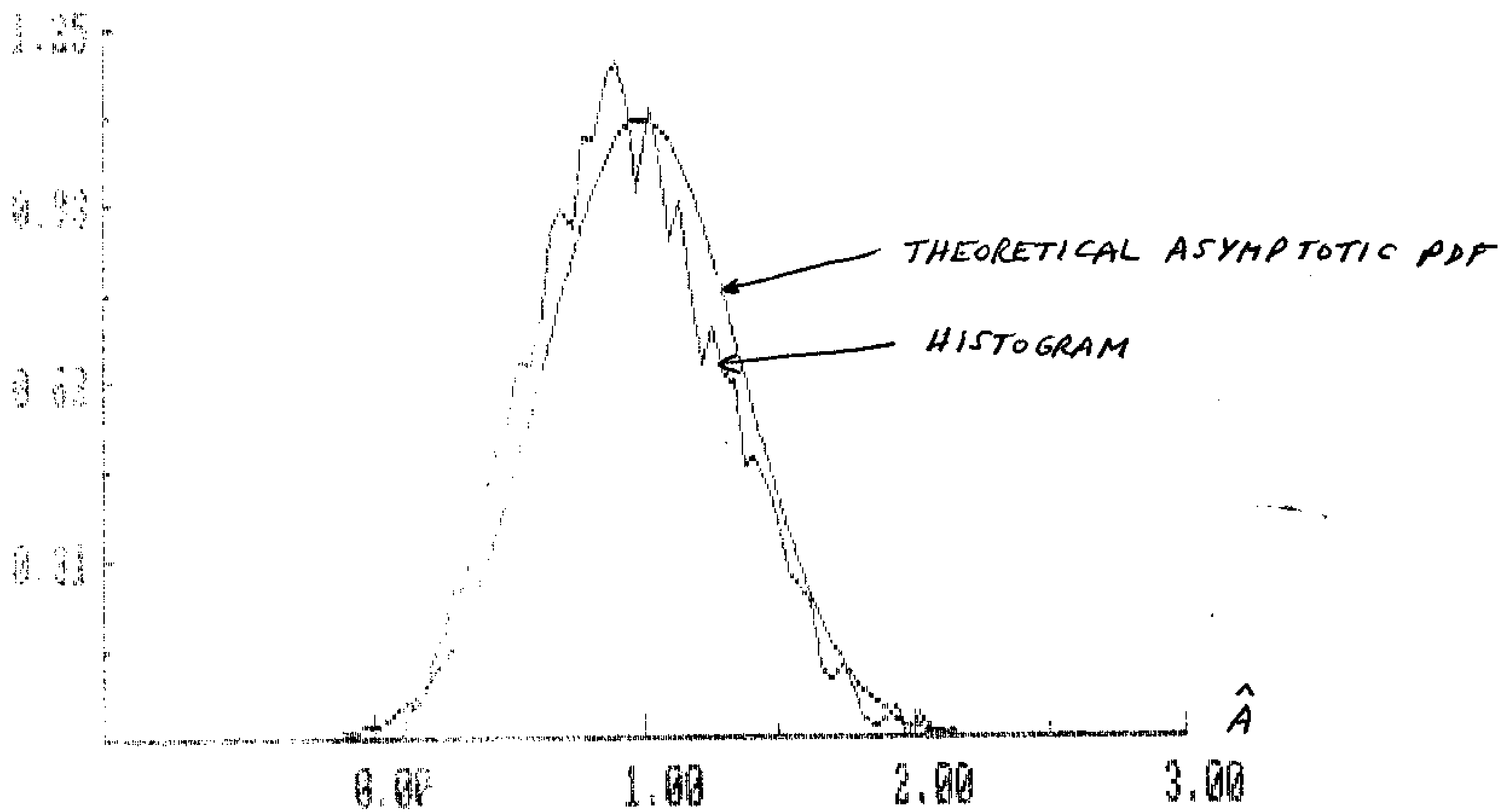
FOR $A = 1$ RESULTS ARE

N	$\widehat{E(\hat{A})}$	$N\,\widehat{\text{VAR}(\hat{A})}$
5	0.954	0.624
10	0.976	0.648
15	0.991	0.696
20	0.996 (0.987)*	0.707 (0.669)*
25	0.994	0.656
THEORETICAL →	1	0.667 $= \frac{A^2}{A + \frac{1}{2}}$

