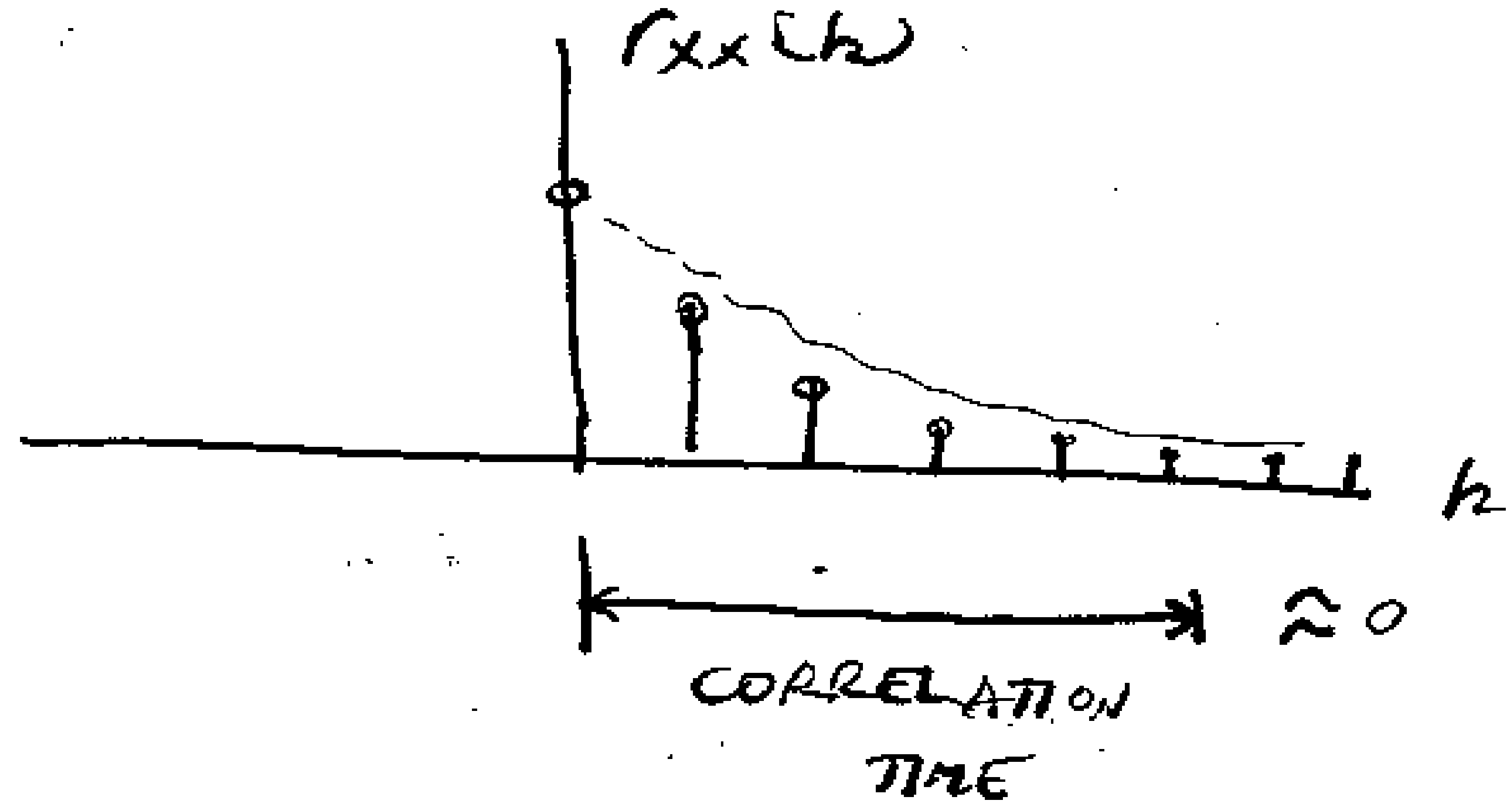
$$r_{xx}[k] = E(x[n]x[n+k])$$

AND $r_{xx}[k] \to 0$ AS $k \to \infty$, WE CAN WRITE DOWN AN APPROXIMATE CRLB.

VALID FOR $N \to \infty$ OR IN PRACTICE FOR $N \gg$ CORRELATION TIME.

CORRELATION TIME IS



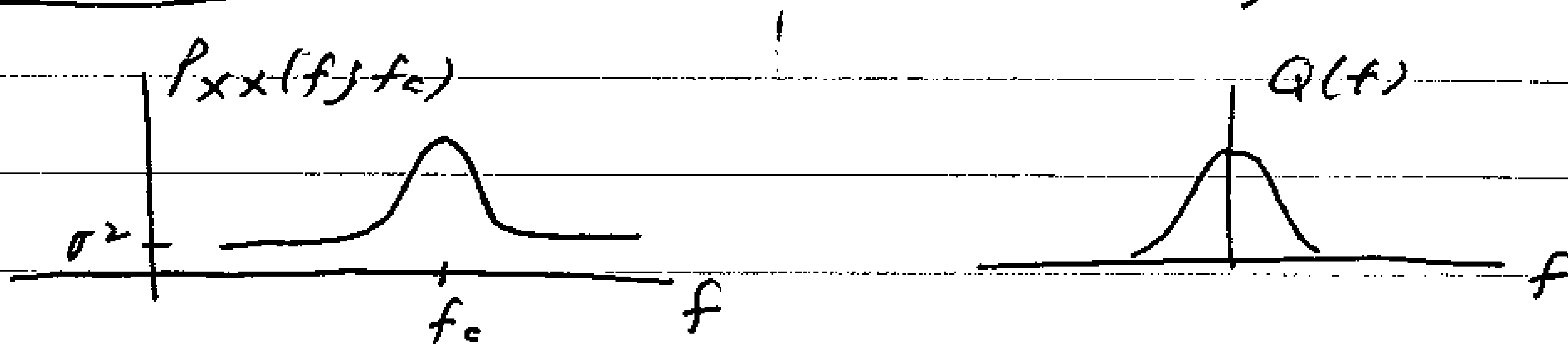
$\Rightarrow$ FOR PROCESSES WITH BROAD PSDs THIS WILL WORK WELL.

$$[I(\theta)]_{ij} = \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \ln P_{xx}(f;\theta)}{\partial \theta_i} \frac{\partial \ln P_{xx}(f;\theta)}{\partial \theta_j}\, df$$

WHERE $P_{xx}(f;\theta) = \mathcal{F}\{r_{xx}[k]\}$
$= PSD$

HOW GOOD IS APPROXIMATION?

