IS SATISFIED FOR SOME g and I, AN ESTIMATOR CAN BE FOUND THAT ATTAINS CRLB. THAT ESTIMATOR IS $\hat{\theta} = g(\underline{x})$ AND THE MIN. VARIANCE IS $1/I(\theta)$.

EXAMPLE: $x[n] = A + w[n] \quad n = 0, 1, \ldots, N-1$

$w[n]$: WGN WITH VARIANCE σ^2

PROBLEM: FIND MVU ESTIMATOR OF A

APPROACH: COMPUTE CRLB, IF ESTIMATOR CAN BE FOUND SATISFYING BOUND, THEN WE HAVE IT.

$$p(\underline{x}; A) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x[n]-A)^2}$$

$$= \frac{1}{(2\pi)^{N/2}\sigma^N} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2}$$

$$\frac{\partial \ln p}{\partial A} = \frac{\partial}{\partial A}\left(-\frac{1}{2\sigma^2}\sum_n (x[n]-A)^2\right)$$

$$= \frac{1}{\sigma^2}\sum_n (x[n]-A)$$

$$= \frac{N}{\sigma^2}(\bar{x}-A)$$

$$\frac{\partial^2 \ln p}{\partial A^2} = -N/\sigma^2$$

$$\Rightarrow \quad \mathrm{VAR}(\hat{A}) \geq \frac{1}{N/\sigma^2} = \sigma^2/N$$

BUT $\bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x(n)$ IS UNBIASED

AND HAS VARIANCE $\sigma^2/N \Rightarrow \bar{x}$ IS MVUE!

WHAT IF WE DID NOT RECALL THAT $\bar{x}$ HAD THESE PROPERTIES?

FROM CRLB THEOREM EQUALITY HOLDS IF AND ONLY IF

$$\frac{\partial \ln p(\underline{x};A)}{\partial A} = I(A)\left(g(\underline{x}) - A\right)$$

BUT
$$\frac{\partial \ln p(\underline{x};A)}{\partial A} = \frac{N}{\sigma^2}\left(\bar{x} - A\right)$$

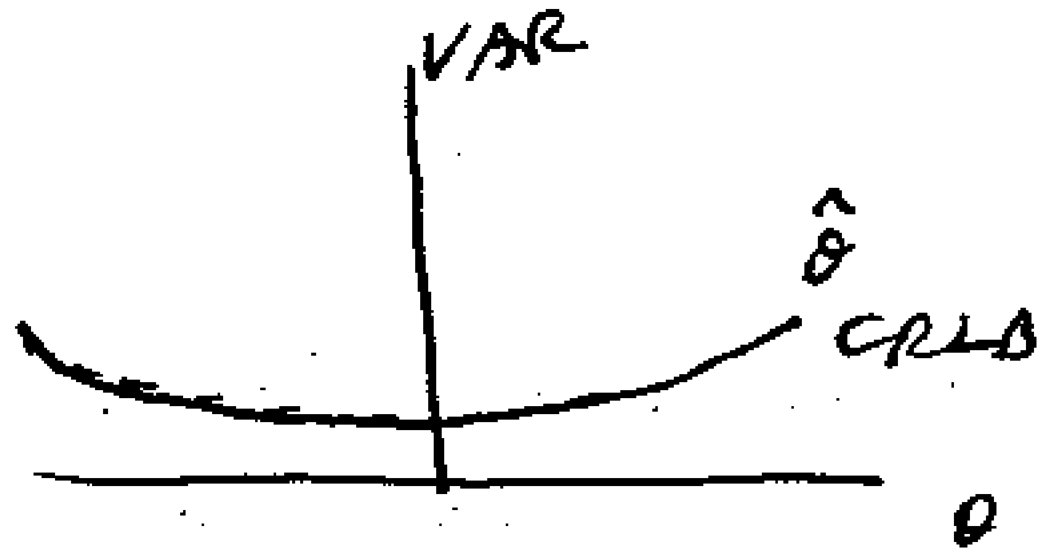
$\Rightarrow$ $\bar{X}$ SATISFIES BOUND ON VARIANCE AND $\mathrm{VAR}(\hat{A}) = 1/I(A) = \sigma^2/N$ AS EXPECTED.

SUMMARY: CRLB THEOREM GIVES US A LOWER BOUND ON VARIANCE OF ANY UNBIASED ESTIMATOR <u>BUT ALSO</u> FINDS ESTIMATOR THAT ATTAINS BOUND (IF IT EXISTS).

