A Brief Historical Review: Laplace, Fourier, Euler

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Laplace Transform

Pierre-Simon Laplace
(1749–1827, France)

The Laplace transform is named after French mathematician and astronomer Pierre-Simon Laplace (1749-1827), who used a similar transform in his work on probability theory. Laplace's use of generating functions was similar to what is now known as the z-transform and he gave little attention to the continuous variable case. The theory was further developed in the 19th and early 20th centuries by Lerch, Heaviside, and Bromwich. The current widespread use of the transform came about during and soon after World War II replacing the earlier Heaviside operational calculus. The advantages of the Laplace transform had been emphasized by Doetsch to whom the name Laplace Transform is apparently due [Wikipedia].

Laplace Transform (LT): $X(s)=\int_0^{\infty} x(t)e^{-st}\,dt$

Inverse LT: $x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$, where $s=\sigma+j\omega,\quad \left|\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right|<\infty$

z-transform

The basic idea now known as the z-transform was known to Laplace, and it was re-introduced in 1947 by W. Hurewicz and others as a way to treat sampled-data control systems used with radar. It gives a tractable way to solve linear, constant-coefficient difference equations. It was later dubbed "the z-transform" by Ragazzini and Zadeh in the sampled-data control group at Columbia University in 1952.

z-Transform (ZT): $X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$

Inverse ZT: $x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz$, where $z=re^{j\omega},\quad \left|\sum_{-\infty}^{\infty} x[n]z^{-n}\right|<\infty$

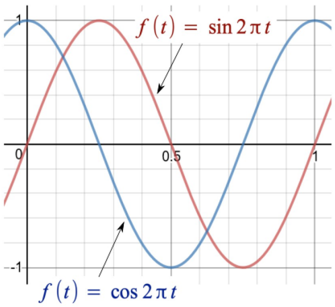
Fourier Analysis

Jean-Baptiste Joseph Fourier
(1768–1830, France)

The representation of a periodic signal by its harmonics was first studied by the French mathematician and physicist Jean-Baptiste Joseph Fourier (1768–1830). The Fourier series analysis was initially concerned with periodic signals. It was later expanded to non-periodic signals by using the Fourier transform. The Fourier transform has many theoretical and practical applications, which provide the foundation for areas such as linear systems and signal processing.

A Fourier series is a way to represent a periodic function as the weighted sum of simple oscillating functions, namely sines and cosines. Why sines and cosines? The answer is related to the phase component (time delay) mentioned previously. As shown by the figure on the right, the sine and the cosine form a so-called orthogonal basis; They are separated by a phase angle of 90 degrees ($\pi/2$). Any angle can be represented by a linear combination of them. By definition a linear combination of sine and cosine is $a\sin 2\pi t+b\cos 2\pi t$, where a and b are constants.

In the following, we will present the formulas for the Fourier series, which utilize the notions of calculus and complex variables. If you don't have these mathematical background,s it's quite alright and please just try to follow the notations.



It is somewhat cumbersome to carry the coefficients for both sine and cosine. Thus, we introduce the complex exponential from another important mathematician Leonhard Euler. The famous Euler's number *e* is an irrational number: *e* = 2.71828182845904523 (and more). The Euler's formula represents sine and cosine with a complex exponential:

$$e^{jx} = \cos x + j \sin x \text{ , where } j = \sqrt{-1} \text{ .}$$


Leonhard Euler
(1707–1783, Switzerland)

A special case of the above formula is known as Euler's identity:

$$e^{j\pi} + 1 = 0 \text{ .}$$

These relationships are illustrated with the unit circle as shown in the figure at the lower-right.

Finally, we present the Fourier series. A *time-domain* periodic signal $f(t)$ with the fundamental frequency of f_0 can be represented by a linear combination of complex exponentials,:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn 2\pi f_0 t}$$

The Fourier coefficients c_n specify the weight on each harmonic in the *frequency-domai*n. The Fourier coefficients are computed according to:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn 2\pi f_0 t} dt \text{ ,}$$

where $T = 1/f_0$ is the period of the signal.