* $M = 5000$

TO ASSESS PDF RECALL

$$\hat{A} \overset{a}{\sim} \mathcal{N}(A, I^{-1}(A)) = \mathcal{N}(1, {}^{2/3}/_{N})$$



(a)

FIGURE 2 - THEORETICAL PDF AND HISTOGRAM

a) $N = 5$

b) $N = 20$

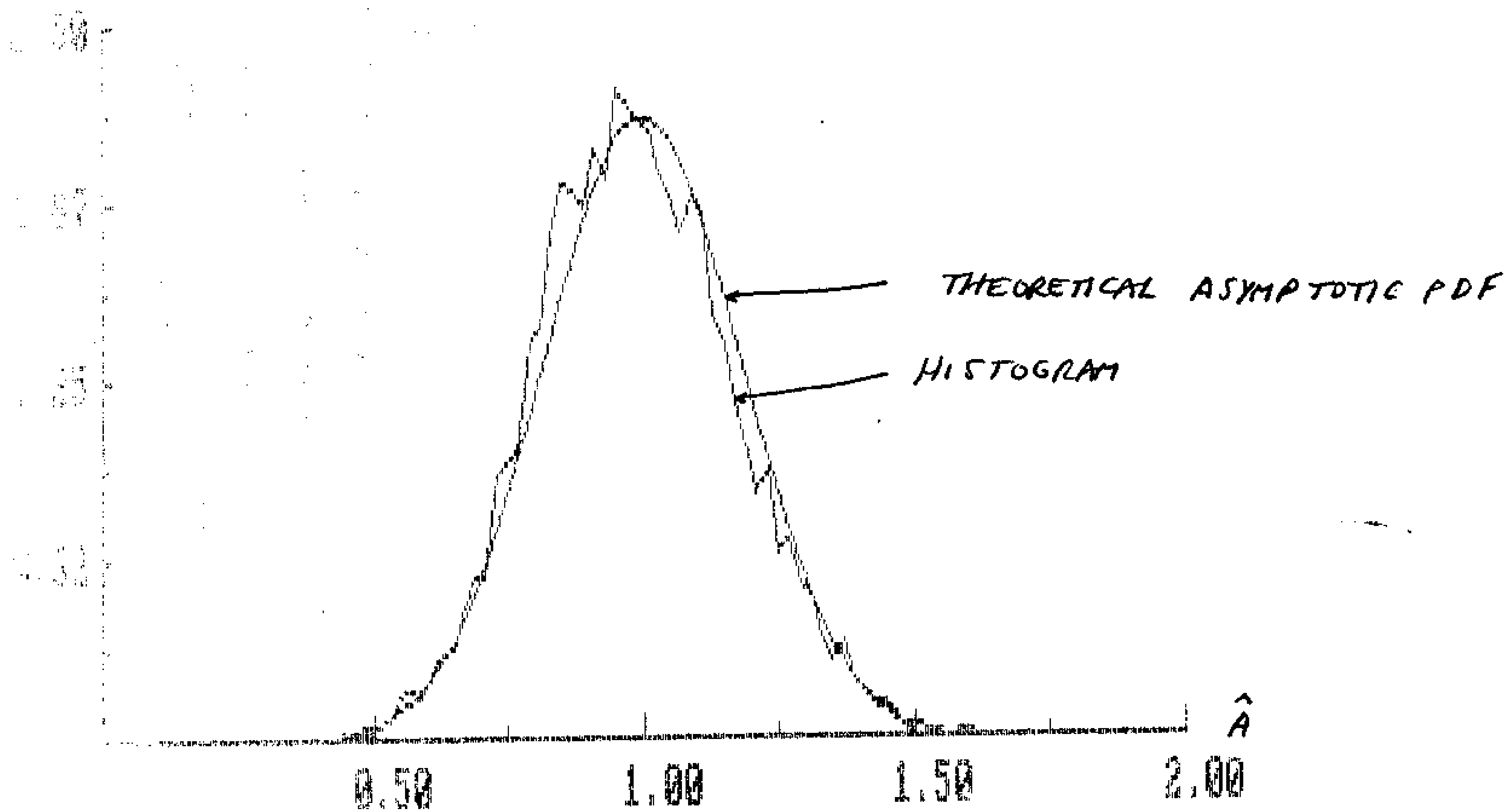


FIGURE 26

SUMMARY

$$\hat{\theta} \overset{a}{\sim} N(\theta, I^{-1}(\theta))$$

$\Rightarrow$ ASYMPTOTICALLY UNBIASED
ASYMPTOTICALLY ATTAINS CRLB

$\Rightarrow$ ASYMPTOTICALLY EFFICIENT $\Rightarrow$
OPTIMAL FOR LARGE DATA RECORDS

