$$\sum_{i=0}^{N-1} (x[i] - s[i]) \left[\frac{\partial \underline{s}(\underline{\theta})}{\partial \underline{\theta}} \right]_{ij} = 0 \qquad j = 1, 2, \ldots, p$$

IN MATRIX FORM

$$\frac{\partial \underline{s}(\underline{\theta})}{\partial \underline{\theta}}^T (\underline{x} - \underline{s}(\underline{\theta})) = \underline{0}$$

WE HAVE p SIMULTANEOUS NONLINEAR EQUATIONS.

USING A NEWTON-RAPHSON ITERATION LET

$$\underline{g}(\underline{\theta}) = \frac{\partial \underline{s}(\underline{\theta})}{\partial \underline{\theta}}^T (\underline{x} - \underline{s}(\underline{\theta}))$$

$$\Rightarrow \quad \underline{\theta}_{k+1} = \underline{\theta}_k - \left(\frac{\partial \underline{g}(\underline{\theta})}{\partial \underline{\theta}} \right)^{-1} \underline{g}(\underline{\theta}) \Big|_{\underline{\theta} = \underline{\theta}_k}$$

CAN BE SHOWN TO REDUCE TO

$$\underline{\theta}_{k+1} = \underline{\theta}_k + \left(\underline{H}^T(\underline{\theta}_k) \underline{H}(\underline{\theta}_k) - \sum_{n=0}^{N-1} \underline{G}_n(\underline{\theta}_k) (x[n] - s[n]) \right)^{-1} \cdot \underline{H}^T(\underline{\theta}_k) (\underline{x} - \underline{s}(\underline{\theta}_k))$$

WHERE $\quad [\underline{H}(\underline{\theta})]_{ij} = \dfrac{\partial s[i]}{\partial \theta_j} \qquad N \times p$

$$[\underline{G}_n(\underline{\theta})]_{ij} = \frac{\partial^2 s[n]}{\partial \theta_i \partial \theta_j} \qquad p \times p$$

IF THE MODEL IS LINEAR OR $\underline{s} = \underline{H}\,\underline{\theta}$

$\Rightarrow$ $\underline{G}_n(\underline{\theta}) = \underline{0}$ AND $\underline{H}(\underline{\theta}) = \underline{H}$.

$$\Rightarrow \quad \underline{\theta}_{k+1} = \underline{\theta}_k + (\underline{H}^T\underline{H})^{-1}\underline{H}^T(\underline{x} - \underbrace{\underline{s}(\underline{\theta}_k)}_{\underline{H}\underline{\theta}_k})$$

$$= (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$$

OR WE ATTAIN CONVERGENCE IN ONE STEP. FOR APPROXIMATELY LINEAR MODELS CONVERGENCE WILL BE RAPID.

SECOND METHOD LINEARIZES SIGNAL MODEL. ASSUME WE WISH TO ESTIMATE θ AND THAT WE KNOW THAT θ IS CLOSE TO θ_0.

$$s(n;\theta) \simeq s(n;\theta_0) + \frac{\partial s[n;\theta]}{\partial \theta_0}(\theta - \theta_0)$$

$$J = \sum_n (x[n] - s[n;\theta])^2$$

$$\simeq \sum_n \left(x[n] - s[n;\theta_0] + \frac{\partial s[n;\theta_0]}{\partial \theta_0}\theta_0 - \frac{\partial s[n;\theta_0]}{\partial \theta_0}\theta\right)^2$$

$$= (\underline{x} - \underline{s}(\underline{\theta}_0) + \underline{H}(\theta_0)\theta_0 - \underline{H}(\theta_0)\theta)^T(\quad)$$

WHERE $[\underline{H}(\theta)]_n = \dfrac{\partial s(n;\theta)}{\partial \theta}$ $\quad n = 0, 1, \ldots, N-1$

BUT $\underline{x} - \underline{s}(\theta_0) + \underline{H}(\theta_0)\theta_0$ IS KNOWN

$$\Rightarrow \quad \theta = \left(\underline{H}^T(\theta_0)\underline{H}(\theta_0)\right)^{-1}\underline{H}^T(\theta_0)\left(\underline{x} - \underline{s}(\theta_0) + \underline{H}(\theta_0)\theta_0\right)$$

$$= \theta_0 + \left(\underline{H}^T(\theta_0)\underline{H}(\theta_0)\right)^{-1}\underline{H}^T(\theta_0)\left(\underline{x} - \underline{s}(\theta_0)\right)$$

OR BY ITERATING

$$\theta_{k+1} = \theta_k + \left(\underline{H}^T(\theta_k)\underline{H}(\theta_k)\right)^{-1}\underline{H}^T(\theta_k)\left(\underline{x} - \underline{s}(\theta_k)\right)$$

THIS IS THE GAUSS METHOD.

