$$= \Pr\{x(n) \le \zeta\}^N \qquad \text{(IDENTICALLY DISTRIBUTED)}$$

TO FIND PDF:

$$p(\zeta) = \frac{d \Pr\{T \le \zeta\}}{d\zeta}$$

$$= N \Pr\{x(n) \le \zeta\}^{N-1} \frac{d \Pr\{x(n) \le \zeta\}}{d\zeta}$$

BUT $\dfrac{d \Pr\{x(n) \le \zeta\}}{d\zeta}$ = PDF OF $x(n)$

$$= \begin{cases} \dfrac{1}{\beta} & 0 < \zeta < \beta \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\Pr\{x(n) \le \zeta\} = \int_{-\infty}^{\zeta} \quad dx(n)$$

$$= \begin{array}{ll} 0 & \zeta < 0 \\ \zeta/\beta & 0 < \zeta < \beta \\ 1 & \zeta > \beta \end{array}$$

$$\therefore \quad p(\zeta) = \begin{cases} 0 & \zeta < 0 \\ N\left(\zeta/\beta\right)^{N-1} \dfrac{1}{\beta} & 0 < \zeta < \beta \\ 0 & \zeta > \beta \end{cases}$$

PDF OF MAX $x(n)$

NOW LET THE PDF BE $p(T)$

$$\Rightarrow \quad E(T) = \int_{-\infty}^{\infty} T p(T)\, dT$$

$$= \int_0^{\beta} T N \left(\frac{T}{\beta}\right)^{N-1} \frac{1}{\beta}\, dT$$

$$= \frac{N}{N+1}\beta = \frac{2N}{N+1}\theta$$

HENCE, TO MAKE STATISTIC UNBIASED

$$\text{LET} \quad \hat{\theta} = \frac{N+1}{2N} T = \frac{N+1}{2N} \text{MAX } x(n)$$

THIS IS MVU ESTIMATOR FOR MEAN OF UNIFORM NOISE. TO ASSESS IMPROVEMENT OVER SAMPLE MEAN:

$$\text{VAR}(\hat{\theta}) = \left(\frac{N+1}{2N}\right)^2 \text{VAR}(T)$$

$$\text{VAR}(T) = E(T^2) - E^2(T)$$

$$= \int_0^{\beta} \xi^2 \frac{N \xi^{N-1}}{\beta^N}\, d\xi - \left(\frac{N\beta}{N+1}\right)^2$$

$$= \frac{N\beta^2}{(N+1)^2(N+2)}$$

$$VAR(\hat{\theta}) = \frac{\beta^2}{4N(N+2)} < \frac{\beta^2}{12N} = VAR(\bar{x})$$

FOR $N \geq 2$.

NOTE VARIANCE DECREASES AS $1/N^2$ VS $1/N$ FOR SAMPLE MEAN - SUBSTANTIAL DIFFERENCE.

THIS APPROACH REQUIRES A SINGLE SS TO WORK. IN PHASE ESTIMATION EXAMPLE HAD T_1, T_2. WE WOULD NEED TO EVALUATE

$$\hat{\phi} = E(\check{\phi} \mid T_1, T_2)$$

OR FIND $g(T_1, T_2)$ THAT WAS UNBIASED. HARD PROBLEM!

VECTOR PARAMETER CASE

$\underline{\theta}$ IS NOW $p \times 1$. A VECTOR STATISTIC $\underline{T}(\underline{x}) = [T_1(\underline{x}) \ldots T_r(\underline{x})]$ IS SUFFICIENT IF

$$p(\underline{x} \mid \underline{T}(\underline{x}); \underline{\theta})$$

DOES NOT DEPEND ON $\underline{\theta}$.