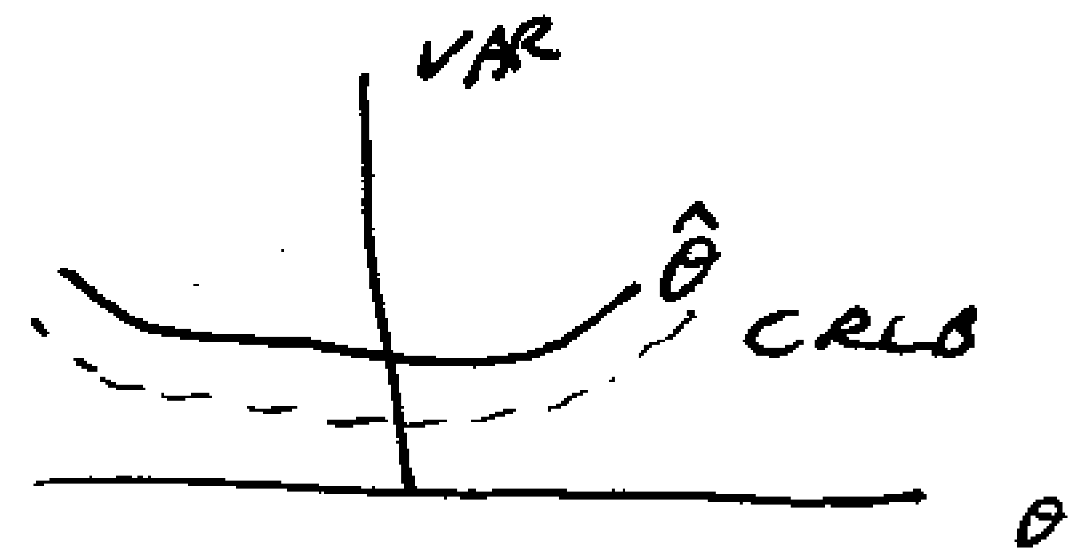
ESTIMATOR THAT ATTAINS BOUND TERMED <u>EFFICIENT</u> (GOOD USE OF DATA)

EFFICIENT $\Rightarrow$ MVUE

MVUE $\not\Rightarrow$ EFFICIENT



$\hat{\theta}$ EFFICIENT & MVU



$\hat{\theta}$ MVU

<u>EXAMPLE</u> : PHASE ESTIMATION

WGN

$$x(n) = A\cos(2\pi f_0 n + \phi) + w(n) \qquad n = 0, 1, \dots, N-1$$

A, f_0 KNOWN

$$p(\underline{x};\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_n (x(n) - A\cos(2\pi f_0 n + \phi))^2}$$

$$\frac{\partial \ln p}{\partial \phi} = -\frac{1}{\sigma^2}\sum_n (x(n) - A\cos(2\pi f_0 n + \phi)) A\sin(2\pi f_0 n + \phi)$$

$$= -\frac{A}{\sigma^2}\sum_n \Big[x(n)\sin(2\pi f_0 n + \phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\Big]$$

$$\frac{\partial^2 \ln p}{\partial \phi^2} = -\frac{A}{\sigma^2} \sum_n \left[x[n] \cos(2\pi f_0 n + \phi) - A \cos(4\pi f_0 n + 2\phi) \right]$$

$$-E\left[\frac{\partial^2 \ln p}{\partial \phi^2}\right] = \frac{A}{\sigma^2} \sum_n \left(A\cos^2(2\pi f_0 n + \phi) - A\cos(4\pi f_0 n + 2\phi) \right)$$

$$= \frac{A^2}{\sigma^2} \sum_n \left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 n + 2\phi) - \cos(4\pi f_0 n + 2\phi) \right)$$

$$= \frac{NA^2}{2\sigma^2} - \frac{A^2}{2\sigma^2} \sum_n \cos(4\pi f_0 n + 2\phi)$$

BUT $\sum_n \cos(4\pi f_0 n + 2\phi) \ll N$ FOR f_0 NOT NEAR 0 OR 1/2.

$$\Rightarrow \quad \mathrm{VAR}(\hat{\phi}) \geq \frac{2\sigma^2}{NA^2}$$

BOUND DECREASES AS $SNR = \frac{A^2}{2\sigma^2}$ INCREASES

AND/OR N INCREASES AS EXPECTED.

DOES EFFICIENT ESTIMATOR EXIST?

MVUE?

WE DEFINE

$$I(\theta) = -E\left(\frac{\partial^2 \ln p}{\partial \theta^2}\right)$$ AS THE FISHER INFORMATION.