EXAMPLE: CENTER FREQUENCY, f_c



$$P_{xx}(f;f_c) = Q(f-f_c) + Q(-f-f_c) + \sigma^2$$

f_c TAKES ON VALUES SO THAT
$Q(f-f_c)$ IS ALWAYS WITHIN $[0, 1/2]$

FOR SCALAR PARAMETER

$$I(\theta) = \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_{xx}(f;\theta)}{\partial \theta}\right)^2 df$$

$$\operatorname{VAR}(\hat{f}_c) \geq \frac{1}{\frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln P_{xx}(f;f_c)}{\partial f_c}\right)^2}$$

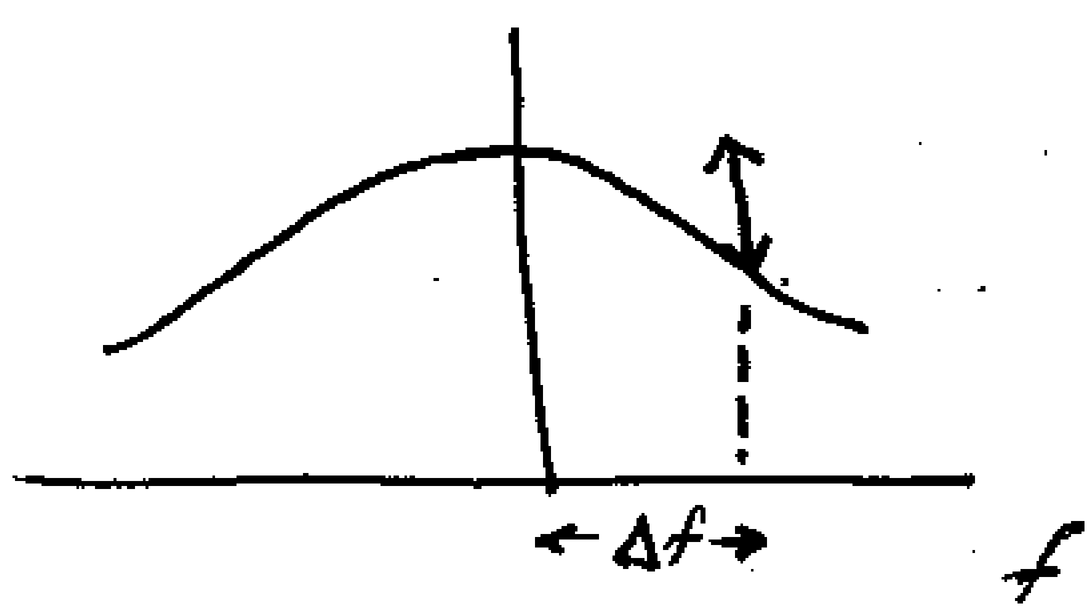
SEE BOOK

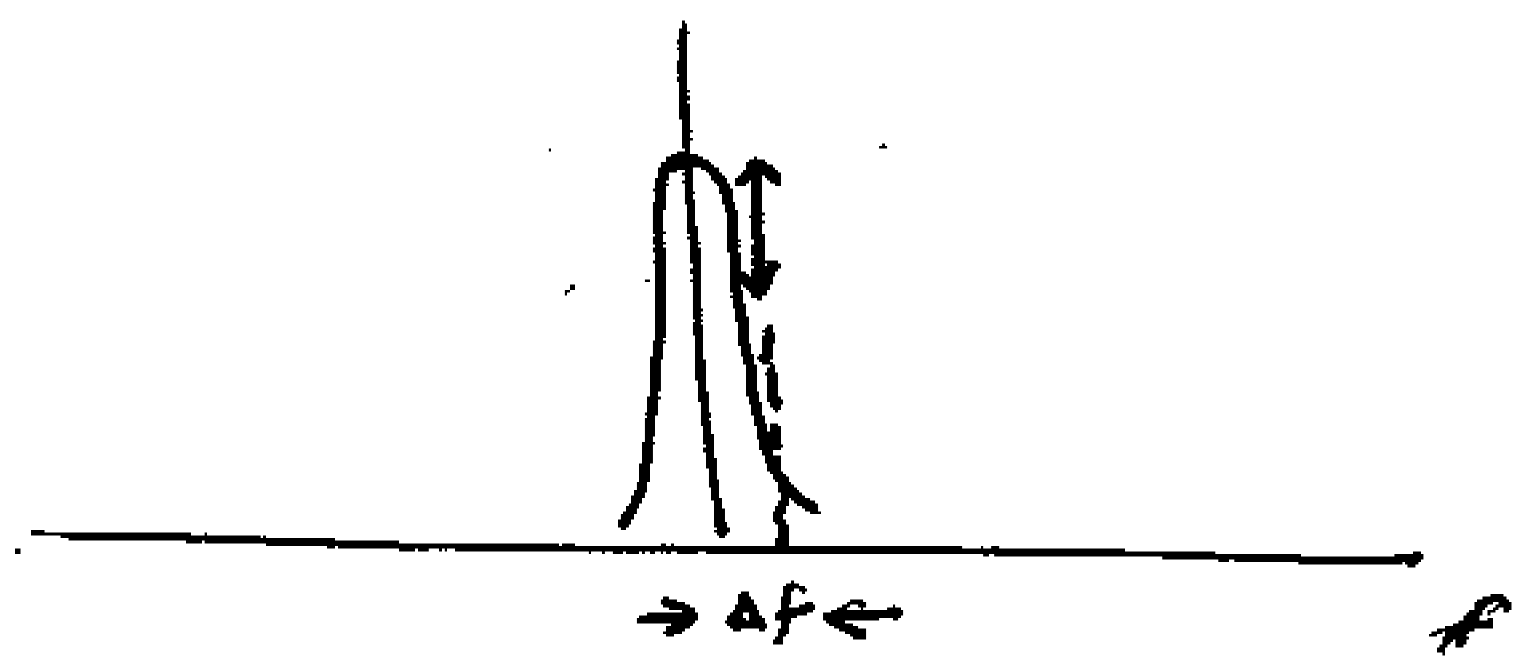
$$= \frac{1}{N\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial \ln(Q(f)+\sigma^2)}{\partial f}\right)^2 df}$$