IT IS POSSIBLE THAT

1) $r > p$ MORE SS THAN PARAMETERS
* 2) $r = p$ EQUAL SS AS PARAMETERS
3) $r < p$ LESS SS THAN PARAMETERS

* INTERESTING CASE — LEADS TO MVU ESTIMATOR (JUST MAKE $\underline{g}(\underline{T}(\underline{x}))$ UNBIASED)

p DIMENSIONAL FUNCTION

NEYMAN-FISHER THEOREM

IF $p(\underline{x};\underline{\theta}) = g(\underline{T}(\underline{x}),\underline{\theta})\,h(\underline{x})$

$\Rightarrow$ $\underline{T}(\underline{x}) = [T_1(\underline{x})\ T_2(\underline{x}) \ldots T_r(\underline{x})]^T$ IS SS.

<u>EXAMPLE</u>: DC LEVEL IN WGN – UNKNOWN σ^2

$x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$

$w[n]$: WGN

$\underline{\theta} = \begin{pmatrix} A \\ \sigma^2 \end{pmatrix}$

$$p(\underline{x};\underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2}$$

BUT $\sum_{n=0}^{N-1}(x[n]-A)^2 = \sum_n x^2[n] - 2AN\bar{x} + NA^2$

$$p(\underline{x};\underline{\theta}) = \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\left[\sum x^2[n]-2AN\bar{x}+NA^2\right]}}_{g(\underline{T}(\underline{x}),\underline{\theta})} \cdot \underbrace{1}_{h(\underline{x})}$$

WHERE $\underline{T}(\underline{x}) = \begin{bmatrix} T_1(\underline{x}) \\ T_2(\underline{x}) \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{N-1} x[n] \\ \sum_{n=0}^{N-1} x^2[n] \end{bmatrix}$

THESE STATISTICS ARE SS — NEED BOTH OF THEM ⇒ JOINTLY SUFFICIENT.

SINCE $r = p = 2$ WE CAN FIND MVU ESTIMATOR.

SAME ARGUMENT AS BEFORE FOR SCALAR CASE

$$\hat{\underline{\theta}} = \underline{g}(\underline{T}(\underline{x}))$$

SO THAT $\hat{\underline{\theta}}$ IS UNBIASED (SEE THEOREM 5.4) — ALSO NEED COMPLETENESS

FOR THIS EXAMPLE

$$E(\underline{T}(\underline{x})) = E\begin{bmatrix} \sum_n x(n) \\ \sum_n x^2(n) \end{bmatrix}$$

$$= \begin{bmatrix} NA \\ N(\sigma^2 + A^2) \end{bmatrix}$$

$\Rightarrow$ $T_1(\underline{x}) = \sum_{n=0}^{N-1} x[n]$ SHOULD BE DIVIDED BY N

$T_2(\underline{x})$ ESTIMATES $NE(x^2)$ $\Rightarrow$

$$\underline{g}(\underline{T}(\underline{x})) = \begin{bmatrix} \frac{1}{N} T_1(\underline{x}) \\ \frac{1}{N} T_2(\underline{x}) - \left[\frac{1}{N} T_1(\underline{x})\right]^2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{x} \\ \frac{1}{N}\sum x^2(n) - \bar{x}^2 \end{bmatrix}$$

$$E\left[\underline{g}(\underline{T}(\underline{x}))\right] = \begin{bmatrix} E(\bar{x}) \\ E\left[\frac{1}{N}\sum x^2(n)\right] - E(\bar{x}^2) \end{bmatrix}$$

$$= \begin{bmatrix} A \\ A^2 + \sigma^2 - E(\bar{x}^2) \end{bmatrix}$$

TO FIND $E(\bar{x}^2)$:

$$\bar{x} \sim N(A, \sigma^2/N)$$

$$\Rightarrow E(\bar{x}^2) = VAR(\bar{x}) + E(\bar{x})^2 = \sigma^2/N + A^2$$

$$E\left[g(\underline{T}(\underline{x}))\right] = \begin{bmatrix} A \\ A^2+\sigma^2 - (\sigma^2/N - A^2) \end{bmatrix}$$

