ELE 314 Linear Systems and Signals – HomeWork #1 (5%) *Ying Sun*

Name: ______________________________ Due date: ______________________

This homework is a practice of the integration techniques for obtaining certain Laplace transforms. Study the examples first and then work on the three questions. You are allowed to use the table of integrals. The Laplace transform (LT) and the inverse Laplace transform (ILT) is given by:

LT: $X(s)=\int_0^{\infty} x(t)e^{-st}\,dt$ -- (1)

ILT: $x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$ -- (2)

Example 1: From the Laplace transforms table, we have $x(t)=e^{at} \rightarrow X(s)=\frac{1}{s-a}$. Prove it.

Using (1), $X(s) = \int_0^{\infty} x(t)e^{-st}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt = \left(\frac{-1}{s-a}e^{-(s-a)t}\right)_0^{\infty} =$

$\left(\frac{-1}{s-a}e^{-(s-a)\infty}\right)-\left(\frac{-1}{s-a}e^{-(s-a)0}\right) = \frac{1}{s-a}$ ------------- QED*

* QED, quod erat demonstrandum in Latin, means "that which was to be demonstrated".
Note: For $e^{-(s-a)\infty}=0$, *Re (s – a)* must be > 0. Thus the region of convergence is $\{s;\ Re\ s > a\}$. Because the Fourier transform (FT) is the Laplace transform evaluated on the $j\omega$ axis. So for FT to exist, a needs to be < 0 so that the $j\omega$ axis falls inside the region of convergence.

Example 2: From the Laplace transforms table, we have $\sin\omega t \rightarrow \frac{\omega}{s^2+\omega^2}$. Prove it.

$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt = \int_0^{\infty} e^{-st}\sin\omega t\,dt$ --- (3)

From the Table of Integrals, formula (105), we have

$\int e^{bx}\sin ax\,dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax-a\cos ax)$ -------------------------------------- (4)

By replacing $a=\omega$ and $b=-s$, we have

$X(s) = \int_0^{\infty} e^{-st}\sin\omega t\,dt = \left[\frac{1}{s^2+\omega^2}e^{-st}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\infty} =$

$\frac{1}{s^2+\omega^2}\{[e^{-\infty}(-s\sin\infty-\omega\cos\infty)]-[e^{0}(-s\sin 0-\omega\cos 0)]\} = \frac{\omega}{s^2+\omega^2}$ ---------- QED

Example 3: Show the derivative property: $x'(t) \;\rightarrow\; sX(s)-x(0)$.

We use integration by parts: $\int_a^b u(t)v'(t)dt \;=\; [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)dt$

Change (1) to fit the form of integration by parts, we have

$$X(s) \;=\; \int_0^\infty x(t)e^{-st}dt \;=\; \int_0^\infty \frac{d}{dt}(\frac{-e^{-st}}{s})\cdot x(t)dt \;=\; \left[(\frac{-e^{-st}}{s})\cdot x(t)\right]_0^\infty - \int_0^\infty (\frac{-e^{-st}}{s})\cdot x'(t)dt \;=$$

$$(\frac{-e^{-s\infty}}{s})\cdot x(\infty) \;-\; (\frac{-e^{0}}{s})\cdot x(0) \;+\; \frac{1}{s}\int_0^\infty x'(t)e^{-st}dt \;=\; \frac{x(0)}{s} \;+\; \frac{1}{s}\int_0^\infty x'(t)e^{-st}dt$$

The region of convergence is $\{s;\, Re\, s > 0\}$. Rearranging the above equation, the LT of x'(t) given by

$\int_0^\infty x'(t)e^{-st}dt \;=\; sX(s)-x(0)$. -------------------------- QED

It is not required, but highly recommended, that you finish this homework by using OpenOffice on this doc file. OpenOffice is a freeware downloadable from <www.openoffice.org>. You can also visit the following <https://wiki.openoffice.org/wiki/Documentation/Reference/Math_commands#Functions> for help with writing the equations. If you don't want to use OpenOffice, you can use another software or simply do the homework on paper.

Homework 1a: Prove $\cos\omega t \;\rightarrow\; \frac{s}{s^2+\omega^2}$. **(2%)**

Homework 1b: Show the time-scaling property: $x(ct) \;\rightarrow\; \frac{1}{c}X(\frac{s}{c})$. **(2%)**

Homework 1c: Show the integration property: $\int_0^t x(\tau)d\tau \;\rightarrow\; \frac{X(s)}{s}$. **(1%)**