NOTE:

1) $I(\theta) \geq 0$

2) ADDITIVE FOR INDEPENDENT OBSERVATIONS

$$\ln p(\underline{x};\theta) = \ln \prod_{n=0}^{N-1} p(x[n];\theta)$$

$$= \sum_{n=0}^{N-1} \ln p(x[n];\theta)$$

$$I(\theta) = -E\left(\frac{\partial^2 \ln p}{\partial \theta^2}\right) = -\sum_{n=0}^{N-1} E\left[\frac{\partial^2 \ln p(x[n];\theta)}{\partial \theta^2}\right]$$

IF OBSERVATIONS ARE ALSO IDENTICALLY DISTRIBUTED (SAME PDF FOR EACH $x[n]$),

$$I(\theta) = N\, i(\theta)$$

$$i(\theta) = -E\left[\frac{\partial^2 \ln p(x[\,];\theta)}{\partial \theta^2}\right]$$

AS $N \to \infty$, FOR IID (INDEPENDENT & IDENTICALLY DISTRIBUTED) $\Rightarrow$ CRLB $\to 0$

WHAT HAPPENS FOR NONINDEPENDENT OBSERVATIONS?

$I(\theta)$ ALSO COMPUTED AS

$$I(\theta) = E\left[\left(\frac{\partial \ln p(\underline{x};\theta)}{\partial\theta}\right)^2\right]$$

USUALLY EASIER TO USE ORIGINAL FORM.

GENERAL CRLB FOR SIGNALS IN WGN

$$x[n] = s[n;\theta] + w[n] \qquad n = 0, 1, \ldots, N-1$$

$s[n;\theta]$ ↑ DEPENDS ON θ ; $w[n]$ ↑ WGN

$$p(\underline{x};\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n;\theta])^2}$$

$$\frac{\partial \ln p}{\partial\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n;\theta])\frac{\partial s[n;\theta]}{\partial\theta}$$

$$\frac{\partial^2 \ln p}{\partial\theta^2} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left[(x[n]-s[n;\theta])\frac{\partial^2 s[n;\theta]}{\partial\theta^2} - \left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2\right]$$

$$E\left(\frac{\partial^2 \ln p}{\partial\theta^2}\right) = -\frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2$$

$$VAR(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta} \right)^2}$$

EXAMPLE: IF $s[n;\theta] = \theta$ $(s[n] = A)$

$$VAR(\hat{A}) \geq \frac{\sigma^2}{\sum_n (1)^2} = \sigma^2/N$$

TRY ALSO FOR PHASE EXAMPLE. $(\theta = \phi)$

NOTE THAT IMPORTANT ASPECT OF SIGNAL IS RATE OF CHANGE WITH θ.

TRANSFORMATION OF PARAMETERS

ASSUME WE WISH TO ESTIMATE $\alpha = g(\theta)$, WHERE PDF IS PARAMETERIZED BY θ. g IS SOME TRANSFORMATION (NONLINEAR IN GENERAL)

CRLB FOR α IS

$$VAR(\hat{\alpha}) \geq \frac{(\partial g/\partial \theta)^2}{-E\left(\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial \theta^2} \right)}$$

EXAMPLE – DC LEVEL IN WGN

$$x[n] = A + w[n]$$

CRLB FOR $\alpha = A^2 \Rightarrow g(A) = A^2$

$$\mathrm{VAR}(\hat{A^2}) \geq \frac{(\partial g/\partial A)^2}{-E\left[\frac{\partial^2 \ln p(\underline{x};A)}{\partial A^2}\right]}$$

$$= \frac{(2A)^2}{N/\sigma^2} = \frac{4A^2\sigma^2}{N}$$

$\hat{A} = \bar{x}$ WAS EFFICIENT FOR A. IS $\hat{A^2} = (\bar{x})^2$ EFFICIENT FOR A^2? NO

$\bar{x} \sim N(A, \sigma^2/N)$

$$E((\bar{x})^2) = E^2(\bar{x}) + \mathrm{VAR}(\bar{x})$$
$$= A^2 + \sigma^2/N \neq A^2$$

$\Rightarrow$ NOT EVEN UNBIASED

$\Rightarrow$ IN GENERAL EFFICIENCY DESTROYED BY NON LINEAR TRANSFORMATION.

MAINTAINED OVER LINEAR (AFFINE) TRANSFORMATIONS.

$\hat{\theta}$ EFFICIENT $\Rightarrow a\hat{\theta}+b$ EFFICIENT FOR $\alpha = a\theta + b$

SINCE

$$E(a\hat{\theta}+b) = aE(\hat{\theta})+b = a\theta+b = \alpha$$

$$\text{VAR}(a\hat{\theta}+b) = a^2\,\text{VAR}(\hat{\theta})$$
$$= \frac{(\partial g/\partial\theta)^2}{-E\left[\frac{\partial^2 \ln p}{\partial\theta^2}\right]}$$