FOR $Q(f) = e^{-\frac{1}{2}(f/\sigma_f)^2}$ AND $Q(f) >> \sigma^2$

$$VAR(\hat{f_c}) \geq \frac{12\sigma_f^4}{N}$$

NARROWER PSDS $\Rightarrow$ BETTER ACCURACY

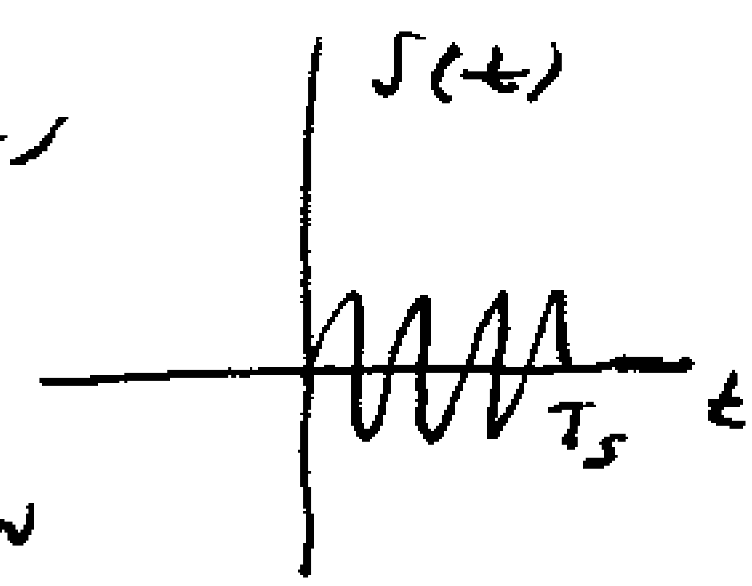




SEE SIGNAL PROCESSING EXAMPLES.

RANGE ESTIMATION

ACTIVE SONAR - TRANSMIT PULSE $s(t)$



RECEIVE $x(t) = s(t-t_0) + w(t)$

ROUND TRIP DELAY

GAUSSIAN NOISE

$P_{ww}(F)$

$\frac{N_0}{2}$

$-B$ B F

CONTINUOUS-TIME FREQ.

$t_0 = 2R/c$ $\quad$ R = RANGE

ASSUME MAX. TIME DELAY = τ_{MAX}

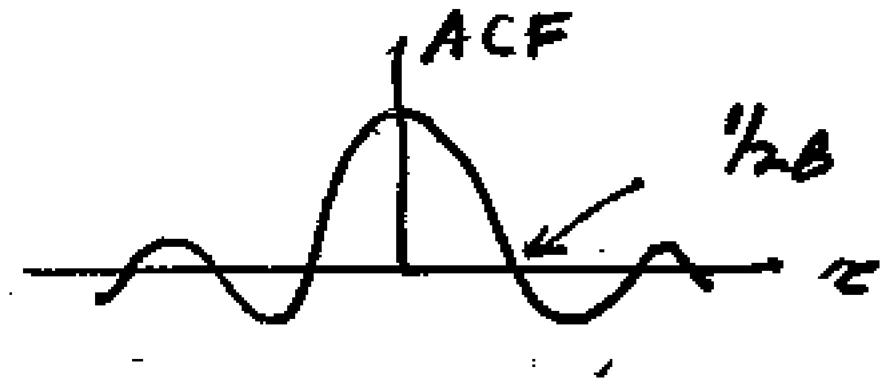
$\Rightarrow$ OBSERVATION INTERVAL $\quad 0 \le t \le T = \tau_{MAX} + T_s$

SAMPLE WAVEFORM AT $t = n\Delta = \frac{n}{2B}$

$$x(n\Delta) = s(n\Delta - t_0) + w[n]$$

$\uparrow$ WGN SINCE

$$\frac{N_0 B \sin 2\pi \tau B}{2\pi \tau B}$$



$\Rightarrow$ NOISE SAMPLES ARE UNCORRELATED WITH $\sigma^2 = N_0 B$.

$N \rightarrow$

$$x[n] = s(n\Delta - t_0) + w[n] = \begin{cases} w[n] & 0 \le n \le n_0 - 1 \\ s(n\Delta - t_0) + w[n] & n_0 \le n \le n_0 + M - 1 \\ w[n] & n_0 + M \le n \le N - 1 \end{cases}$$

M = LENGTH OF SIGNAL IN SAMPLES

n_0 = DELAY IN SAMPLES

WE USE $\quad \mathrm{VAR}(\hat{\theta}) \ge \dfrac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta} \right)^2}$

$$VAR(\hat{t}_0) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial S(n; t_0)}{\partial t_0} \right)^2}$$

$$= \frac{\sigma^2}{\sum_{n=n_0}^{n_0+M-1} \left(\frac{\partial S(n\Delta - t_0)}{\partial t_0} \right)^2}$$

$$= \frac{\sigma^2}{\sum_{n=n_0}^{n_0+M-1} \left(\frac{\partial S(t)}{\partial t}\Big|_{t=n\Delta - t_0} \right)^2}$$

$$= \frac{\sigma^2}{\sum_{n=0}^{M-1} \left(\frac{\partial S(t)}{\partial t}\Big|_{t=n\Delta} \right)^2}$$

SINCE $t_0 = n_0 \Delta$.