IN GENERAL,

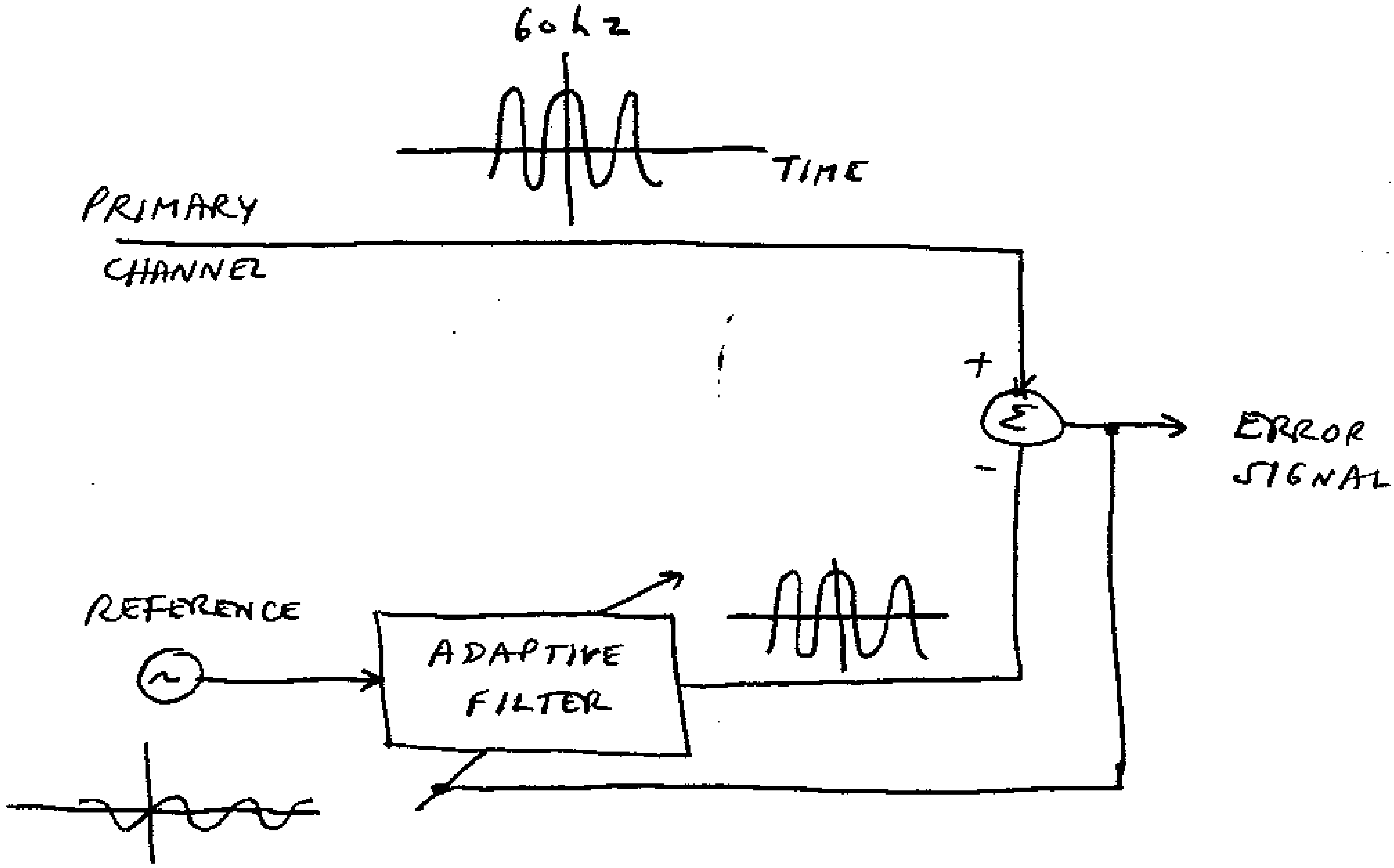
$$\underline{\theta}_{k+1} = \underline{\theta}_k + \left(\underline{H}^T(\underline{\theta}_k)\underline{H}(\underline{\theta}_k)\right)^{-1}\underline{H}^T(\underline{\theta}_k)\left(\underline{x} - \underline{s}(\underline{\theta}_k)\right)$$

WHERE $[\underline{H}(\underline{\theta})]_{ij} = \dfrac{\partial s[i]}{\partial \theta_j}$ $\quad N \times p$

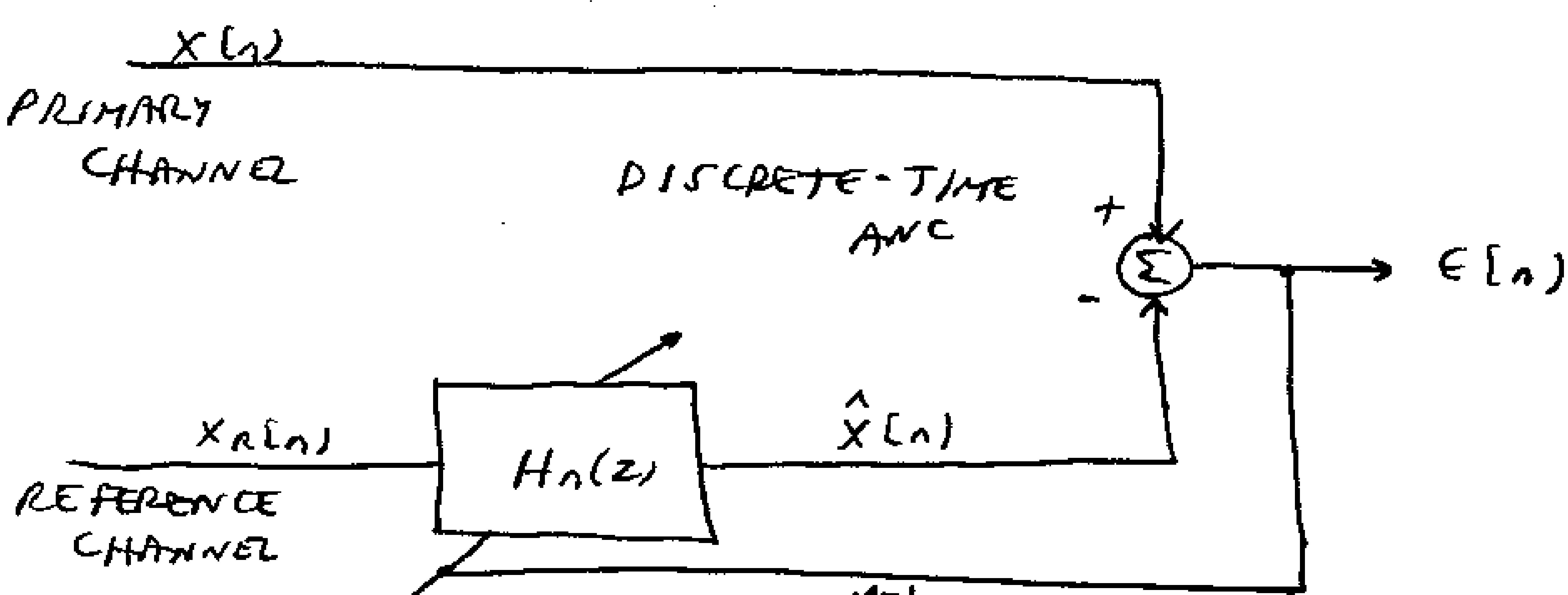
SIGNAL PROCESSING EXAMPLE

ADAPTIVE NOISE CANCELER

USES A REFERENCE SIGNAL TO CANCEL OUT INTERFERENCE. CONSIDER 60 hz INTERFERENCE



AMPLITUDE AND PHASE OF REFERENCE DOES NOT MATCH INTERFERENCE. ADAPTIVE FILTER MODIFIES THESE BEFORE SUBTRACTION.



$$H_n(z) = \sum_{l=0}^{p-1} h_n[l]\, z^{-l}$$

↑ WEIGHTS CHANGE WITH TIME

CHOOSE WEIGHTS AT TIME n SO THAT
$\hat{x}[n] \approx x[n] \Rightarrow \epsilon[n] \approx 0$. TO DO SO
MINIMIZE