CRLB FOR VECTOR PARAMETER

$$\underline{\theta} = [\theta_1\ \theta_2\ \dots\ \theta_p]^T$$

ASSUME $E(\hat{\underline{\theta}}) = \underline{\theta}$ UNBIASED

$$\text{VAR}(\hat{\theta}_i) \geq [\underline{I}^{-1}(\underline{\theta})]_{ii}$$

WHERE $[\underline{I}(\underline{\theta})]_{ij} = -E\left(\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial\theta_i\partial\theta_j}\right)$

$\underline{I}(\underline{\theta})$ IS $p \times p$ FISHER INFORMATION MATRIX

EXAMPLE : LINE FITTING

$$x[n] = A + Bn + \underbrace{w[n]}_{\text{WGN}}$$

$\underline{\theta} = \begin{bmatrix} A \\ B \end{bmatrix}$ (DC LEVEL HAD $B=0$)

TO FIND CRLB COMPUTE $\underline{I}(\underline{\theta})$ FIRST.

$$\underline{I}(\underline{\theta}) = \begin{bmatrix} -E\left[\frac{\partial^2 \ln p}{\partial A^2}\right] & -E\left[\frac{\partial^2 \ln p}{\partial A \partial B}\right] \\ -E\left[\frac{\partial^2 \ln p}{\partial B \partial A}\right] & -E\left[\frac{\partial^2 \ln p}{\partial B^2}\right] \end{bmatrix}$$

$$p(\underline{x};\underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A-Bn)^2}$$

$$\frac{\partial \ln p}{\partial A} = \frac{1}{\sigma^2}\sum_n (x[n]-A-Bn)$$

$$\frac{\partial \ln p}{\partial B} = \frac{1}{\sigma^2}\sum_n (x[n]-A-Bn)n$$

$$\frac{\partial^2 \ln p}{\partial A^2} = -N/\sigma^2$$

$$\frac{\partial^2 \ln p}{\partial A \partial B} = -\frac{1}{\sigma^2}\sum_n n$$

$$\frac{\partial^2 \ln p}{\partial B^2} = -\frac{1}{\sigma^2}\sum_n n^2$$

$$\Rightarrow \quad \underline{I}(\underline{\theta}) = \frac{1}{\sigma^2}\begin{pmatrix} N & \sum n \\ \sum n & \sum n^2 \end{pmatrix}$$

BUT $\displaystyle\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2}$ $\quad \displaystyle\sum_{n=0}^{N-1} n^2 = \frac{N(N-1)(2N-1)}{6}$

$$\underline{I}(\underline{\theta}) = \frac{1}{\sigma^2}\begin{bmatrix} N & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{N(N-1)(2N-1)}{6} \end{bmatrix}$$

(ALWAYS SYMMETRIC - WHY ?)

$$\underline{I}^{-1}(\underline{\theta}) = \sigma^2\begin{bmatrix} \frac{2(2N-1)}{N(N+1)} & \frac{-6}{N(N+1)} \\ \frac{-6}{N(N+1)} & \frac{12}{N(N^2-1)} \end{bmatrix}$$

$$VAR(\hat{A}) \geq [\underline{I}^{-1}(\underline{\theta})]_{11} = \frac{2(2N-1)\sigma^2}{N(N+1)}$$

$$VAR(\hat{B}) \geq [\underline{I}^{-1}(\underline{\theta})]_{22} = \frac{12\sigma^2}{N(N^2-1)}$$

OBSERVATIONS:

1) RECALL FOR $x[n] = A + w[n]$

$VAR(\hat{A}) \geq \sigma^2/N$

NOW $VAR(\hat{A}) \geq \frac{\sigma^2}{N}\underbrace{\left(\frac{2(2N-1)}{N+1}\right)}_{>1 \text{ FOR } N \geq 2}$

