Fourier & Euler

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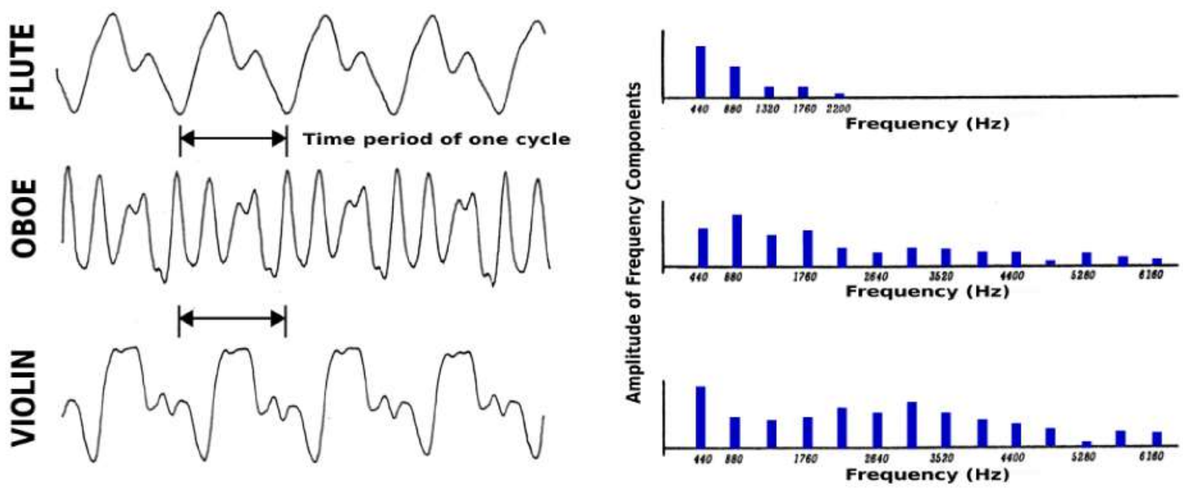
Harmonics and Timbre

Figure below shows recorded waveforms from three instruments (flute, oboe, and violin) playing the A4 note (440 Hz) and their harmonic components. The flute is an aerophone or reedless wind instrument. The oboe is a double reed woodwind instrument. The violin is a string instrument. Although all three instruments play the same note, the harmonic contents are quite different to give each instrument a unique timbre (tone quality).

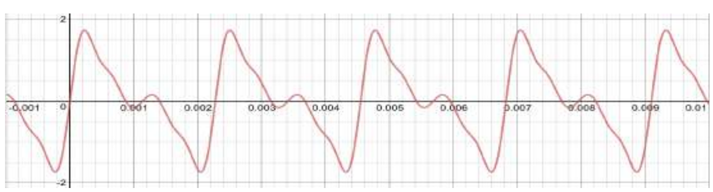


The time period of one cycle shown in the figure is the reciprocal of the fundamental frequency:

$$T = \frac{1}{f_0} = \frac{1}{440\ \text{Hz}} = 0.00227\ \text{s}$$

We now use the graphing calculator technique from the previous section to simulate the waveform of the flute. The resulting waveform is shown below.

$$y(t) = \sin 2\pi\cdot 440\cdot t + \frac{4}{5}\sin 2\pi\cdot 880\cdot t + \frac{1}{4}\sin 2\pi\cdot 1320\cdot t + \frac{1}{4}\sin 2\pi\cdot 1760\cdot t + \frac{1}{20}\sin 2\pi\cdot 2200\cdot t$$



While the simulated waveform bears the general shape of the actual waveform, there is some degree of discrepancy. This may be due to unrepresented *phase* components, which are time delays among the different harmonics. The effect of the phase will be further discussed later.

Just intonation

Just intonation or pure intonation is the tuning musical intervals as small integer ratios of frequencies. Any interval tuned in this way is called a just interval. In just intonation the diatonic scale may be easily constructed using the three simplest intervals within the octave, the perfect fifth (3/2), perfect fourth (4/3), and the major third (5/4). As forms of the fifth and third are naturally

C	D	E	F	G	A	B	C
1	9/8	5/4	4/3	3/2	5/3	15/8	2

present in the overtone series of harmonic resonators, this is a very simple process. The table shows the harmonic fractions between the frequencies of the just intonation for the C major scale.