USUALLY N DOES NOT HAVE TO BE VERY LARGE

EXAMPLE : PHASE ESTIMATION

$$x[n] = A\cos(2\pi f_0 n + \phi) + \underset{\uparrow \text{ WGN}}{w[n]}$$

A, f_0 KNOWN

RECALL NO SINGLE SS FOR THIS PROBLEM

$$\left.\begin{aligned} T_1(\underline{x}) &= \sum_n x[n]\cos 2\pi f_0 n \\ T_2(\underline{x}) &= \sum_n x[n]\sin 2\pi f_0 n \end{aligned}\right\} \text{SS}$$

TO FIND MLE :

$$p(\underline{x};\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n [x(n) - A\cos(2\pi f_0 n + \phi)]^2}$$

MUST MINIMIZE

$$J(\phi) = \sum_n [x(n) - A\cos(2\pi f_0 n + \phi)]^2$$

$$\frac{\partial J}{\partial \phi} = 2 \sum_n [x(n) - A\cos(2\pi f_0 n + \phi)] A\sin(2\pi f_0 n + \phi)$$

$$= 0$$

$$\Rightarrow \quad \sum_n x(n)\sin(2\pi f_0 n + \hat{\phi}) = A \underbrace{\sum_n \sin(2\pi f_0 n + \hat{\phi}) \cdot \cos(2\pi f_0 n + \hat{\phi})}$$

$$\sum_n \sin(2\pi f_0 n + \hat{\phi}) \cos(2\pi f_0 n + \hat{\phi})$$

$$= \frac{1}{2} \sum_n \sin(4\pi f_0 n + 2\hat{\phi}) << N \quad \Rightarrow \text{IGNORE}$$

$$\Rightarrow \quad \sum_n x(n)\sin(2\pi f_0 n + \hat{\phi}) = 0$$

OR

$$\sum x(n)\sin(2\pi f_0 n)\cos\hat{\phi} = -\sum_n x(n)\cos(2\pi f_0 n) \cdot \sin\hat{\phi}$$

$$\therefore \quad \hat{\phi} = -\text{ARCTAN} \frac{\sum_n x(n)\sin 2\pi f_0 n}{\sum_n x(n)\cos 2\pi f_0 n}$$

NOTE THAT $\hat{\phi} = g(T_1, T_2)$ = FUNCTION OF SS.
THIS WILL ALWAYS BE THE CASE. IF SS'S EXIST, MLE WILL BE A FUNCTION OF THEM. WHY?

TO ASSESS PERFORMANCE OF MLE:

$$\hat{\phi} \overset{a}{\sim} N(\phi, I^{-1}(\phi))$$

RECALL THAT $I(\phi) = NA^2/2\sigma^2$

$$\rightarrow \mathrm{VAR}(\hat{\phi}) = \frac{1}{NA^2/2\sigma^2} = \frac{1}{N\underset{\uparrow \text{SNR}}{\rho}}$$

HOW LARGE DOES N HAVE TO BE?

FOR $A = 1$, $f_0 = 0.08$, $\phi = \pi/4$, $\sigma^2 = 0.05$

	N	$E(\hat{\phi})$	$N\,\mathrm{VAR}(\hat{\phi})$
MONTE CARLO SIMULATION	20	0.732	0.0978
	40	0.746	0.108
	60	0.774	0.110
	80	0.789	0.099
THEORETICAL →		$\phi = 0.785$	$N\,\mathrm{VAR}(\hat{\phi}) = \frac{1}{\rho} = 0.1$