WE CAN EXPRESS THIS APPROXIMATELY AS

$$VAR(\hat{t}_0) \geq \frac{\sigma^2}{\frac{1}{\Delta} \sum_{n=0}^{M-1} \left(\frac{ds(t)}{dt} \right)^2 \Big|_{t=n\Delta} \Delta}$$

$$\approx \frac{\sigma^2}{\frac{1}{\Delta} \int_0^{T_s} \left(\frac{ds(t)}{dt} \right)^2 dt}$$

$$= \frac{N_0/2}{\int_0^{T_s} \left(\frac{ds(t)}{dt} \right)^2 dt}$$

SINCE $\sigma^2 \Delta = \frac{N_0 B}{2B} = N_0/2$

ALSO, $\mathcal{E} = \text{ENERGY} \triangleq \int_0^{T_s} s^2(t)\,dt$

$$VAR(\hat{t}_0) \geq \frac{1}{\frac{\mathcal{E}}{N_0/2}\,\overline{F^2}}$$

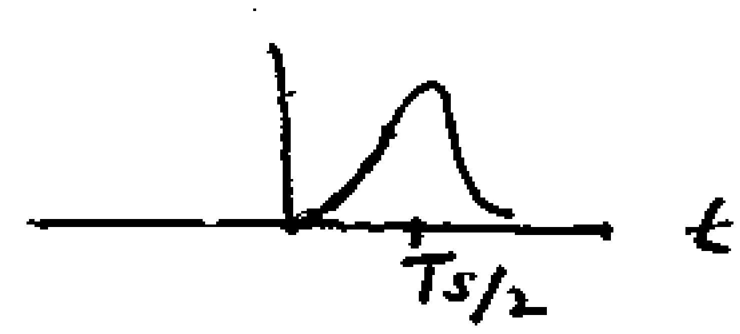
WHERE
$$\overline{F^2} = \frac{\int_0^{T_s} \left(\frac{ds(t)}{dt}\right)^2 dt}{\int_0^{T_s} s^2(t)\,dt}$$

$$= \frac{\int_{-\infty}^{\infty} (2\pi F)^2 \,|S(F)|^2\, dF}{\int_{-\infty}^{\infty} |S(F)|^2\, dF}$$

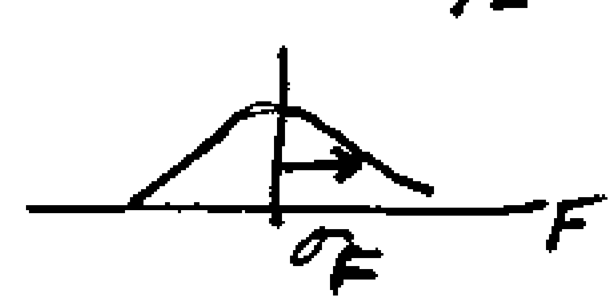
$S(F) = \mathcal{F}\{s(t)\}$

$\overline{F^2}$ IS THE MEAN SQUARE BANDWIDTH

IF $s(t) = e^{-\frac{1}{2}\sigma_F^2 (t - T_s/2)^2}$



$|S(F)|^2 = \frac{\sigma_F}{\sqrt{2\pi}}\, e^{-2\pi^2 F^2/\sigma_F^2}$



$\Rightarrow \overline{F^2} = \sigma_F^2/2$

TO FIND CRLB FOR RANGE

$R = c\, t_0/2$

$$\Rightarrow \quad \mathrm{VAR}(\hat{R}) \geq \frac{c^2/4}{\frac{\mathcal{E}}{N_0/2}\,\overline{F^2}}$$

FOR GOOD RANGE ESTIMATION

1) LARGE $\overline{F^2}$ $\Rightarrow$ LARGE BANDWIDTH (WHY?)

2) LARGE $\mathcal{E}/N_0/2$ OR SNR

LINEAR MODELS

RECALL LINE FITTING

$$x[n] = A + Bn + w[n] \qquad n = 0, 1, \ldots, N-1$$

$\uparrow$ WGN

ESTIMATE SLOPE B AND INTERCEPT A

IN MATRIX FORM

$$\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} A \\ B \end{bmatrix}}_{\underline{\theta}} + \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}}_{\underline{w}}$$

$\underline{H}$ CALLED OBSERVATION MATRIX

$\underline{\theta}$ = VECTOR PARAMETER TO BE ESTIMATED

$$\underset{\text{NORMAL OR GAUSSIAN}}{\underline{w} \sim \mathcal{N}}(\underbrace{\underline{0}}_{\text{MEAN}}, \underbrace{\sigma^2 \underline{I}}_{\text{COVARIANCE}})$$

$\underline{x}$ = DATA

IN GENERAL

$$\underbrace{\underline{x}}_{N\times 1} = \underbrace{\underline{H}}_{\substack{N\times p\\ \text{KNOWN}}}\ \underbrace{\underline{\theta}}_{p\times 1} + \underbrace{\underline{w}}_{N\times 1} \sim \mathcal{N}(\underline{0}, \sigma^2 \underline{I})$$

TERMED THE LINEAR MODEL. IMPORTANT BECAUSE WE CAN EASILY FIND MVU ESTIMATOR.