$$= \begin{bmatrix} A \\ \frac{N-1}{N}\sigma^2 \end{bmatrix}$$

MUST MULTIPLY $\frac{1}{N}\sum x^2(n) - \bar{x}^2$ BY $\frac{N}{N-1}$ OR

$$g'(\underline{T}(\underline{x})) = \begin{bmatrix} \bar{x} \\ \underbrace{\frac{1}{N-1}\sum_n x^2(n) - \frac{N}{N-1}\bar{x}^2} \end{bmatrix}$$

$$\frac{1}{N-1}\left(\sum_n x^2(n) - N\bar{x}^2\right) = \frac{1}{N-1}\sum_n (x[n]-\bar{x})^2$$

$$\therefore \quad \hat{\underline{\theta}} = \begin{bmatrix} \bar{x} \\ \frac{1}{N-1}\sum_{n=0}^{N-1}(x[n]-\bar{x})^2 \end{bmatrix}$$ IS MVU ESTIMATOR

NOTE THAT WE DIVIDE SAMPLE VARIANCE BY N−1 DUE TO LOSS OF DEGREE OF FREEDOM

IN ESTIMATING A.

ALSO, CAN BE SHOWN THAT

$$\underline{C}_{\hat{\theta}} = \begin{bmatrix} \sigma^2/N & 0 \\ 0 & \frac{2\sigma^4}{N-1} \end{bmatrix}$$

BUT CRLB IS (SEE EXAMPLE 3.6)

$$\underline{I}^{-1}(\underline{\theta}) = \begin{bmatrix} \sigma^2/N & 0 \\ 0 & \frac{2\sigma^4}{N} \end{bmatrix}$$

$\Rightarrow$ NOT EFFICIENT $\Rightarrow$ COULD NOT HAVE BEEN FOUND USING CRLB THEOREM.

FINALLY NEED TO VERIFY COMPLETENESS (FOLLOWS FROM THEORY OF EXPONENTIAL FAMILY OF PDF'S)

CHAPTER 6 – BEST LINEAR UNBIASED ESTIMATORS (BLUE)

TRIED 1) CRLB THEORY

2) RBLS THEORY

WHAT DO WE TRY NEXT?

RESTRICT ESTIMATOR TO BE LINEAR

$$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n]$$

AND FIND MVU ESTIMATOR WITHIN THIS CLASS.

LINEAR MVU ESTIMATOR $\equiv$ BLUE

OPTIMALITY OF BLUE?

RECALL $\hat{\theta} = \bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$ (DC LEVEL IN WGN)

$\Rightarrow a_n = 1/N$

$\Rightarrow$ MVU ESTIMATOR = BLUE IF MVU ESTIMATOR HAPPENS TO BE LINEAR

COUNTEREXAMPLE: MEAN OF UNIFORM NOISE (EXAMPLE 5.8)

MVU ESTIMATOR = $\hat{\theta} = \frac{N+1}{2N} \max x[n]$

$\Rightarrow$ NOT LINEAR

ANY LINEAR ESTIMATOR IS SUBOPTIMAL, EVEN BLUE.

SOMETIMES BLUE IS TOTALLY WRONG.

EXAMPLE: $x[n] = w[n]$

$\uparrow$ WGN WITH VARIANCE σ^2

MVU ESTIMATOR $= \hat{\sigma}^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$

(SEE EXAMPLE 3.6)

BLUE $= \hat{\sigma}^2 = \sum_{n=0}^{N-1} a_n x[n]$

TO BE UNBIASED $E(\hat{\sigma}^2) = \sigma^2$

BUT $E(\hat{\sigma}^2) = \sum_{n=0}^{N-1} a_n E(x[n]) = 0$

IMPOSSIBLE TO FIND a_n'S TO MAKE $\hat{\sigma}^2$ UNBIASED.