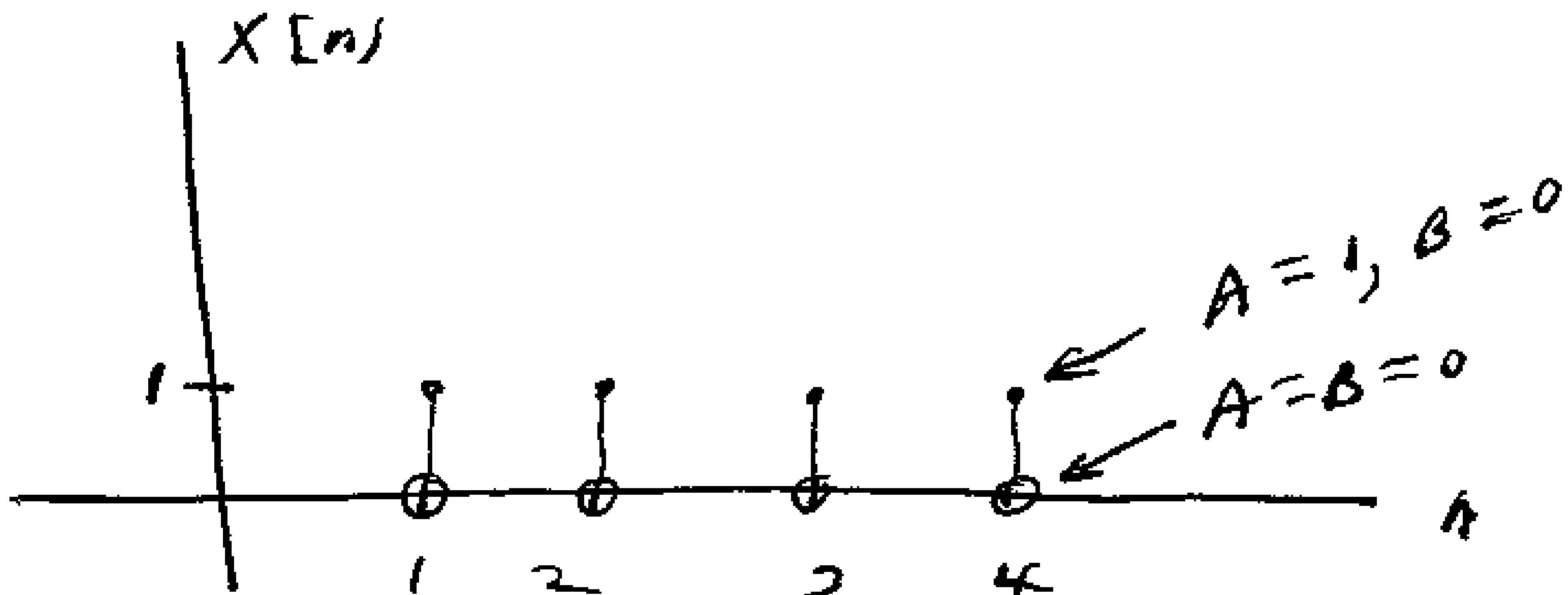
CRLB ALWAYS INCREASES AS WE ESTIMATE MORE PARAMETERS

$$\frac{CRLB(\hat{A})}{CRLB(\hat{B})} = \frac{(2N-1)(N-1)}{6} > 1 \qquad N \geq 3$$

B IS EASIER TO ESTIMATE. REASON IS SLIGHT CHANGE IN B CAUSES LARGE CHANGE IN $s[n]$ OR

$$\Delta x[n] \approx \frac{\partial x[n]}{\partial A}\Delta A = \frac{\partial s[n]}{\partial A}\Delta A = \Delta A$$

$$\Delta x[n] \approx \frac{\partial x[n]}{\partial B}\Delta B = \frac{\partial s[n]}{\partial B}\Delta B = n\,\Delta B$$



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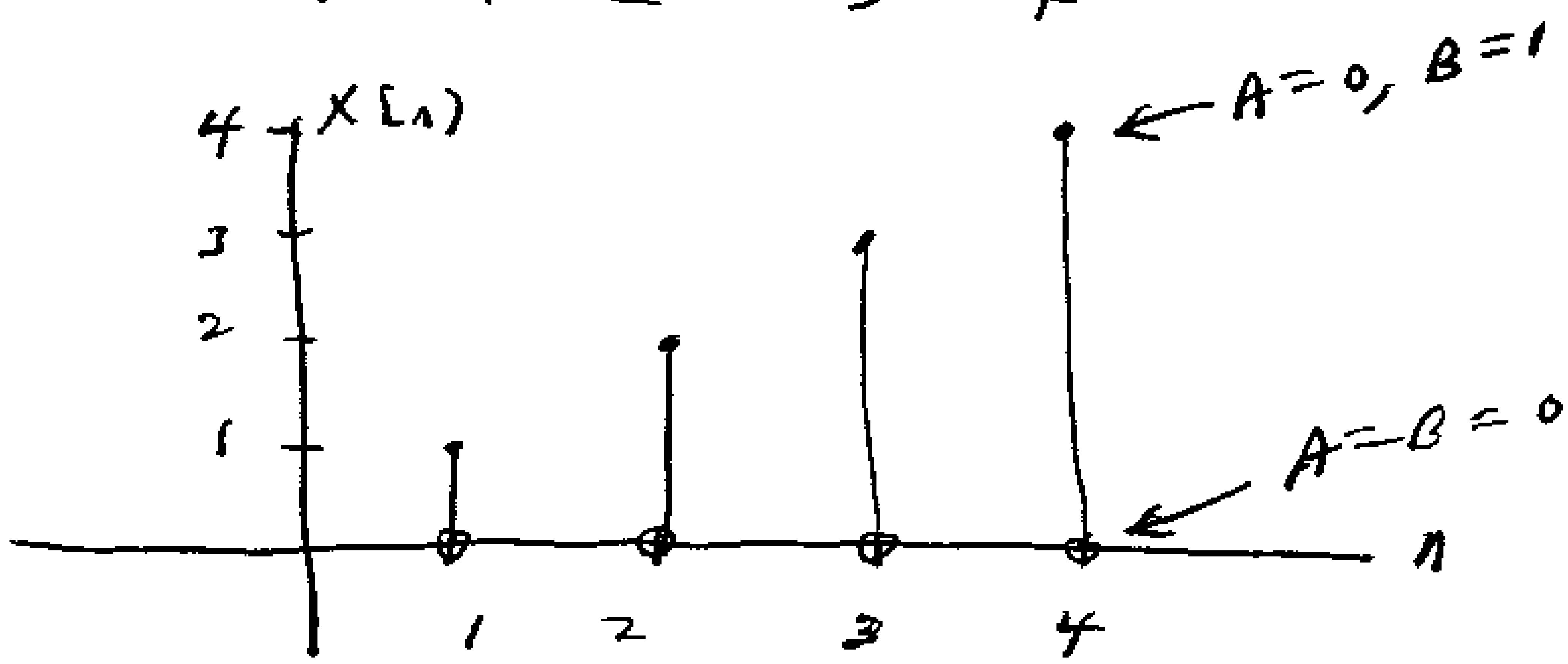