NOW $\underline{x} \sim \mathcal{N}(\underline{H}\underline{\theta}, \sigma^2 \underline{I})$

$$p(\underline{x};\underline{\theta}) = \frac{1}{(2\pi)^{N/2} |\sigma^2 \underline{I}|^{1/2}} e^{-\frac{1}{2\sigma^2}(\underline{x}-\underline{H}\underline{\theta})^T(\underline{x}-\underline{H}\underline{\theta})}$$

$$\frac{\partial \ln p}{\partial \underline{\theta}} = -\frac{1}{2\sigma^2}\frac{\partial}{\partial \underline{\theta}}\left[\underline{x}^T\underline{x} - 2\underline{x}^T\underline{H}\underline{\theta} + \underline{\theta}^T\underline{H}^T\underline{H}\underline{\theta}\right]$$

BUT $\dfrac{\partial \underline{b}^T\underline{\theta}}{\partial \underline{\theta}} = \underline{b}$ $\qquad \dfrac{\partial \underline{\theta}^T\underline{A}\underline{\theta}}{\partial \underline{\theta}} = 2\underline{A}\underline{\theta}$ (SHOW THIS)

FOR $\underline{A}$ SYMMETRIC

$$\Rightarrow \quad \frac{\partial \ln p}{\partial \underline{\theta}} = \frac{1}{\sigma^2}\left(\underline{H}^T\underline{x} - \underline{H}^T\underline{H}\underline{\theta}\right)$$

NOW ASSUME $\underline{H}^T\underline{H}$ IS INVERTIBLE (WILL BE IFF $\underline{H}$ IS FULL RANK $= p$)

$$\frac{\partial \ln p}{\partial \underline{\theta}} = \underbrace{\frac{\underline{H}^T\underline{H}}{\sigma^2}}_{\underline{I}(\underline{\theta})}\left(\underbrace{(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}}_{\hat{\underline{\theta}}} - \underline{\theta}\right)$$

BY CRLB THEOREM!

$$\Rightarrow \boxed{\begin{array}{l} \hat{\underline{\theta}} = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x} \quad \text{- IS MVU ESTIMATOR (AND EFFICIENT)} \\ \underline{C}_{\hat{\theta}} = \underline{I}^{-1}(\underline{\theta}) = \sigma^2(\underline{H}^T\underline{H})^{-1} \end{array}}$$

CAN NOW VERIFY LINE FITTING RESULTS GIVEN PREVIOUSLY.

<u>ASIDE</u> : FOR $\underline{H}$ TO BE FULL RANK COLUMNS MUST BE LINEARLY INDEPENDENT

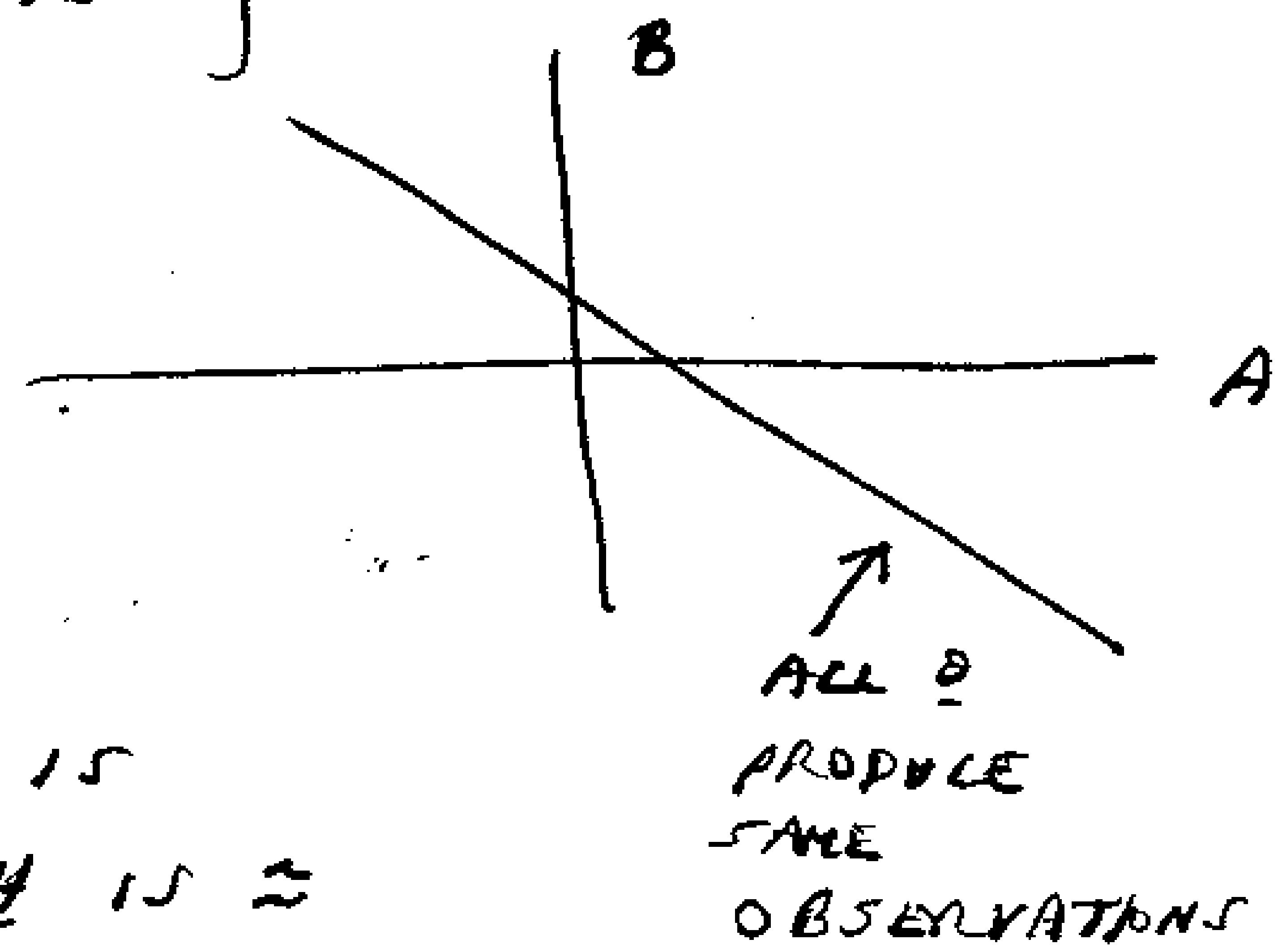
$$\underline{H} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}$$

NOT L. I. $\Rightarrow$ $(\underline{H}^T\underline{H})^{-1}$ DOES NOT EXIST

FOR NO NOISE CANNOT IDENTIFY PARAMETERS.

$$\underline{x} = \underline{H}\,\underline{\theta} = \begin{bmatrix} A+B \\ A+B \\ \vdots \\ A+B \end{bmatrix}$$

IF $\underline{x} = [2\ 2 \ldots 2]^T$



IN PRACTICE, IF $\underline{H}^T\underline{H}$ IS ILL CONDITIONED OR $\underline{H}$ IS $\doteq$ RANK DEFICIENT

$$\underline{C}_{\hat{\theta}} = \sigma^2 (\underline{H}^T\underline{H})^{-1} \to \infty$$

CANNOT ESTIMATE $\underline{\theta}$ RELIABLY.