CAN'T USE LINEAR ESTIMATOR

FINDING THE BLUE

UNBIASED CONSTRAINT: $E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$

VARIANCE: $VAR(\hat{\theta}) = E\left[\left(\sum_n a_n x[n] - E\left(\sum_n a_n x[n]\right)\right)^2\right]$

LET $\underline{a} = [a_0\ a_1 \ldots a_{N-1}]^T$

$$VAR(\hat{\theta}) = E\left[\left(\underline{a}^T \underline{x} - \underline{a}^T E(\underline{x})\right)^2\right]$$

$$= E\left[\left(\underline{a}^T (\underline{x} - E(\underline{x})\right)^2\right]$$

$$= E\left[\underline{a}^T (\underline{x} - E(\underline{x}))(\underline{x} - E(\underline{x}))^T \underline{a}\right]$$

$$= \underline{a}^T \underline{C}\, \underline{a}$$

WHERE $\underline{C} = E\left[(\underline{x} - E(\underline{x}))(\underline{x} - E(\underline{x}))^T\right]$
= COVARIANCE MATRIX OF $\underline{x}$

FOR UNBIASED CONSTRAINT TO HOLD
$$\sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

REQUIRE $E(x[n])$ TO BE LINEAR IN θ OR
$$E(x[n]) = s[n]\theta$$
$\uparrow$ KNOWN

SINCE $x[n] = E(x[n]) + [x[n] - E(x[n])]$
$$= s[n]\theta + w[n]$$

$\Rightarrow$ BLUE APPLICABLE TO AMPLITUDE ESTIMATION OF SIGNALS IN NOISE

SUMMARY: MUST MINIMIZE OVER $\underline{a}$

$VAR(\hat{\theta}) = \underline{a}^T \underline{C} \underline{a}$ SUBJECT TO

CONSTRAINT $\sum_{n=0}^{N-1} a_n E(x(n)) = \theta$

WHERE $E(x(n)) = s(n)\theta$.

$$\sum_n a_n E(x(n)) = \sum_n a_n s(n)\theta = \theta$$

$$\Rightarrow \sum_n a_n s(n) = 1 \Rightarrow \underline{a}^T \underline{s} = 1$$

WHERE $\underline{s} = [s(0)\ s(1) \ldots s(N-1)]^T$

<u>SOLUTION</u>: USE LAGRANGIAN MULTIPLIERS

$$F = \underline{a}^T \underline{C} \underline{a} + \lambda(\underline{a}^T \underline{s} - 1)$$

QUADRATIC FORM

USING $\dfrac{\partial \underline{x}^T \underline{A} \underline{x}}{\partial \underline{x}} = 2\underline{A}\underline{x}$ FOR $\underline{A}^T = \underline{A}$

$$\frac{\partial \underline{b}^T \underline{x}}{\partial \underline{x}} = \underline{b}$$

WHERE $\dfrac{\partial g}{\partial \underline{x}} = \left[\dfrac{\partial g}{\partial x_1}\ \dfrac{\partial g}{\partial x_2} \ldots \dfrac{\partial g}{\partial x_N}\right]^T$ = GRADIENT

$$\frac{\partial F}{\partial \underline{a}} = 2\underline{C}\underline{a} + \lambda \underline{s} = \underline{0}$$

$$\Rightarrow \underline{a} = -\frac{\lambda}{2} \underline{C}^{-1} \underline{s}$$

TO FIND λ (LAGRANGIAN MULTIPLIER) USE

CONSTRAINT $\Rightarrow \underline{a}^T \underline{s} = -\frac{\lambda}{2} \underline{s}^T \underline{C}^{-1} \underline{s} = 1$

$$\Rightarrow -\lambda/2 = \frac{1}{\underline{s}^T \underline{C}^{-1} \underline{s}}$$

$$\therefore \underline{a}_{OPT} = \frac{\underline{C}^{-1} \underline{s}}{\underline{s}^T \underline{C}^{-1} \underline{s}}$$