TO COMPUTE $\underline{I}(\underline{\theta})$ CAN ALSO USE

$$[\underline{I}(\underline{\theta})]_{ij} = -E\left[\frac{\partial^2 \ln p}{\partial\theta_i \partial\theta_j}\right] \leftarrow \text{EASIER TO USE}$$

$$= E\left[\frac{\partial \ln p}{\partial\theta_i}\frac{\partial \ln p}{\partial\theta_j}\right]$$

CRLB THEOREM - VECTOR PARAMETER

DENOTE COVARIANCE MATRIX OF $\hat{\underline{\theta}}$ AS $\underline{C}_{\hat{\theta}}$

IF $\underline{\theta} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $(p=3)$,

$$\underline{C}_{\hat{\theta}} = E\left((\hat{\underline{\theta}} - E(\hat{\underline{\theta}}))(\hat{\underline{\theta}} - E(\hat{\underline{\theta}}))^T\right)$$

$$= E\left(\underbrace{\left[\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} - \begin{bmatrix} E(\hat{x}) \\ E(\hat{y}) \\ E(\hat{z}) \end{bmatrix}\right]}_{\text{COLUMN VECTOR}} \underbrace{\left[\quad\right]^T}_{\text{ROW VECTOR}}\right)$$

$$= \begin{bmatrix} \mathrm{VAR}(\hat{x}) & \mathrm{COV}(\hat{x},\hat{y}) & \mathrm{COV}(\hat{x},\hat{z}) \\ ? & \mathrm{VAR}(\hat{y}) & \mathrm{COV}(\hat{y},\hat{z}) \\ ? & ? & \mathrm{VAR}(\hat{z}) \end{bmatrix}$$

IF PDF SATISFIES REGULARITY CONDITIONS, THEN

$$\underline{C}_{\hat{\theta}} - \underline{I}^{-1}(\underline{\theta}) \quad \text{IS POSITIVE SEMIDEFINITE}$$

(EVALUATE $\underline{I}(\underline{\theta})$ AT TRUE VALUE OF $\underline{\theta}$). AN UNBIASED ESTIMATOR ATTAINS CRLB IFF

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \underline{I}(\underline{\theta})(\underline{g}(\underline{x}) - \underline{\theta})$$

FOR SOME $\underline{g}$ AND $\underline{I}$. THAT ESTIMATOR IS

$$\hat{\underline{\theta}} = \underline{g}(\underline{x})$$

AND ITS COVARIANCE IS $\underline{C}_{\hat{\theta}} = \underline{I}^{-1}(\underline{\theta})$.