PRESENT AND PAST SAMPLES

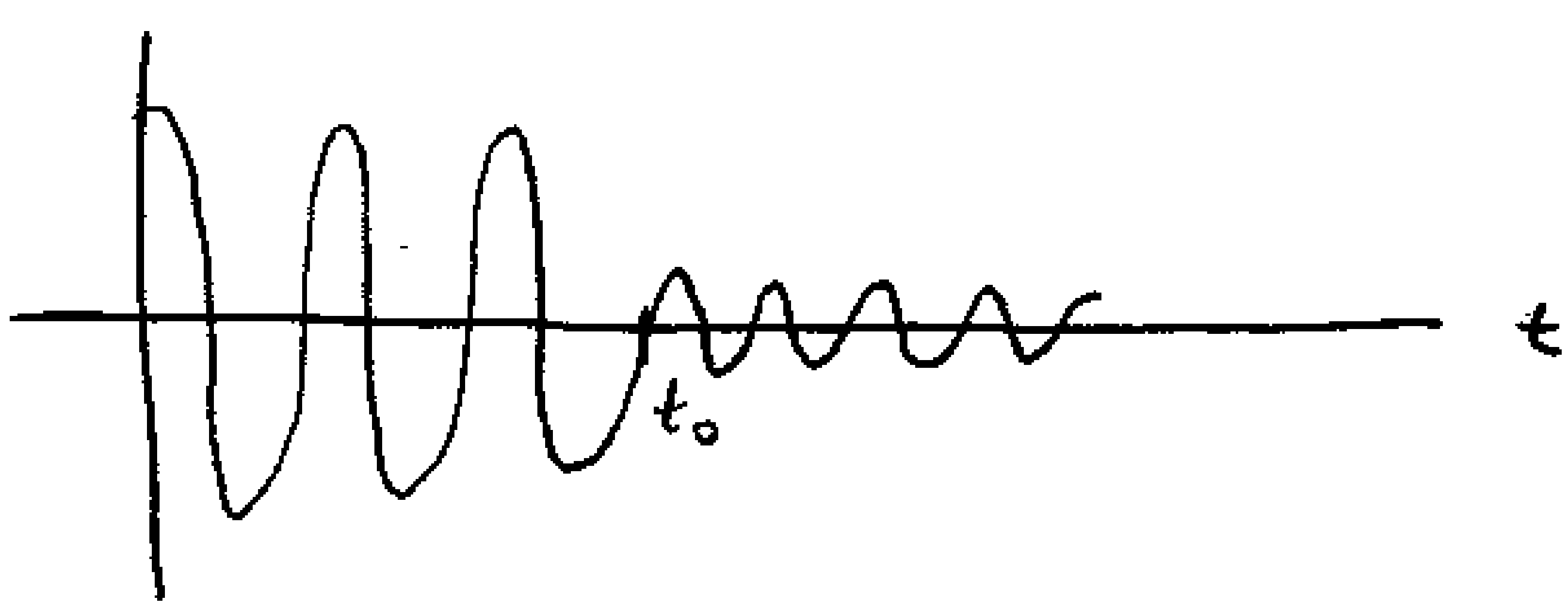
$$J[n] = \sum_{k=0}^{n} \epsilon^2[k]$$

$$= \sum_{k=0}^{n} \left(x[k] - \hat{x}[k] \right)^2$$

$$= \sum_{k=0}^{n} \left(x[k] - \sum_{l=0}^{p-1} h_n[l]\, x_R[k-l] \right)^2$$

OVER $\{ h_n[0], h_n[1], \ldots, h_n[p-1] \}$. $h_n[l]$ IS IMPULSE RESPONSE AT TIME n (TIME-VARYING FILTER). $\Rightarrow$ LEAST SQUARES!

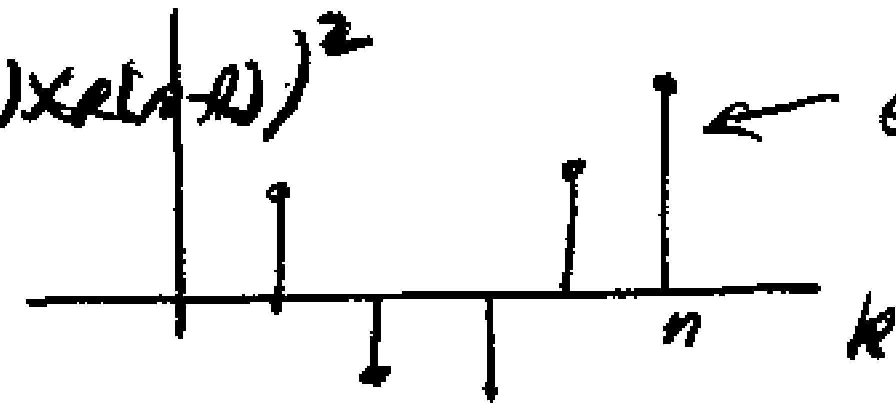
IN PRACTICE, INTERFERENCE MAY BE NONSTATIONARY



TO ALLOW FILTER TO ADAPT QUICKLY WE SHOULD DOWNWEIGHT ERRORS IN $J[n]$ PRIOR TO TIME t_0. OTHERWISE, IF n IS LARGE THE FILTER WILL TAKE A LONG TIME TO CHANGE ITS IMPULSE RESPONSE.

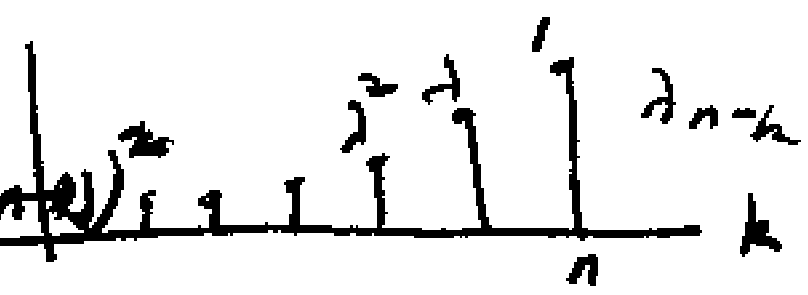
USE A FORGETTING FACTOR λ, $0<\lambda<1$

$$J[n] = \sum_{k=0}^{n} \lambda^{n-k}\left(x[k]-\sum_{n} h_n[l]\,x_R[k-l]\right)^2$$



WE WILL HAVE SAME SOLUTION IF

$$J'[n] = \sum_{k=0}^{n} \frac{1}{\lambda^k}\left(x[k]-\sum_{n} h_n[l]\,x_R[k-l]\right)^2$$



$\Rightarrow$ WEIGHTED LS PROBLEM $\left(\sigma_k^2 = \lambda^k\right)$.