EXAMPLES

FOURIER ANALYSIS

SUSPECT DATA CONSISTS OF STRONG CYCLICAL

COMPONENTS IN NOISE

$$x[n] = \sum_{k=1}^{M} a_k \cos \frac{2\pi k n}{N} + \sum_{k=1}^{M} b_k \sin \frac{2\pi k n}{N} + w[n]$$

$$n = 0, 1, \ldots, N-1$$

FREQUENCIES ARE $f_k = k/N$, HARMONICALLY RELATED. AMPLITUDES a_k, b_k TO BE ESTIMATED.

$$\underline{\theta} = [a_1 \ldots a_M \; b_1 \ldots b_M]^T$$

AS A LINEAR MODEL WE HAVE

$$\underline{x} = \underline{H}\underline{\theta} + \underline{w}$$

$$\underline{H} = \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ \cos 2\pi/N & & \cos \frac{2\pi}{N} M & \sin \frac{2\pi}{N} & \ldots & \sin \frac{2\pi}{N} M \\ \vdots & & \vdots & & & \\ \cos \frac{2\pi}{N}(N-1) & \ldots & \cos \frac{2\pi}{N} M(N-1) & \sin \frac{2\pi}{N}(N-1) & \ldots & \sin \frac{2\pi}{N} \cdot M(N-1) \end{bmatrix}$$

$$N \times 2M = p \qquad (p < N)$$

TO FIND $\underline{H}^T \underline{H}$ LET $\underline{H} = [\underline{h}_1 \; \underline{h}_2 \; \ldots \; \underline{h}_{2M}]$

COLUMNS

$$\underline{H}^T\underline{H} = \begin{bmatrix} \underline{h}_1^T \\ \vdots \\ \underline{h}_{2M}^T \end{bmatrix} [\underline{h}_1\ \underline{h}_2 \ldots \underline{h}_{2M}]$$

$$= \begin{bmatrix} \underline{h}_1^T\underline{h}_1 & \underline{h}_1^T\underline{h}_2 & \ldots & \underline{h}_1^T\underline{h}_{2M} \\ \underline{h}_2^T\underline{h}_1 & \underline{h}_2^T\underline{h}_2 & \ldots & \underline{h}_2^T\underline{h}_{2M} \\ & & & \\ \underline{h}_{2M}^T\underline{h}_1 & \underline{h}_{2M}^T\underline{h}_2 & & \underline{h}_{2M}^T\underline{h}_{2M} \end{bmatrix}$$

BUT $\underline{h}_i^T\underline{h}_j = 0 \quad i \neq j$ (COLUMNS ARE ORTHOGONAL)

SINCE

$$\sum_{n=0}^{N-1} \cos f_i n \cos f_j n = \frac{N}{2}\delta_{ij}$$

$= 0 \quad i \neq j$
$1 \quad i = j$
(KRONECKER DELTA)

$$\sum_{n=0}^{N-1} \sin f_i n \sin f_j n = \frac{N}{2}\delta_{ij}$$

$$\sum_{n=0}^{N-1} \cos f_i n \sin f_j n = 0 \quad \text{ALL } i, j$$

DUE TO CHOICE OF FREQUENCIES.

$$\Rightarrow \quad \underline{H}^T\underline{H} = \frac{N}{2}\underline{I}$$

$$\hat{\underline{\theta}} = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{x}$$

$$= \frac{2}{N}\begin{bmatrix} \underline{h}_1^T \\ \vdots \\ \underline{h}_{2M}^T \end{bmatrix}\underline{x} = \begin{bmatrix} \frac{2}{N}\underline{h}_1^T\underline{x} \\ \vdots \\ \frac{2}{N}\underline{h}_{2M}^T\underline{x} \end{bmatrix}$$

OR

$$\hat{a}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi}{N} kn$$

$$\hat{b}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \sin \frac{2\pi}{N} kn$$

USUAL DISCRETE FOURIER TRANSFORM COEFFICIENTS

TO FIND PDF OF MVU ESTIMATOR

$$\underline{\hat{\theta}} = (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{x}$$

$$E(\underline{\hat{\theta}}) = (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underbrace{E(\underline{x})}_{\underline{H}\underline{\theta}} = \underline{\theta} \qquad \text{UNBIASED}$$

$$\underline{C}_{\hat{\theta}} = E\left[(\underline{\hat{\theta}} - \underline{\theta})(\underline{\hat{\theta}} - \underline{\theta})^T\right]$$

$$= E\left[\left((\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{x} - (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{H}\,\underline{\theta}\right)\left(\qquad\right)^T\right]$$

$$= E\left[\left((\underline{H}^T \underline{H})^{-1} \underline{H}^T (\underline{x} - \underline{H}\underline{\theta})\right)\left(\qquad\right)^T\right]$$

$$= E\left[(\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{w}\, \underline{w}^T \underline{H} (\underline{H}^T \underline{H})^{-1}\right]$$

$$= (\underline{H}^T \underline{H})^{-1} \underline{H}^T \sigma^2 \underline{I}\, \underline{H} (\underline{H}^T \underline{H})^{-1}$$

$$= \sigma^2 (\underline{H}^T \underline{H})^{-1}$$ AS EXPECTED

$\hat{\underline{\theta}} \sim$ GAUSSIAN SINCE IT IS LINEAR FUNCTION OF $\underline{x}$

$$\Rightarrow \quad \hat{\underline{\theta}} \sim N(\underline{\theta}, \sigma^2 (\underline{H}^T \underline{H})^{-1})$$