EXPLANATION

$\underline{A}$ IS POSITIVE SEMIDEFINITE IF IT IS SYMMETRIC AND $\underline{x}^T \underline{A}\underline{x} \geq 0$ FOR ALL $\underline{x}$

$$\Rightarrow \quad \underline{x}^T(\underline{C}_{\hat{\theta}} - \underline{I}^{-1}(\underline{\theta}))\underline{x} \geq 0$$

$$\text{LET } \underline{x} = [0\,0 \ldots 0\,1\,0 \ldots 0]^T = \underline{e}_i$$

i^{th} PLACE

BUT $\underline{e}_i^T \underline{A}\, \underline{e}_i = (\underline{A})_{ii}$

$$\Rightarrow \quad \underline{e}_i^T \left(\underline{C}_{\hat{\theta}} - \underline{I}^{-1}(\underline{\theta}) \right) \underline{e}_i \geq 0$$

$$(\underline{C}_{\hat{\theta}})_{ii} \geq [\underline{I}^{-1}(\underline{\theta})]_{ii}$$

$$\mathrm{VAR}(\hat{\theta}_i) \geq (\underline{I}^{-1}(\underline{\theta}))_{ii} \quad \text{. AS BEFORE.}$$

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \text{GRADIENT} = \begin{bmatrix} \frac{\partial \ln p}{\partial \theta_1} \\ \vdots \\ \frac{\partial \ln p}{\partial \theta_p} \end{bmatrix}$$

FOR EQUALITY TO HOLD REQUIRE

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \underline{I}(\theta)\left(\underline{g}(\underline{x}) - \underline{\theta}\right)$$

$\uparrow$ $\hat{\underline{\theta}}$ = EFFICIENT

<u>EXAMPLE</u> : LINE FITTING

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \begin{bmatrix} \frac{\partial \ln p}{\partial A} \\ \frac{\partial \ln p}{\partial B} \end{bmatrix}$$

$$= \frac{1}{\sigma^2} \begin{bmatrix} \sum_n (x[n] - A - Bn) \\ \sum_n (x[n] - A - Bn)\, n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} N/\sigma^2 & \frac{N(N-1)}{2\sigma^2} \\ \frac{N(N-1)}{2\sigma^2} & \frac{N(N-1)(2N-1)}{6\sigma^2} \end{bmatrix}}_{\underline{I}(\underline{\theta})} \underbrace{\begin{bmatrix} \hat{A} - A \\ \hat{B} - B \end{bmatrix}}_{\hat{\underline{\theta}} - \underline{\theta}}$$

$$\hat{A} = \frac{2(2N-1)}{N(N+1)} \sum_{n=0}^{N-1} x[n] - \frac{6}{N(N+1)} \sum_{n=0}^{N-1} n\, x[n]$$

$$\hat{B} = -\frac{6}{N(N+1)} \sum_{n=0}^{N-1} x[n] + \frac{12}{N(N^2-1)} \sum_{n=0}^{N-1} n\, x[n]$$

(NOT OBVIOUS!)

SPECIAL CASE OF <u>LINEAR</u> <u>MODEL</u>.

<u>VECTOR CRLB – TRANSFORMATIONS</u>

WISH TO ESTIMATE $\underset{r \times 1}{\underline{\alpha}} = \underline{g}(\underset{p \times 1}{\underline{\theta}})$

CRLB IS

$$\mathbf{C}_{\hat{\alpha}} - \frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}}^T \geq 0$$

OR

$$\mathrm{VAR}(\hat{\alpha}_i) \geq \left[\frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}}^T \right]_{ii}$$

WHERE $\frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}}$ IS JACOBIAN OR

$$\frac{\partial \mathbf{g}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} & \cdots & \frac{\partial g_1}{\partial \theta_p} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} & \cdots & \frac{\partial g_2}{\partial \theta_p} \\ \frac{\partial g_r}{\partial \theta_1} & \frac{\partial g_r}{\partial \theta_2} & \cdots & \frac{\partial g_r}{\partial \theta_p} \end{bmatrix} \quad r \times p$$

EXAMPLE : $x(n) = A + w(n)$

$\uparrow$ WGN - VARIANCE $= \sigma^2$

$\boldsymbol{\theta} = [A \ \sigma^2]^T$ UNKNOWN

FIND CRLB FOR $\alpha = A^2/\sigma^2$ = SNR

$$g(\boldsymbol{\theta}) = \theta_1^2/\theta_2 \qquad r = 1,\ p = 2$$

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} N/\sigma^2 & 0 \\ 0 & \frac{N}{2\sigma^4} \end{bmatrix}$$ SHOW THIS

$$\frac{\partial \underline{g}}{\partial \underline{\theta}} = \left[\frac{\partial g_1}{\partial \theta_1} \quad \frac{\partial g_1}{\partial \theta_2} \right]$$

$$= \left[\frac{2\theta_1}{\theta_2} \quad -\frac{\theta_1^2}{\theta_2^2} \right]$$

$$= \left[\frac{2A}{\sigma^2} \quad -\frac{A^2}{\sigma^4} \right]$$

$$\mathrm{VAR}(\hat{\alpha}) \geq \frac{\partial \underline{g}}{\partial \underline{\theta}} \, \underline{I}^{-1}(\theta) \, \frac{\partial \underline{g}}{\partial \underline{\theta}}^T$$

$$= \left[\frac{2A}{\sigma^2} \quad -\frac{A^2}{\sigma^4} \right] \begin{bmatrix} \sigma^2/N & 0 \\ 0 & \frac{2\sigma^4}{N} \end{bmatrix} \begin{bmatrix} \frac{2A}{\sigma^2} \\ -\frac{A^2}{\sigma^4} \end{bmatrix}$$

$$= \frac{4A^2}{N\sigma^2} + \frac{2A^4}{N\sigma^4} = \frac{2}{N}\left(2\alpha + \alpha^2\right)$$