CAN USE SEQUENTIAL LS

$$\hat{\underline{\theta}}(n) = \left[\hat{h}_n[0]\ \hat{h}_n[1]\ \ldots\ \hat{h}_n[p-1]\right]^T$$

$$\underline{h}[n] = \left[x_R[n]\ x_R[n-1]\ \ldots\ x_R[n-(p-1)]\right]^T$$

NEW ROW OF $\underline{H}$ MATRIX

$$\sigma_n^2 = \lambda^n$$

ALSO, PREDICTION ERROR IS

$$e[n] = x[n] - \sum_{l=0}^{p-1} \hat{h}_{n-1}[l]\,x_R[n-l]$$

$$= x[n] - \underline{h}^T[n]\,\hat{\underline{\theta}}[n-1]$$

BASED ON FILTER WEIGHTS AT <u>PREVIOUS</u> TIME (NOT $\epsilon[n]$)

IN SUMMARY, WE HAVE

$$\hat{\underline{\theta}}[n] = \hat{\underline{\theta}}[n-1] + \underline{K}[n]\, e[n]$$

WHERE $$e[n] = x[n] - \sum_{\ell=0}^{p-1} \hat{h}_{n-1}[\ell]\, x_R[n-\ell]$$

$$\underline{K}[n] = \frac{\underline{\Sigma}[n-1]\, \underline{h}[n]}{\lambda^n + \underline{h}^T[n]\, \underline{\Sigma}[n-1]\, \underline{h}[n]}$$

WHERE $$\underline{h}[n] = [\, x_R[n] \;\; x_R[n-1] \;\; \ldots \;\; x_R[n-(p-1)]\,]^T$$

$$\underline{\Sigma}[n] = (\underline{I} - \underline{K}[n]\, \underline{h}^T[n])\, \underline{\Sigma}[n-1]$$

EXAMPLE : $x[n] = 10 \cos(2\pi(0.1)n + \pi/4)$
$x_R[n] = \cos 2\pi(0.1)n$

NEED TWO FILTER COEFFICIENTS
USE $\lambda = 0.99$
INITIALIZE WITH $\hat{\underline{\theta}}[-1] = \underline{0}$
$\underline{\Sigma}[-1] = 10^5\, \underline{I}$

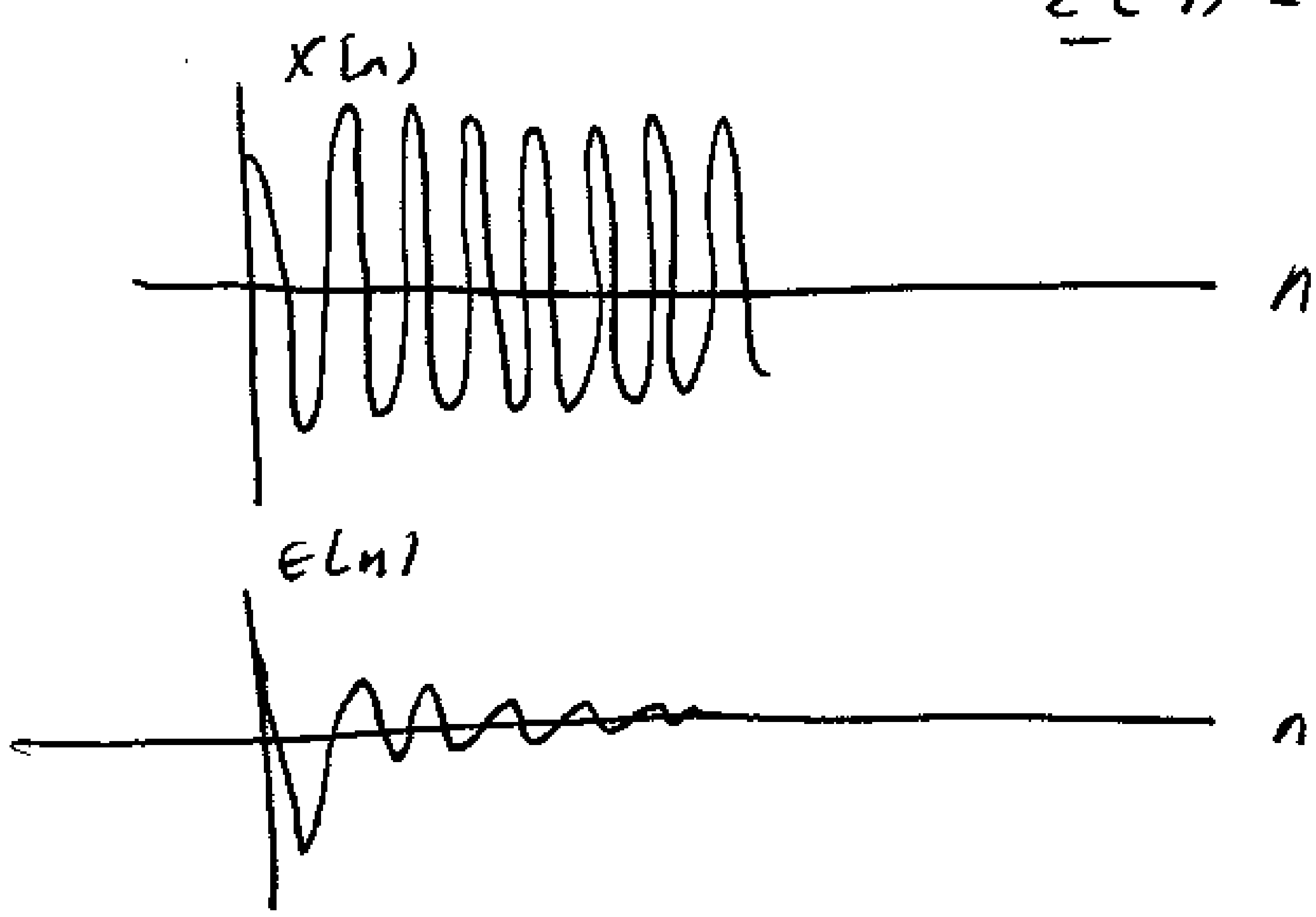
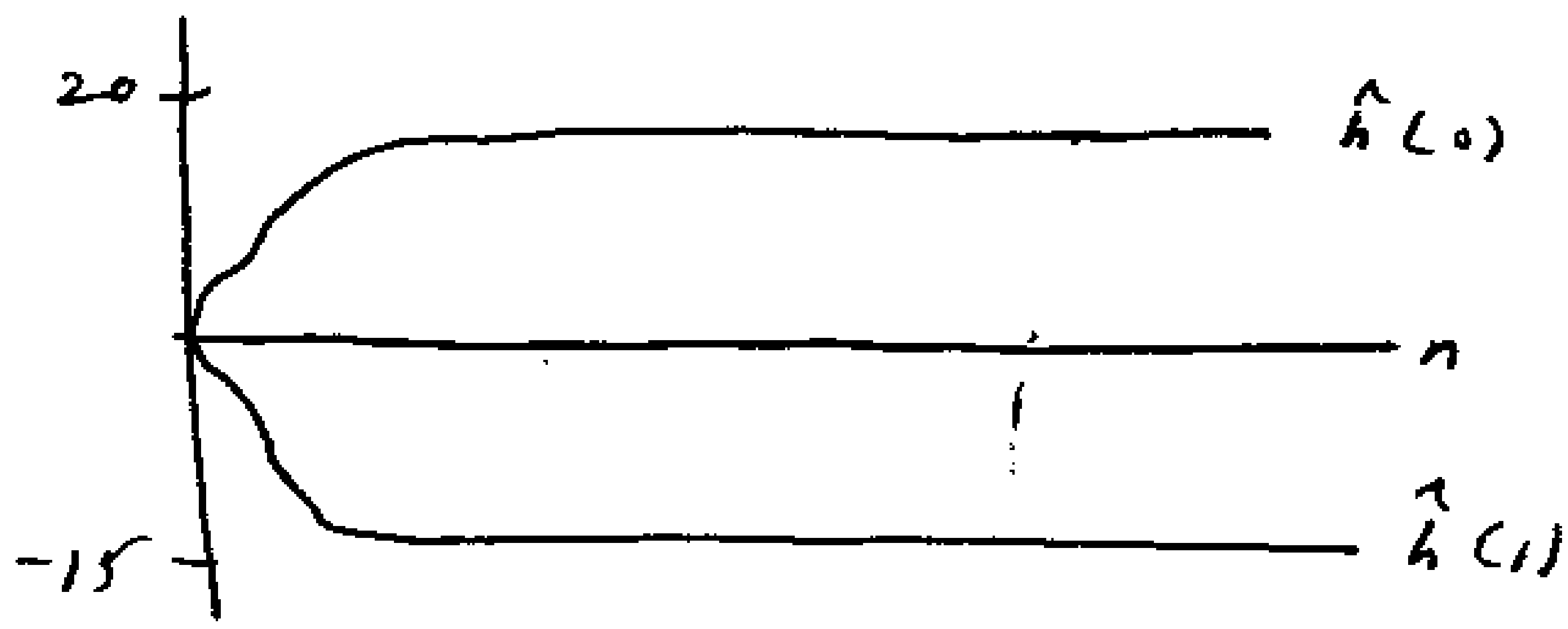