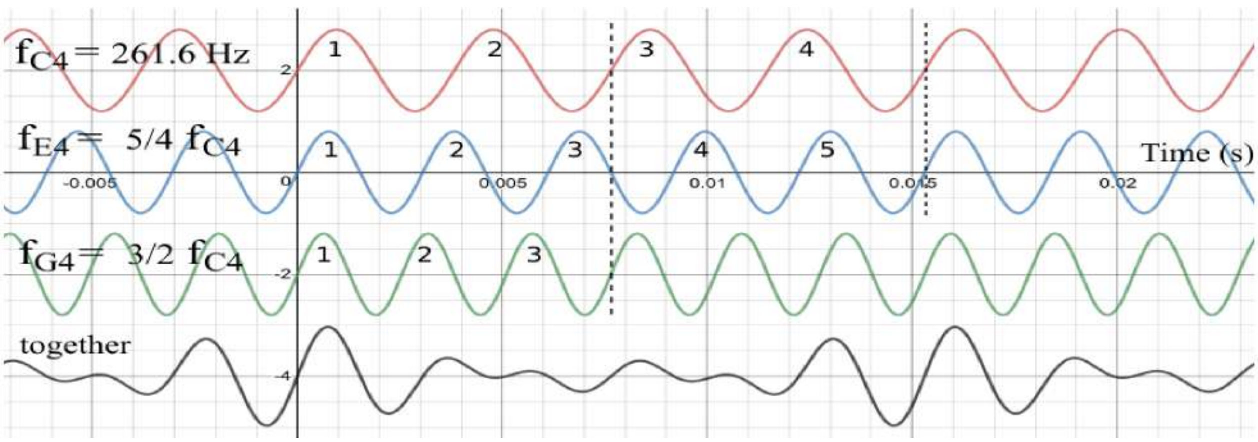
An example is generated by using an online graphing calculator <https://www.desmos.com/calculator>. The note C4 is represented by a pure sine wave at 261.6 Hz. Based on the harmonic fractions, the note E4 is 5/4 times higher and the note G4 is 3/2 times higher than C4. Together the three waves form a stable, periodic oscillation. The equations entries are shown on the right with the waveforms shown below.

$$y = 0.8\sin(6.28 \cdot 261.6\,t) + 2$$

$$y = 0.8\sin\left(6.28 \cdot \frac{5}{4} \cdot 261.6\,t\right)$$

$$y = 0.8\sin\left(6.28 \cdot \frac{3}{2} \cdot 261.6\,t\right) - 2$$

$$y = \frac{\left(\sin(6.28 \cdot 261.6\,t) + \sin\left(6.28 \cdot \frac{5}{4} \cdot 261.6\,t\right) + \sin\left(6.28 \cdot \frac{3}{2} \cdot 261.6\,t\right)\right)}{3} - 4$$

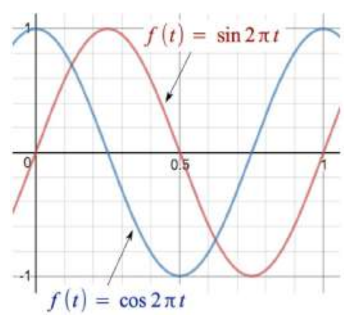


Fourier Analysis

To probe further, the representation of a periodic signal by its harmonics was first studied by the French mathematician and physicist Jean-Baptiste Joseph Fourier (1768–1830). The Fourier series analysis was initially concerned with periodic signals. It was later expanded to non-periodic signals by using the Fourier transform. The Fourier transform has many theoretical and practical applications, which provide the foundation for areas such as linear systems and signal processing.

Jean-Baptiste Joseph Fourier (1768–1830, France)

A Fourier series is a way to represent a periodic function as the weighted sum of simple oscillating functions, namely sines and cosines. Why sines and cosines? The answer is related to the phase component (time delay) mentioned previously. As shown by the figure on the right, the sine and the cosine form a so-called orthogonal basis; They are separated by a phase angle of 90 degrees ($\pi/2$). Any angle can be represented by a linear combination of them. By definition a linear combination of sine and cosine is $a\sin 2\pi t + b\cos 2\pi t$, where a and b are constants.



In the following, we will present the formulas for the Fourier series, which utilize the notions of calculus and complex variables. If you don't have these mathematical background,s it's quite alright and please just try to follow the notations.

It is somewhat cumbersome to carry the coefficients for both sine and cosine. Thus, we introduce the complex exponential from another important mathematician Leonhard Euler. The famous Euler's number e is an irrational number: e = 2.71828182845904523 (and more). The Euler's formula represents sine and cosine with a complex exponential:

$$e^{jx} = \cos x + j \sin x \text{, where } j = \sqrt{-1} \text{.}$$

Leonhard Euler
(1707–1783, Switzerland)

A special case of the above formula is known as Euler's identity:

$$e^{j\pi} + 1 = 0 \text{.}$$

These relationships are illustrated with the unit circle as shown in the figure at the lower-right.