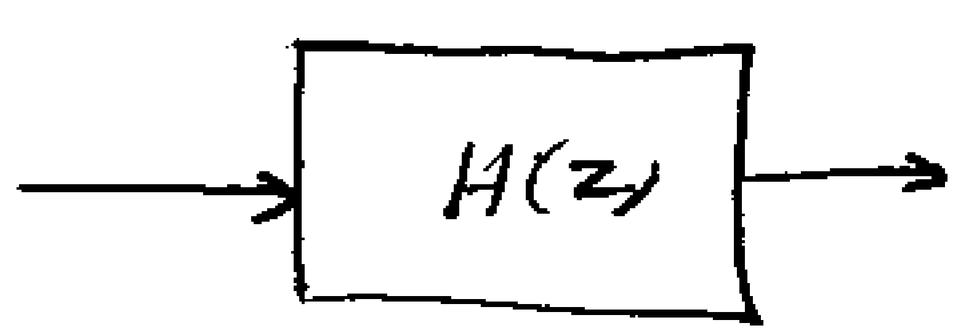
FOR FOURIER ANALYSIS

$$\hat{\underline{\theta}} \sim N(\underline{\theta}, \tfrac{2\sigma^2}{N} \underline{I})$$

OR $\left.\begin{array}{l} \hat{a}_k \sim N(a_k, 2\sigma^2/N) \\ \hat{b}_k \sim N(b_k, 2\sigma^2/N) \end{array}\right\}$ ALL INDEPENDENT

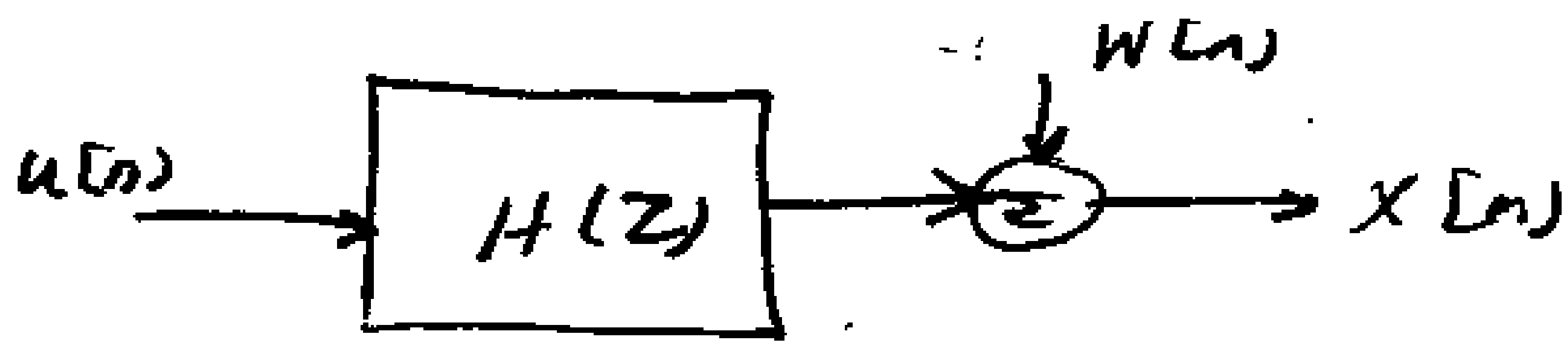
SYSTEM IDENTIFICATION

SIMPLE MODEL FOR LINEAR SYSTEM IS TAPPED DELAY LINE OR FIR FILTER



$$H(z) = \sum_{k=0}^{p-1} h(k) z^{-k}$$

ACCOUNTING FOR NOISE AT OUTPUT



WE PROVIDE $u[n]$ AND MEASURE $x[n]$. WISH TO ESTIMATE TAP WEIGHTS $h[k]$

IF $u[n]$ IS PROVIDED FOR $n = 0, 1, \ldots, N-1$ AND $u[n] = 0$ $n < 0$, WE OBSERVE

$$x[n] = \sum_{k=0}^{p-1} h[k]\, u[n-k] + w[n] \qquad n = 0, 1, \ldots, N-1$$

OR

$$\underline{x} = \underbrace{\begin{bmatrix} u[0] & 0 & \ldots & 0 \\ u[1] & u[0] & \ldots & 0 \\ & & \ddots & \\ u[N-1] & u[N-2] & \ldots & u[N-1-p] \end{bmatrix}}_{\underline{H}} \underbrace{\begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[p-1] \end{bmatrix}}_{\underline{\theta}} + \underline{w}$$

IF $w[n]$ IS WGN WITH VARIANCE σ^2, $\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$ AND WE HAVE LINEAR MODEL

$$\Rightarrow \quad \hat{\underline{\theta}} = (\underline{H}^T \underline{H})^{-1} \underline{H}^T \underline{x}$$

IS MVU ESTIMATOR AND EFFICIENT

$$\underline{C}_{\hat{\theta}} = \sigma^2 (\underline{H}^T \underline{H})^{-1}$$

NOTE THAT HOW WELL WE CAN ESTIMATE THE TAP WEIGHTS DEPENDS ON $\underline{C}_{\hat{\theta}}$ OR $\underline{H}$ OR $u[n]$ SEQUENCE CHOSEN.

PROBLEM: CHOOSE $u[n]$ TO MINIMIZE
$VAR(\hat{\theta}_i) = [\underline{C}_{\hat{\theta}}]_{ii} \quad i = 1, 2, \ldots, p$
SUBJECT TO CONSTRAINT $\sum_{n=0}^{N-1} u^2[n]$ IS FIXED (WHY?)