FIG. 8.17



TO FIND STEADY STATE FILTER WEIGHTS NOTE THAT

$$H(e^{j2\pi f}) = 10\, e^{j\pi/4} \quad \text{AT} \quad f = 0.1$$

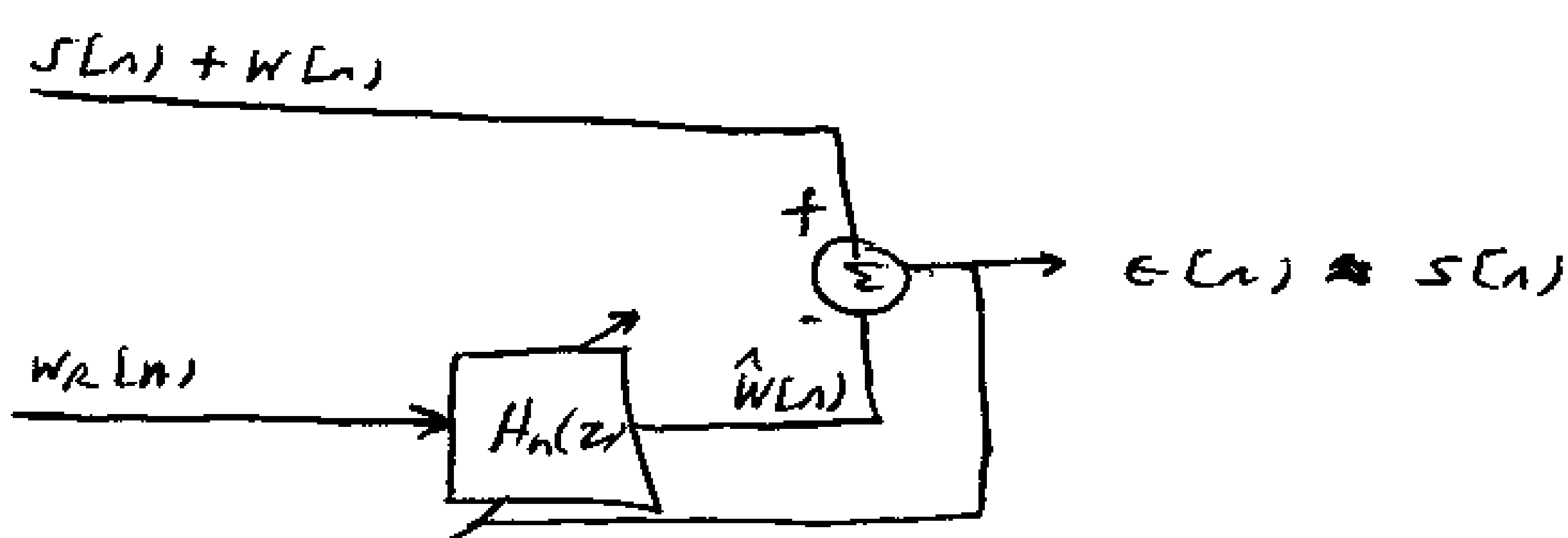
OR ADAPTIVE FILTER MUST INCREASE GAIN BY 10 AND SHIFT PHASE BY $\pi/4$.

$$h[0] + h[1]\, e^{-j2\pi(0.1)} = 10\, e^{j\pi/4}$$

SOLVING $\Rightarrow$ $h[0] = 16.8$

$h[1] = -12.0$

IN PRACTICE, THE DESIRED SIGNAL IN PRIMARY CHANNEL MUST BE UNCORRELATED WITH REFERENCE SIGNAL TO AVOID SIGNAL CANCELATION.