NEED ABOUT $N \geq 80$

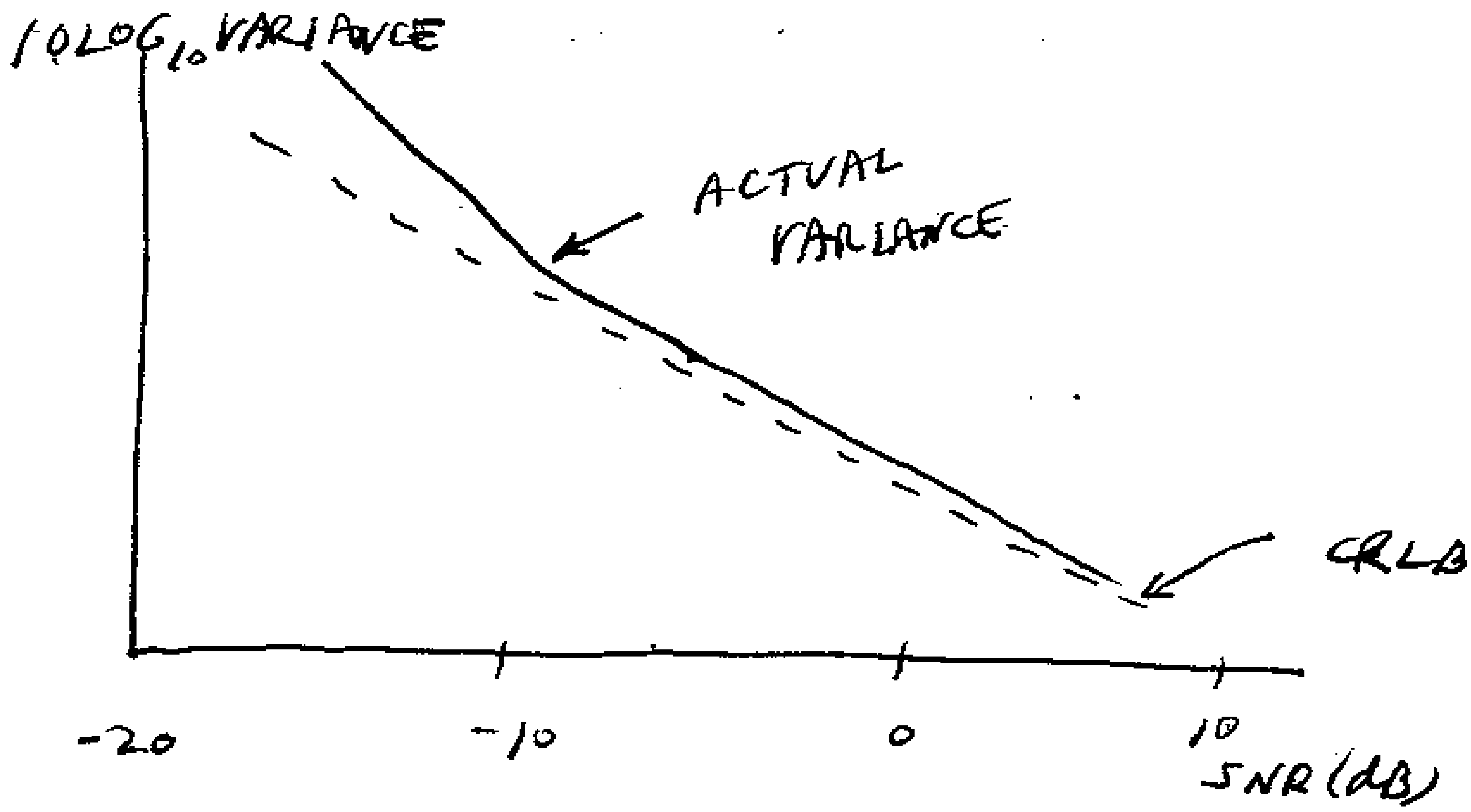
ALSO INTERESTING TO ASSESS PERFORMANCE VS SNR. PLOT $10 \log_{10} \mathrm{VAR}(\hat{\phi})$ VS SNR, AS WELL AS CRLB.

$$\text{CRLB}: \quad 10 \log_{10} \mathrm{VAR}(\hat{\phi}) = 10 \log_{10} \frac{1}{N\rho}$$

$$= -10 \log_{10} N \underbrace{- 10 \log_{10} \rho}_{\text{SNR IN dB}}$$

$\Rightarrow$ CRLB BECOMES LINEAR

SEE FIG. 7.3, MEAN ATTAINS ASYMPTOTIC VALUE FOR SNR ≥ -10 dB.



FOR SNR ≥ -10 dB MLE ATTAINS CRLB OR ASYMPTOTIC VARIANCE