TO FIND THE MINIMUM VARIANCE

$$\mathrm{VAR}(\hat{\theta}) = \underline{a}_{OPT}^T \underline{C}\, \underline{a}_{OPT}$$

$$= \frac{\underline{s}^T \underline{C}^{-1^T} \underline{C}\, \underline{C}^{-1} \underline{s}}{(\underline{s}^T \underline{C}^{-1} \underline{s})^2}$$

$$= \frac{\underline{s}^T \underline{C}^{-1} \underline{s}}{(\underline{s}^T \underline{C}^{-1} \underline{s})^2} = \frac{1}{\underline{s}^T \underline{C}^{-1} \underline{s}}$$

$$\therefore \hat{\theta} = \underline{a}^T \underline{x}$$

$$\boxed{\begin{aligned} \hat{\theta} &= \frac{\underline{s}^T \underline{C}^{-1} \underline{x}}{\underline{s}^T \underline{C}^{-1} \underline{s}} \\ \mathrm{VAR}(\hat{\theta}) &= \frac{1}{\underline{s}^T \underline{C}^{-1} \underline{s}} \end{aligned}} \longleftarrow \text{BLUE}$$

TO SHOW THAT $\hat{\theta}$ IS UNBIASED

$$E(\hat{\theta}) = \frac{\underline{s}^T \underline{C}^{-1} E(\underline{x})}{\underline{s}^T \underline{C}^{-1} \underline{s}} = \frac{\underline{s}^T \underline{C}^{-1} \underline{s}\, \theta}{\underline{s}^T \underline{C}^{-1} \underline{s}} = \theta$$

TO FIND BLUE WE NEED

1) $\underline{s}$ (SCALED MEAN) - FIRST MOMENT

2) $\underline{C}$ (COVARIANCE OF $\underline{x}$) - SECOND "

BUT NOT ENTIRE PDF.

$\Rightarrow$ GET SAME ESTIMATOR FOR GAUSSIAN DATA AS UNIFORM DATA.

EXAMPLE: DC LEVEL IN WHITE NOISE

$$x[n] = A + w[n]$$

$\uparrow$ WHITE NOISE WITH VARIANCE σ^2

DON'T KNOW PDF OF $w[n]$. SAMPLES OF $w[n]$ ARE NOT NECESSARILY INDEPENDENT, ONLY UNCORRELATED.

BUT $E(x[n]) = A \Rightarrow s[n] = 1 \Rightarrow \underline{s} = \underline{1}$

$$\hat{A} = \frac{\underline{s}^T \underline{C}^{-1} \underline{x}}{\underline{s}^T \underline{C}^{-1} \underline{s}} = \frac{\underline{1}^T \frac{1}{\sigma^2} \underline{I}\, \underline{x}}{\underline{1}^T \frac{1}{\sigma^2} \underline{I}\, \underline{1}} = \frac{\underline{1}^T \underline{x}}{\underline{1}^T \underline{1}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\mathrm{VAR}(\hat{A}) = \frac{1}{\underline{s}^T \underline{C}^{-1} \underline{s}} = \frac{1}{\underline{1}^T \frac{1}{\sigma^2} \underline{I}\, \underline{1}} = \sigma^2/N$$

SAMPLE MEAN IS BLUE INDEPENDENT OF PDF OF DATA, OPTIMAL?

EXAMPLE : DC LEVEL IN COLORED NOISE

NOW $w[n]$ IS COLORED OR $\underline{C} \neq \sigma^2 \underline{I}$.

$$\hat{A} = \frac{\underline{1}^T \underline{C}^{-1} \underline{x}}{\underline{1}^T \underline{C}^{-1} \underline{1}}$$

NEED TO ASSUME SOME $\underline{C}$. CONSIDER

$$\underline{C} = \begin{bmatrix} \sigma_0^2 & & & \underline{0} \\ & \sigma_1^2 & & \\ & & \ddots & \\ \underline{0} & & & \sigma_{N-1}^2 \end{bmatrix}$$

SAMPLES ARE UNCORRELATED BUT HAVE DIFFERENT VARIANCES.