METHOD OF MOMENTS

NO OPTIMALITY PROPERTIES BUT

1) EASY TO IMPLEMENT

2) USUALLY CONSISTENT

EXAMPLE : GAUSSIAN MIXTURE

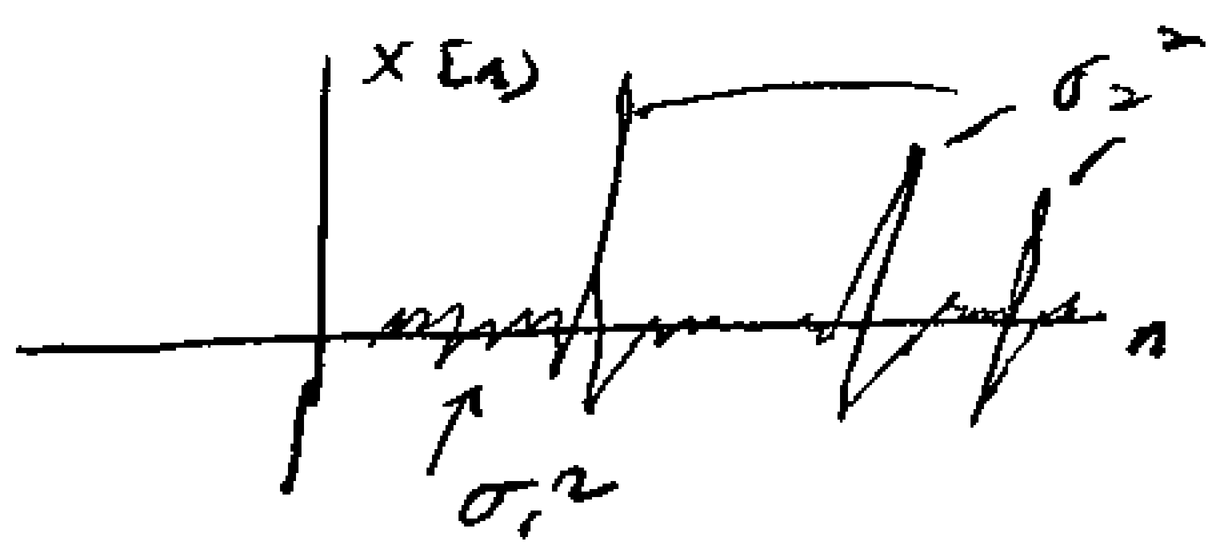
$x[n] \quad n = 0, 1, \ldots, N-1 \quad$ IID

$$p(x[n];\epsilon) = \frac{1-\epsilon}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{x^2[n]}{\sigma_1^2}} + \frac{\epsilon}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{x^2[n]}{\sigma_2^2}}$$

$$= (1-\epsilon)\phi_1(x[n]) + \epsilon\, \phi_2(x[n])$$

$0 < \epsilon < 1 \Rightarrow$ MIXTURE PARAMETER

USUALLY ϵ SMALL



$\sigma_2^2 >> \sigma_1^2$

CAN THINK OF THIS AS CONTAMINATED GAUSSIAN PDF OR GAUSSIAN WITH σ_1^2 $1-\epsilon$ OF TIME AND GAUSSIAN WITH σ_2^2

ASSUME σ_1^2, σ_2^2 KNOWN. ESTIMATE ϵ.
- COULD USE MLE $\Rightarrow$ GRID SEARCH OVER $0 < \epsilon < 1$.

<u>METHOD OF MOMENTS</u> - 1) FIND MOMENT AS FUNCTION OF ϵ

$$E(x^2[n]) = \int_{-\infty}^{\infty} x^2[n] \left[(1-\epsilon)\phi_1(x[n]) + \epsilon\phi_2(x[n])\right] dx[n]$$

$$= (1-\epsilon)\sigma_1^2 + \epsilon\sigma_2^2$$

2) REPLACE MOMENT BY ESTIMATOR

$$E(x^2[n]) \rightarrow \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

3) SOLVE FOR $\hat{\epsilon}$

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = (1-\hat{\epsilon})\sigma_1^2 + \hat{\epsilon}\sigma_2^2$$

$$\Rightarrow \hat{\epsilon} = \frac{\frac{1}{N}\sum x^2[n] - \sigma_1^2}{\sigma_2^2 - \sigma_1^2}$$

EASY TO SOLVE SINCE MOMENT EQUATION IS <u>LINEAR</u> IN ϵ.

HOW GOOD IS $\hat{\epsilon}$?

$$E(\hat{\epsilon}) = \frac{E\left(\frac{1}{N}\sum x^2[n]\right) - \sigma_1^2}{\sigma_2^2 - \sigma_1^2}$$

$$= \frac{E(x^2[n]) - \sigma_1^2}{\sigma_2^2 - \sigma_1^2} = \epsilon$$

$\Rightarrow$ UNBIASED

$$\mathrm{VAR}(\hat{\epsilon}) = \frac{1}{(\sigma_2^2-\sigma_1^2)^2}\,\mathrm{VAR}\left(\frac{1}{N}\sum x^2[n]\right)$$

$$= \frac{1}{N(\sigma_2^2-\sigma_1^2)^2}\,\mathrm{VAR}(x^2[n])$$

BUT $E(x^4[n]) = (1-\epsilon)3\sigma_1^4 + \epsilon 3\sigma_2^4$

$$\Rightarrow \mathrm{VAR}(\hat{\epsilon}) = \frac{3(1-\epsilon)\sigma_1^4 + 3\epsilon\sigma_2^4 - [(1-\epsilon)\sigma_1^2 + \epsilon\sigma_2^2]^2}{N(\sigma_2^2-\sigma_1^2)^2}$$

CONSISTENT SINCE $\mathrm{VAR}(\hat{\epsilon}) \to 0$ AS $N \to \infty$

NEED TO FIND CRLB AND EVALUATE.