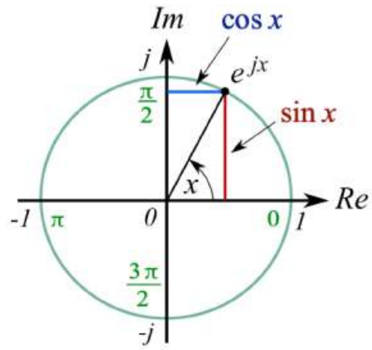
Finally, we present the Fourier series. A *time-domain* periodic signal $f(t)$ with the fundamental frequency of f_0 can be represented by a linear combination of complex exponentials,:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn 2\pi f_0 t}$$

The Fourier coefficients c_n specify the weight on each harmonic in the *frequency-domai*n. The Fourier coefficients are computed according to:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn 2\pi f_0 t} dt \text{ ,}$$

where $T = 1/f_0$ is the period of the signal.



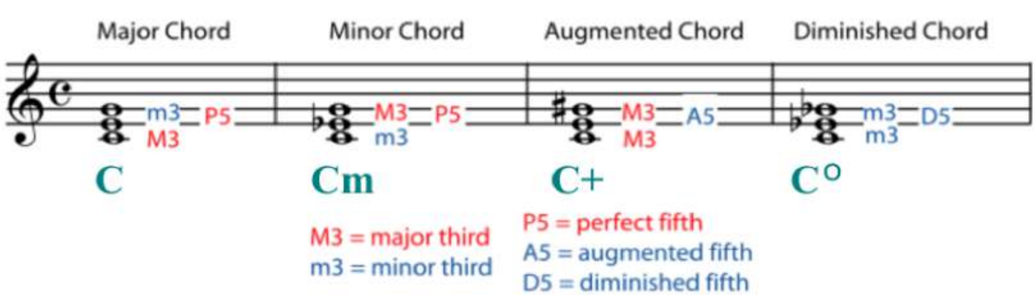
Harmony

In music, harmony considers the process by which the composition of individual sounds, or superpositions of sounds, is analyzed by hearing. Usually, this means simultaneously occurring frequencies, pitches, or chords. The study of harmony involves chords and their construction and chord progressions and the principles of connection that govern them. Harmony is often said to refer to the "vertical" aspect of music, as distinguished from melodic line, or the "horizontal" aspect [Wikipedia].

A chord is a group of three or more notes sounded together, as a basis of harmony. A triad is a a three-note chord consisting of:

- the root – this note specifying the name of the chord;
- the third – its interval above the root being a minor third (3 semitones) or a major third (4 semitones);
- the fifth – its interval above the third being a minor third or a major third.

With the choice of minor third and major third for two intervals, there are a total of 4 possible combinations. Using C as the root note, the four chords are shown below.



The diagrams below show all the common triads belong to each major keys (left chart) and minor keys (right chart). Roman numerals indicate each chord position relative to the scale.

Chords In All Major Keys

Major Keys	I	ii	iii	IV	V	vi	vii°
C	C	Dm	Em	F	G	Am	B°
C♯	C♯	D♯m	E♯m	F♯	G♯	A♯m	B♯°
D♭	D♭	E♭m	Fm	G♭	A♭	B♭m	C°
D	D	Em	F♯m	G	A	Bm	C♯°
E♭	E♭	Fm	Gm	A♭	B♭	Cm	D°
E	E	F♯m	G♯m	A	B	C♯m	D♯°
F	F	Gm	Am	B♭	C	Dm	E°
F♯	F♯	G♯m	A♯m	B	C♯	D♯m	E♯°
G♭	G♭	A♭m	B♭m	C♭	D♭	E♭m	F°
G	G	Am	Bm	C	D	Em	F♯°
A♭	A♭	B♭m	Cm	D♭	E♭	Fm	G°
A	A	Bm	C♯m	D	E	F♯m	G♯°
B♭	B♭	Cm	Dm	E♭	F	Gm	A°
B	B	C♯m	D♯m	E	F♯	G♯m	A♯°

Chords In All Minor Keys

Minor Keys	i	ii°	III	iv	V	VI	VII
Cm	Cm	D°	E♭	Fm	G	A♭	B♭
C♯m	C♯m	D♯°	E	F♯m	G♯	A	B
Dm	Dm	E°	F	Gm	A	B♭	C
D♯m	D♯m	E♯°	F♯	G♯m	A♯	B	C♯
E♭m	E♭m	F°	G♭	A♭m	B♭	C♭	D♭
Em	Em	F♯°	G	Am	B	C	D
Fm	Fm	G°	A♭	B♭m	C	D♭	E♭
F♯m	F♯m	G♯°	A	Bm	C♯	D	E
Gm	Gm	A°	B♭	Cm	D	E♭	F
G♯m	G♯m	A♯°	B	C♯m	D♯	E	F♯
A♭m	A♭m	B♭°	C♭	D♭m	E♭	F♭	G♭
Am	Am	B°	C	Dm	E	F	G
A♯m	A♯m	B♯°	C♯	D♯m	E♯	F♯	G♯
B♭m	B♭m	C°	D♭	E♭m	F	G♭	A♭
Bm	Bm	C♯°	D	Em	F♯	G	A