$$\underline{C}^{-1} = \begin{bmatrix} 1/\sigma_0^2 & & & \underline{0} \\ & 1/\sigma_1^2 & & \\ & & \ddots & \\ \underline{0} & & & 1/\sigma_{N-1}^2 \end{bmatrix}$$

$$\hat{A} = \frac{\underline{1}^T \begin{bmatrix} 1/\sigma_0^2 & & \underline{0} \\ & \ddots & \\ \underline{0} & & 1/\sigma_{N-1}^2 \end{bmatrix} \underline{x}}{\underline{1}^T \begin{bmatrix} 1/\sigma_0^2 & & \underline{0} \\ & \ddots & \\ \underline{0} & & 1/\sigma_{N-1}^2 \end{bmatrix} \underline{1}} = \frac{\underline{1}^T \begin{bmatrix} x[0]/\sigma_0^2 \\ \vdots \\ x[N-1]/\sigma_{N-1}^2 \end{bmatrix}}{\underline{1}^T \begin{bmatrix} 1/\sigma_0^2 \\ \vdots \\ 1/\sigma_{N-1}^2 \end{bmatrix}}$$

$$= \frac{\sum_{n=0}^{N-1} x[n]/\sigma_n^2}{\sum_{n=0}^{N-1} 1/\sigma_n^2}$$

$$\text{VAR}(\hat{A}) = \frac{1}{\underline{1}^T \underline{C}^{-1} \underline{1}} = \frac{1}{\sum_{n=0}^{N-1} 1/\sigma_n^2}$$

BLUE WEIGHTS SAMPLES MORE HEAVILY IF THEY HAVE LESS VARIANCE.

IF $\sigma_0^2 \to 0$, WHAT IS $\hat{A}$?

OUR MODEL

$$E(x[n]) = s[n]\theta$$

IS ACTUALLY THE GENERAL LINEAR MODEL SINCE

$$x[n] = s[n]\theta + \underbrace{(x[n] - E(x[n]))}_{w[n] \text{ (ZERO MEAN)}}$$

$$\Rightarrow \underline{x} = \underline{s}\theta + \underline{w}$$

ALTHOUGH WITHOUT $\underline{w} \sim$ GAUSSIAN ASSUMPTION.

IF $\underline{w}$ IS GAUSSIAN OR $\underline{x}$ IS GAUSSIAN, BLUE $\equiv$ MVU ESTIMATOR.

BLUE FOR VECTOR PARAMETER

$$\underline{\theta} = [\theta_1\, \theta_2 \ldots \theta_p]^T$$

PROPOSE $\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$

OR $\hat{\underline{\theta}} = \underline{A}\,\underline{x} \qquad \underline{A}$ IS $p \times N$

TO BE UNBIASED

$$E(\hat{\theta}_i) = \sum_{n=0}^{N-1} a_{in} E(x[n]) = \theta_i \qquad i = 1, 2, \ldots, p$$

OR $\qquad E(\hat{\underline{\theta}}) = \underline{A}\,E(\underline{x}) = \underline{\theta}$

TO SATISFY CONSTRAINT

$E(\underline{x}) = \underset{\uparrow \text{KNOWN}}{\underline{H}}\underline{\theta}$ MUST BE SATISFIED.

IN SCALAR CASE WE ASSUMED

$$E(\underline{x}) = \underbrace{\begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix}}_{\underline{H} \;\; N \times 1} \theta$$

NOW $\underline{H}$ IS $N \times p$.

$$\underline{A}\,E(\underline{x}) = \underline{\theta} \Rightarrow \underline{A}\,\underline{H}\,\underline{\theta} = \underline{\theta} \Rightarrow \underline{A}\,\underline{H} = \underline{I}$$

NOW LET $\underline{A}$ BE WRITTEN AS

$$\underline{A} = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_p^T \end{bmatrix} \qquad \underline{a}_i^T \text{ IS } i^{th} \text{ ROW}$$