SINCE MOMENT ESTIMATOR IS GENERALLY CONSISTENT, WHEN WE PLUG INTO EQUATION AND LET $N \to \infty$ $\Rightarrow$ THEORETICAL EQUATION
$\Rightarrow$ SOLUTION GIVES TRUE VALUE

SUMMARY : SCALAR PARAMETER θ

$$\mu_k = E(x^k[n]) \quad \text{DEPENDS ON } \theta$$

$$= h(\theta)$$

SOLVE FOR θ

$$\Rightarrow \quad \theta = h^{-1}(\mu_k)$$

REPLACE μ_k BY ESTIMATOR

$$\hat{\mu}_k = \frac{1}{N} \sum_{n=0}^{N-1} x^k[n]$$

$$\hat{\theta} = h^{-1}\left(\frac{1}{N} \sum_{n=0}^{N-1} x^k[n] \right)$$

<u>EXAMPLE</u> : EXPONENTIAL PDF

$$p(x[n];\lambda) = \begin{cases} \lambda e^{-\lambda x[n]} & x[n] > 0 \\ 0 & x[n] < 0 \end{cases}$$

IID

ESTIMATE λ $(\lambda > 0)$

$$\mu_1 = E(x[n]) = \int_0^\infty x[n] \lambda e^{-\lambda x[n]} dx[n]$$

$$= \frac{1}{\lambda} \int_0^\infty \xi e^{-\xi} d\xi = 1/\lambda$$

$\lambda = 1/\mu_1$

$$\Rightarrow \hat{\lambda} = \frac{1}{\frac{1}{N}\sum_{n=0}^{N-1} x[n]}$$

VECTOR PARAMETER

SUPPOSE

$$\mu_1 = h_1(\theta_1, \theta_2, \ldots, \theta_p)$$
$$\mu_2 = h_2(\theta_1, \theta_2, \ldots, \theta_p)$$
$$\vdots$$
$$\mu_p = h_p(\theta_1, \theta_2, \ldots, \theta_p)$$

OR $\underline{\mu} = \underline{h}(\underline{\theta})$

$$\underline{\theta} = \underline{h}^{-1}(\underline{\mu})$$

$$\Rightarrow \hat{\underline{\theta}} = \underline{h}^{-1}(\hat{\underline{\mu}})$$

WHERE $\hat{\underline{\mu}} = \begin{bmatrix} \frac{1}{N}\sum_n x[n] \\ \frac{1}{N}\sum_n x^2[n] \\ \vdots \\ \frac{1}{N}\sum_n x^p[n] \end{bmatrix}$

WOULD LIKE EQUATIONS $\underline{\mu} = \underline{h}(\underline{\theta})$ TO BE LINEAR OR ELSE EASILY SOLVABLE.
OTHERWISE, NEED NONLINEAR EQUATION SOLVER

FOR $\underline{h}(\hat{\underline{\theta}}) = \hat{\underline{\mu}}$,
DEFEATING PURPOSE OF SIMPLE MOMENT ESTIMATOR.

EXAMPLE - GAUSSIAN MIXTURE

NOW ASSUME σ_1^2, σ_2^2 UNKNOWN AS WELL.

$\Rightarrow \underline{\theta} = [\epsilon\ \sigma_1^2\ \sigma_2^2]^T$

NEED 3 MOMENT EQUATIONS

BUT PDF IS EVEN IN $x[n]$ $\Rightarrow$ ODD ORDER MOMENTS ARE ZERO

$$\mu_2 = E(x^2[n]) = (1-\epsilon)\sigma_1^2 + \epsilon\sigma_2^2$$
$$\mu_4 = E(x^4[n]) = 3(1-\epsilon)\sigma_1^4 + 3\epsilon\sigma_2^4$$
$$\mu_6 = E(x^6[n]) = 15(1-\epsilon)\sigma_1^6 + 15\epsilon\sigma_2^6$$

NEED TO SOLVE FOR $\epsilon, \sigma_1^2, \sigma_2^2$

LET $u = \sigma_1^2 + \sigma_2^2$

$v = \sigma_1^2\sigma_2^2$

$$\Rightarrow \quad u = \frac{\mu_6 - 5\mu_4\mu_2}{5\mu_4 - 15\mu_2^2}$$

$$v = \mu_2 u - \frac{\mu_4}{3}$$

FIND u (DEPENDS ONLY ON MOMENTS)
$\Rightarrow$ DETERMINE v

SOLVING FOR σ_1^2, σ_2^2

$$\sigma_1^2 = \frac{u + \sqrt{u^2 - 4v}}{2}$$

$$\sigma_2^2 = \frac{v}{\sigma_1^2}$$

FINALLY, $\epsilon = \dfrac{\mu_2 - \sigma_1^2}{\sigma_2^2 - \sigma_1^2}$

ESTIMATOR OBTAINED BY REPLACING μ_2, μ_4, μ_6 BY THEIR ESTIMATORS.

STATISTICAL EVALUATION OF ESTIMATORS

FOR MOMENT ESTIMATORS

$$\hat{\underline{\theta}} = \underline{h}^{-1}(\hat{\underline{\mu}})$$

$$= \underline{g}(\underline{x})$$

HOW DO WE FIND PDF OF $\hat{\underline{\theta}}$?

IN GENERAL, QUITE DIFFICULT. ALTERNATIVELY, WE SHOW HOW TO OBTAIN APPROXIMATE EXPRESSIONS FOR MEAN AND VARIANCE. THESE ARE ASYMPTOTIC RESULTS (AS $N \to \infty$).

RESULTS ARE VALID FOR ANY ESTIMATOR.

ASSUME THAT $\hat{\theta}$ DEPENDS ON $r \ll N$ STATISTICS $\{T_1(\underline{x}), T_2(\underline{x}), \ldots, T_r(\underline{x})\}$ WHOSE VARIANCES AND COVARIANCES ARE SMALL.

PDF OF $[T_1 \; T_2 \ldots T_r]^T$ CONCENTRATED ABOUT MEAN.

<u>EXAMPLE</u>: EXPONENTIAL PDF

$$\hat{\lambda} = \frac{1}{\frac{1}{N}\sum_n x(n)} = \frac{1}{T_1(\underline{x})}$$

AS $N \to \infty$ T_1 BECOMES CONCENTRATED ABOUT MEAN SINCE $\mathrm{VAR}(T_1) = \frac{1}{N\lambda^2}$.