CRLB FOR GENERAL GAUSSIAN CASE

$$\underline{x} \sim N\left(\underline{\mu}(\underline{\theta}), \underline{C}(\underline{\theta})\right)$$

DEPENDS ON $\underline{\theta}$

$$[\underline{I}(\underline{\theta})]_{ij} = \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_i}\right]^T \underline{C}^{-1}(\underline{\theta}) \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_j}\right]$$

$$+ \frac{1}{2} \mathrm{TR}\left[\underline{C}^{-1}(\underline{\theta}) \frac{\partial \underline{C}(\theta)}{\partial \theta_i} \underline{C}^{-1}(\underline{\theta}) \frac{\partial \underline{C}(\theta)}{\partial \theta_j} \right]$$

WHERE
$$\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [\underline{\mu}(\underline{\theta})]_1}{\partial \theta_i} \\ \vdots \\ \frac{\partial [\underline{\mu}(\underline{\theta})]_N}{\partial \theta_i} \end{bmatrix} \quad N \times 1$$

$$\frac{\partial \underline{C}(\underline{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [\underline{C}]_{11}}{\partial \theta_i} & \frac{\partial [\underline{C}]_{12}}{\partial \theta_i} & \cdots & \frac{\partial [\underline{C}]_{1N}}{\partial \theta_i} \\ \vdots & & & \\ \frac{\partial [\underline{C}]_{N1}}{\partial \theta_i} & & \cdots & \frac{\partial [\underline{C}]_{NN}}{\partial \theta_i} \end{bmatrix}$$

$$N \times N$$

FOR SCALAR CASE :

$$\underline{X} \sim N(\underline{\mu}(\theta), \underline{C}(\theta))$$

$$I(\theta) = \left[\frac{\partial \underline{\mu}(\theta)}{\partial \theta}\right]^T \underline{C}^{-1}(\theta) \left[\frac{\partial \underline{\mu}(\theta)}{\partial \theta}\right] + \frac{1}{2} \, TR\left[\left(\underline{C}^{-1}(\theta) \frac{\partial \underline{C}(\theta)}{\partial \theta}\right)^2\right]$$

<u>EXAMPLE</u> : IF $x[n] = s[n; \underline{\theta}] + w[n]$

↑ WGN WITH VARIANCE σ^2

$$\Rightarrow \quad \underline{X} = \underline{s}(\underline{\theta}) + \underline{w}$$

$$\sim N(\underline{s}(\underline{\theta}), \sigma^2 \underline{I})$$

$\underline{C}$ DOES NOT DEPEND ON $\underline{\theta}$. $\underline{C}^{-1} = 1/\sigma^2 \, \underline{I}$

$$\Rightarrow \quad [\underline{I}(\underline{\theta})]_{ij} = \frac{1}{\sigma^2} \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_i} \right]^T \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_j} \right]$$

$$= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \frac{\partial s[n;\underline{\theta}]}{\partial \theta_i} \frac{\partial s[n;\underline{\theta}]}{\partial \theta_j}$$

EXTENDS RESULT FOR SCALAR PARAMETER.

<u>EXAMPLE</u>: $x[n] = w[n]$

↑ WGN WITH VARIANCE σ^2

IF $\theta = \sigma^2 \Rightarrow \underline{x} \sim N(\underline{0}, \sigma^2 \underline{I})$

$$\underline{C}(\theta) = \theta \, \underline{I} = \sigma^2 \underline{I}$$

$$I(\sigma^2) = \frac{1}{2} \operatorname{TR} \left[\left(\underline{C}^{-1}(\sigma^2) \frac{\partial \underline{C}(\sigma^2)}{\partial \sigma^2} \right)^2 \right]$$

$$= \frac{1}{2} \operatorname{TR} \left[\left(\frac{1}{\sigma^2} \underline{I} \, \underline{I} \right)^2 \right]$$

$$= \frac{1}{2\sigma^4} \operatorname{TR}(\underline{I}) = \frac{N}{2\sigma^4}$$

<u>ASYMPTOTIC CRLB - WSS PROCESSES</u>

IF $x[n]$ IS ZERO MEAN, GAUSSIAN AND HAS AUTOCORRELATION FUNCTION