ALSO $\underline{H} = [\underline{h}_1 \; \underline{h}_2 \ldots \underline{h}_p]$

$\uparrow$ COLUMNS

$$\underline{A}\,\underline{H} = \underline{I} \Rightarrow \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_p^T \end{bmatrix} [\underline{h}_1 \ldots \underline{h}_p] = \underline{I}$$

$$\begin{bmatrix} \underline{a}_1^T\underline{h}_1 & \underline{a}_1^T\underline{h}_2 & \cdots & \underline{a}_1^T\underline{h}_p \\ \underline{a}_2^T\underline{h}_1 & \underline{a}_2^T\underline{h}_2 & \cdots & \underline{a}_2^T\underline{h}_p \\ & & \ddots & \underline{a}_p^T\underline{h}_p \end{bmatrix} = \underline{I}$$

THE UNBIASED CONSTRAINT IS

$$\underline{a}_i^T\underline{h}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

FOR $i = 1, 2, \ldots, p;\; j = 1, 2, \ldots, p$

THE VARIANCE IS AS BEFORE

$$\text{var}(\hat{\theta}_i) = \underline{a}_i^T \underline{C}\, \underline{a}_i \qquad i = 1, 2, \ldots, p$$

RESULT (SEE APPENDIX 6B)

$$\hat{\underline{\theta}} = (\underline{H}^T\underline{C}^{-1}\underline{H})^{-1}\underline{H}^T\underline{C}^{-1}\underline{x}$$

$$\underline{C}_{\hat{\theta}} = (\underline{H}^T\underline{C}^{-1}\underline{H})^{-1}$$

SAME RESULT AS GENERAL LINEAR MODEL WHERE

$$\underline{x} = \underline{H}\underline{\theta} + \underline{w} \qquad \underline{w} \sim N(\underline{0}, \underline{C})$$

$\Rightarrow$ IF $\underline{x}$ IS GAUSSIAN, BLUE IS MVU ESTIMATOR

SUMMARY

GAUSS-MARKOV THEOREM: IF DATA HAS FORM

$$\underbrace{\underline{x}}_{} = \underset{\substack{\uparrow \\ N \times p \text{ AND} \\ \text{KNOWN}}}{\underline{H}}\ \underset{\substack{\uparrow \\ p \times 1}}{\underline{\theta}} + \underline{w} \leftarrow N \times 1 \quad E(\underline{w}) = \underline{0}$$

COVARIANCE = $\underline{C}$

(PDF OF $\underline{w}$ IS ARBITRARY), THEN BLUE IS

$$\hat{\underline{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1} \underline{H}^T \underline{C}^{-1} \underline{x}$$