$\Rightarrow$ CAN USE TAYLOR EXPANSION ABOUT MEAN OF T_1 (RECALL STATISTICAL LINEARIZATION)

IN GENERAL, $\hat{\theta} = g(\underline{T})$

WHERE $\underline{T} = (T_1 \; T_2 \ldots T_r)^T$

USE FIRST ORDER TAYLOR EXPANSION OF g

ABOUT $\underline{T} = E(\underline{T}) = \underline{\mu}$

$$\hat{\theta} = g(\underline{T}) \approx g(\underline{\mu}) + \sum_{k=1}^{r} \left.\frac{\partial g}{\partial T_k}\right|_{\underline{T}=\underline{\mu}} (T_k - \mu_k)$$

$$\Rightarrow E(\hat{\theta}) = g(\underline{\mu})$$

(ASYMPTOTICALLY, EXPECTED VALUE COMMUTES OVER g SINCE IT IS LINEAR).

$$\mathrm{VAR}(\hat{\theta}) = E\left[(\hat{\theta} - E(\hat{\theta}))^2\right]$$

$$= E\left[\Big(g(\underline{\mu}) + \sum_{k=1}^{r} \left.\frac{\partial g}{\partial T_k}\right|_{\underline{T}=\underline{\mu}} (T_k - \mu_k) - \underbrace{E(\hat{\theta})}_{g(\underline{\mu})}\Big)^2\right]$$

$$= E\left(\left(\left.\frac{\partial g}{\partial \underline{T}}\right|_{\underline{T}=\underline{\mu}}^{T} (\underline{T} - \underline{\mu})\right)^2\right)$$

$$= E\left[\left.\frac{\partial g}{\partial \underline{T}}\right|_{\underline{T}=\underline{\mu}}^{T} (\underline{T} - \underline{\mu})(\underline{T} - \underline{\mu})^T \left.\frac{\partial g}{\partial \underline{T}}\right|_{\underline{T}=\underline{\mu}}\right]$$

$$= \left.\frac{\partial g}{\partial \underline{T}}\right|_{\underline{T}=\underline{\mu}}^{T} \underline{C}_T \left.\frac{\partial g}{\partial \underline{T}}\right|_{\underline{T}=\underline{\mu}}$$

$\underline{C}_T$ = COVARIANCE MATRIX OF $\underline{T}$

<u>EXAMPLE</u> - EXPONENTIAL PDF

$$\hat{\lambda} = \frac{1}{\frac{1}{N}\sum_{n=0}^{N-1} x[n]}$$

FIND ASYMPTOTIC MEAN & VARIANCE.

$$\hat{\lambda} = g(T_1) \quad \text{WHERE} \quad g(T_1) = 1/T_1$$

$$\text{AND} \quad T_1 = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

$$E(\hat{\lambda}) = E\{g(T_1)\} = g[E(T_1)] = g(\mu_1) = 1/\mu_1$$

$$\text{BUT} \quad E(T_1) = \frac{1}{N}\sum_n E(x(n)) = E(x(n)) = 1/\lambda$$

$$\Rightarrow E(\hat{\lambda}) = \lambda$$

$$\text{VAR}(\hat{\lambda}) = \left.\frac{\partial g}{\partial T_1}\right|_{T_1=\mu_1} \text{VAR}(T_1) \left.\frac{\partial g}{\partial T_1}\right|_{T_1=\mu_1}$$

$$\left.\frac{\partial g}{\partial T_1}\right|_{T_1=\mu_1} = -\left.\frac{1}{T_1^2}\right|_{T_1=\mu_1} = -\frac{1}{\mu_1^2} = -\lambda^2$$

$$\text{VAR}(T_1) = \text{VAR}\left(\frac{1}{N}\sum_n x(n)\right)$$

$$= \frac{\text{VAR}(x(n))}{N}$$

$$\text{VAR}(x(n)) = \int_0^\infty x^2(n)\,\lambda e^{-\lambda x(n)}\,dx(n) - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\Rightarrow \text{VAR}(T_1) = \frac{1}{N\lambda^2}$$

$$\text{VAR}(\hat{\lambda}) = \left(\left.\frac{\partial g}{\partial T_1}\right|_{T_1=\mu}\right)^2 \text{VAR}(T_1)$$

$$= (-\lambda^2)^2 \frac{1}{N\lambda^2} = \frac{\lambda^2}{N}$$

ASYMPTOTICALLY,

$$E(\hat{\lambda}) = \lambda$$

$$\mathrm{VAR}(\hat{\lambda}) = \lambda^2/N$$

(ALSO VERIFIED SINCE $\hat{\lambda}$ IS MLE - USE ASYMPTOTIC PROPERTIES)

CANNOT SAY HOW LARGE N MUST BE FOR THESE TO HOLD. REQUIRE g TO BE APPROXIMATELY LINEAR OVER RANGE OF $\underline{T}$ FOR $p(\underline{T};\theta)$ NONZERO APPROXIMATELY. SATISFIED IF

1) DATA RECORD LARGE $\Rightarrow$ $p(\underline{T};\theta)$ CONCENTRATED ABOUT MEAN
2) SIGNAL IN NOISE PROBLEMS AT HIGH SNR $\Rightarrow$ PDF CONCENTRATED ABOUT SIGNAL

$$x[n] = s[n;\theta] + w[n] \qquad n = 0, 1, \ldots, N-1$$

ZERO MEAN, COVARIANCE $\underline{C}$

$$\hat{\theta} = g(\underline{x}) = g(\underline{s}(\theta) + \underline{w}) = h(\underline{w})$$

NOW WE CHOOSE STATISTIC T AS $\underline{x}$ SINCE AS SNR BECOMES LARGE PDF OF $\underline{x}$ BECOMES CONCENTRATED ABOUT MEAN OR $\underline{s}(\theta)$.

EXPAND $\hat{\theta} = g(\underline{s}(\theta) + \underline{w})$ ABOUT MEAN $\underline{s}(\theta)$ OR EQUIVALENTLY EXPAND $\hat{\theta} = h(\underline{w})$ ABOUT $\underline{w} = \underline{0}$.

$$\hat{\theta} \approx h(\underline{0}) + \sum_{n=0}^{N-1} \frac{\partial h}{\partial w[n]}\bigg|_{\underline{w}=\underline{0}} w[n]$$

$$E(\hat{\theta}) = h(\underline{0}) = g(\underline{s}(\theta))$$

$$\mathrm{VAR}(\hat{\theta}) = \frac{\partial h}{\partial \underline{w}}\bigg|_{\underline{w}=\underline{0}}^{T} \underline{C} \frac{\partial h}{\partial \underline{w}}\bigg|_{\underline{w}=\underline{0}}$$

USUALLY, IN THE ABSENCE OF NOISE ESTIMATOR YIELDS TRUE VALUE OR $\hat{\theta} = g(\underline{s}(\theta)) = \theta$ $\Rightarrow$ $\hat{\theta}$ IS UNBIASED AT HIGH SNR.

<u>EXAMPLE</u> - EXPONENTIAL SIGNAL

$$x[n] = r^n + w[n] \qquad n = 0, 1, 2$$

$\uparrow$ WHITE NOISE
VARIANCE $= \sigma^2$

CONSIDER

$$\hat{r} = \frac{x[2] + x[1]}{x[1] + x[0]} \qquad \text{AD HOC!}$$