SOLUTION: SEE TEXT

CHOOSE $u[n]$ SO THAT

$$r_{uu}[k] = \frac{1}{N} \sum_{n=0}^{N-1-|k|} u[n]\, u[n+|k|]$$

$$= 0 \quad k \neq 0$$

OR $u[n]$ SHOULD BE WHITE NOISE, ACTUALLY WE USE PSEUDORANDOM NOISE. UNDER THESE CONDITIONS WE HAVE

$$\hat{h}[i] = \frac{\frac{1}{N} \sum_{n=0}^{N-1-i} u[n]\, x[n+i]}{r_{uu}[0]}$$

OR IF $r_{ux}[i] = \frac{1}{N} \sum_{n=0}^{N-1-i} u[n]\, x[n+i]$

$$\hat{h}[i] = \frac{r_{ux}[i]}{r_{uu}[0]} \qquad i = 0, 1, \ldots, p-1$$

$$= \frac{\text{CROSSCORRELATION}}{\text{AUTOCORRELATION}}$$

ALSO, $\text{VAR}(\hat{h}[i]) = \dfrac{1}{N r_{uu}[0]/\sigma^2}$

$$= \frac{1}{\mathcal{E}/\sigma^2} \qquad \mathcal{E} = \text{ENERGY}$$

EXTENSIONS TO LINEAR MODEL

A MORE GENERAL MODEL ASSUMES

$$\underline{x} = \underline{H}\,\underline{\theta} + \underline{w}$$

$$\uparrow N(\underline{0}, \underline{C}) \qquad \underline{C} \neq \sigma^2 \underline{I}$$

TO FIND MVU ESTIMATOR

1) REPEAT $\frac{\partial \ln p}{\partial \underline{\theta}}$ DERIVATION

2) WHITENING APPROACH

BY WHITENING APPROACH,

FACTOR $\underline{C}^{-1}$ AS $\underline{D}^T \underline{D}$

$\underline{D}$ IS NxN INVERTIBLE MATRIX

(ALWAYS POSSIBLE FOR

POSITIVE DEFINITE $\underline{C}^{-1}$ $\Rightarrow$

CHOLESKY DECOMPOSITION

AS EXAMPLE)

$\underline{D}$ IS A WHITENING MATRIX SINCE

IF $\underline{w}' = \underline{D}\,\underline{w}$

$$
\begin{aligned}
E(\underline{w}'\underline{w}'^T) &= E(\underline{D}\,\underline{w}\,\underline{w}^T\underline{D}^T) \\
&= \underline{D}\,\underline{C}\,\underline{D}^T \\
&= \underline{D}\,(\underline{D}^T\underline{D})^{-1}\underline{D}^T \\
&= \underline{D}\,\underline{D}^{-1}\,\underline{D}^{T^{-1}}\,\underline{D}^T = \underline{I}
\end{aligned}
$$

LET $\underline{x}' = \underline{D}\,\underline{x}$

$$
\begin{aligned}
\Rightarrow \quad \underline{x}' &= D(\underline{H}\,\underline{\theta} + \underline{w}) \\
&= \underline{D}\,\underline{H}\,\underline{\theta} + \underline{w}' \\
&= \underline{H}'\underline{\theta} + \underline{w}'
\end{aligned}
$$

$\uparrow N(\underline{0}, \underline{I})$

THIS IS THE LINEAR MODEL. THUS,

$$\hat{\underline{\theta}} = (\underline{H}'^T \underline{H}')^{-1} \underline{H}'^T \underline{x}'$$

$$= \left[(\underline{D}\underline{H})^T \underline{D}\underline{H}\right]^{-1} (\underline{D}\underline{H})^T \underline{D}\, \underline{x}$$

$$= (\underline{H}^T \underline{D}^T \underline{D} \underline{H})^{-1} \underline{H}^T \underline{D}^T \underline{D}\, \underline{x}$$

$$= (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1} \underline{H}^T \underline{C}^{-1} \underline{x}$$

ALSO, $\underline{C}_{\hat{\theta}} = (\underline{H}'^T \underline{H}')^{-1}$

$$= (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1}$$

CAN NOW HANDLE ESTIMATION OF SIGNAL PARAMETERS IN <u>COLORED</u> NOISE ENVIRONMENT.

<u>EXAMPLE</u> : $x(n) = A + w(n) \qquad n = 0, 1, \ldots, N-1$

$w(n)$ IS COLORED GAUSSIAN NOISE WITH $N \times N$ COVARIANCE MATRIX $\underline{C}$. THE MVU ESTIMATOR (AND EFFICIENT) OF A IS

$$\hat{A} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1} \underline{H}^T \underline{C}^{-1} \underline{x}$$

WHERE $\underline{H} = [1\ 1 \ldots 1]^T = \underline{1}^T$ SINCE

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} A + \underline{w}$$

OR $\hat{A} = \dfrac{\underline{1}^T \underline{C}^{-1} \underline{x}}{\underline{1}^T \underline{C}^{-1} \underline{1}}$

THE VARIANCE IS

$$\underline{C}_{\hat{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1}$$

$$VAR(\hat{A}) = \frac{1}{\underline{1}^T \underline{C}^{-1} \underline{1}}$$

WE NOW NO LONGER HAVE $\hat{A} = \bar{x}$ DUE TO NOISE CORRELATION.

RECALL $\underline{C}^{-1} = \underline{D}^T \underline{D}$ $\quad \underline{D}$ = WHITENER

$$\hat{A} = \frac{\underline{1}^T \underline{D}^T \underline{D} \underline{x}}{\underline{1}^T \underline{D}^T \underline{D} \underline{1}} = \frac{(\underline{D}\underline{1})^T \underline{x}'}{\underline{1}^T \underline{D}^T \underline{D} \underline{1}} \leftarrow \underline{d}^T$$

LET $d_n = \dfrac{(\underline{D}\underline{1})_n}{\underline{1}^T \underline{D}^T \underline{D} \underline{1}}$

$$\hat{A} = \sum_{n=0}^{N-1} d_n \, x'[n]$$

WE FIRST WHITEN AND THEN "AVERAGE"
(WHITENING CHANGES SIGNAL AS WELL
AS NOISE $\Rightarrow \hat{A} \neq 1/N \sum_{n=0}^{N-1} x'[n]$)