THE MINIMUM VARIANCE IS

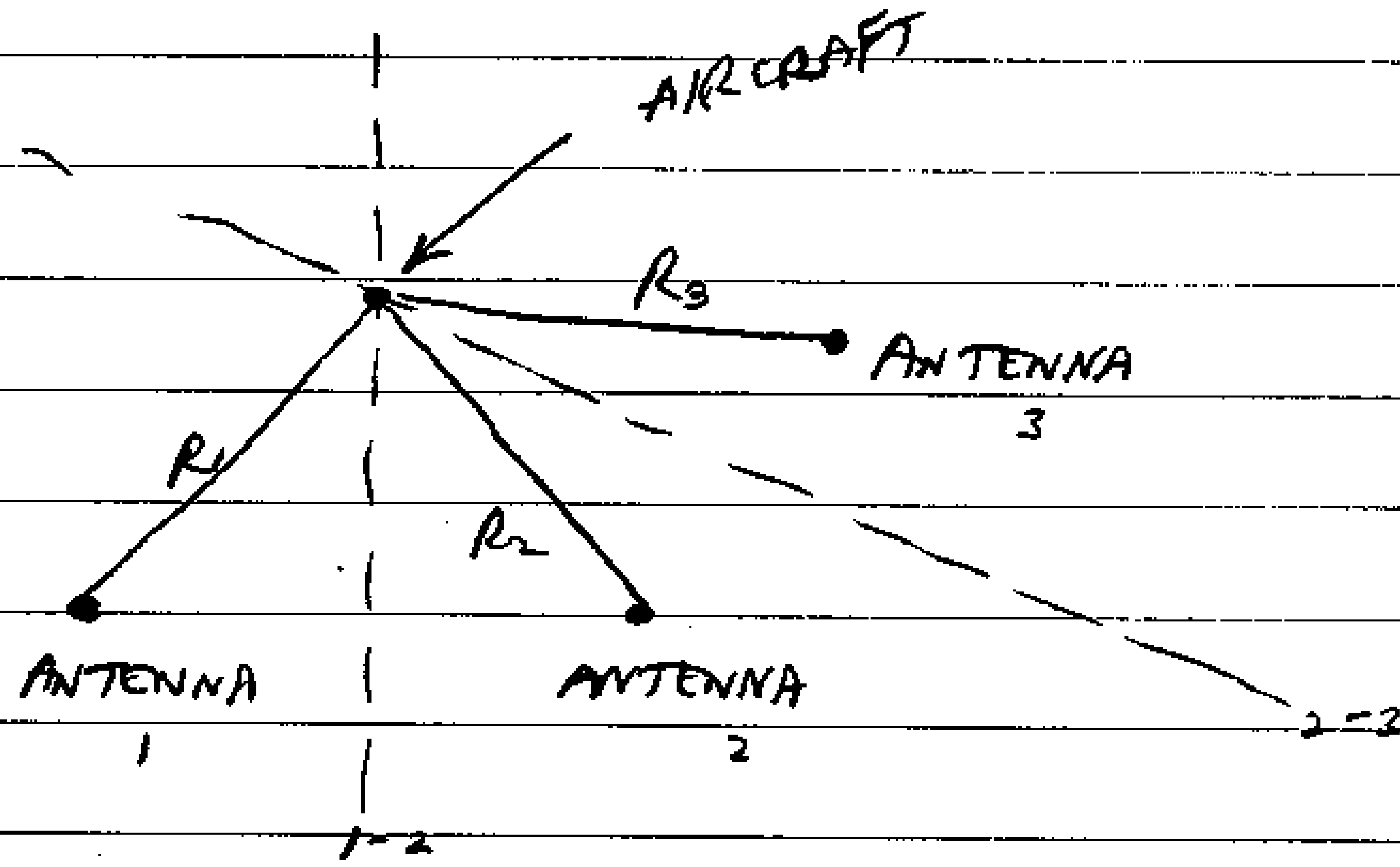
$$\text{VAR}(\hat{\theta}_i) = \left[(\underline{H}^T \underline{C}^{-1} \underline{H})^{-1}\right]_{ii}$$

AND COVARIANCE IS

$$\underline{C}_{\hat{\theta}} = (\underline{H}^T \underline{C}^{-1} \underline{H})^{-1}.$$

SIGNAL PROCESSING EXAMPLE

SOURCE LOCALIZATION — PROBLEM IS TO DETERMINE POSITION OF SOURCE BASED ON EMITTED SIGNAL.



IF $R_1 = R_2 = R_3 \Rightarrow t_2 - t_1 = 0, \quad t_3 - t_2 = 0$

t_i = TIME OF ARRIVAL OF SIGNAL.

$\Rightarrow$ CAN FIND POSITION FROM t_i's. HERE WE MEASURE $t_2 - t_1 = 0 \Rightarrow$ POSITION IS ALONG LINE 1-2, MEASURE $t_3 - t_2 = 0 \Rightarrow$ POSITION IS ALONG LINE 2-3 $\Rightarrow$ KNOW POSITION (TRIANGULATION METHOD)

IN GENERAL, FOR TIME DIFFERENCES $\neq 0$, LINES BECOME HYPERBOLAS.

ASSUME WE HAVE N ANTENNAS.