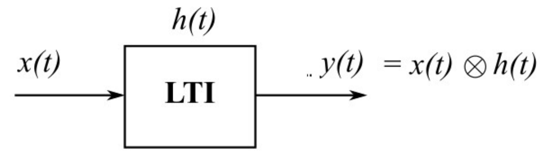
ELE 314 Linear Systems and Signals – Introduction

Ying Sun

A linear time-invariant (LTI) system is a system that satisfies the properties of linearity and time-invariance. A **linear** system supports superposition: A linear combination of individual inputs results in the same linear combination of the corresponding individual outputs. As shown in the figure, if $y_1(t)$ and $y_2(t)$ are the outputs of $x_1(t)$ and $x_2(t)$, respectively, the input $ax_1(t) +bx_2(t)$ results in the output $ay_1(t)+by_2(t)$, where a and b are scaler constants. A **time-invariant** system holds the same input-output relationships over time. If the input is delayed by a time period of T, the output remains the same but with the same amount of delay. In other words, input $x(t+T)$ *results in output* $y(t+T)$.



Linearity: $x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

then $ax_1(t)+bx_2(t) \longrightarrow ay_1(t)+by_2(t)$

Time-invariance:

$x(t-T) \longrightarrow y(t-T)$

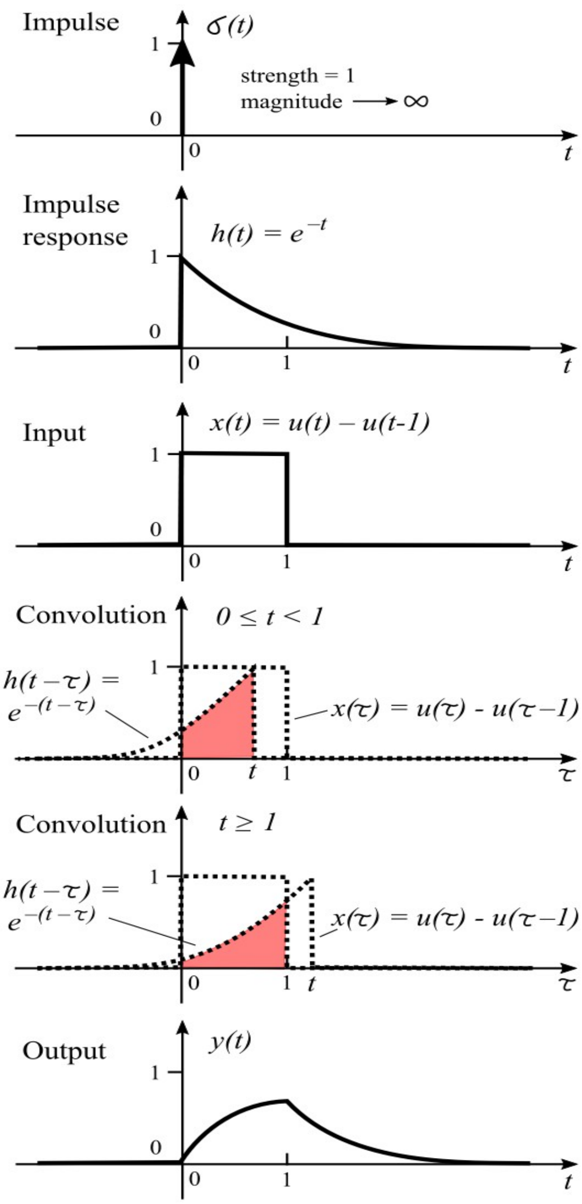
A LTI system is completely characterized by its impulse response $h(t)$, which is the output corresponding to the impulse input $\delta(t)$. The convolution is a linear operation that defines the input-output relationship:

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

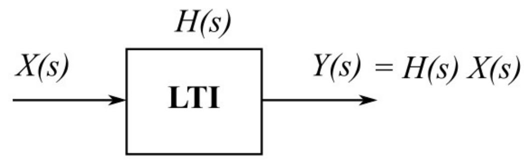
Example 1: The impulse response of a LTI system is: $h(t)=e^{-t}u(t)$, where $u(t)$ is the unit step function. If the input is a square function $x(t)=u(t)-u(t-1)$, determine the output $y(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} [u(\tau)-u(\tau-1)]e^{-(t-\tau)}d\tau = \begin{cases} 0, & t<0 \\ \int_0^t e^{-(t-\tau)}d\tau, & 0\le t<1 \\ \int_0^1 e^{-(t-\tau)}d\tau, & t\ge 1 \end{cases}$$

$$y(t) = \begin{cases} 0, & t<0 \\ 1-e^{-t}, & 0\le t<1 \\ (e-1)e^{-t}, & t\ge 1 \end{cases}$$



Convolution in the time domain is equivalent to multiplication in the frequency domain. The figure shows the LTI system in the frequency domain. *H(s)* is the transfer function, which is the Laplace transform of the impulse response *h(t)*. *H(s)* is the ratio of the LT of the output over the LT of the input:



$$H(s) = \frac{Y(s)}{X(s)} \quad \text{and} \quad Y(s) = H(s)X(s)$$

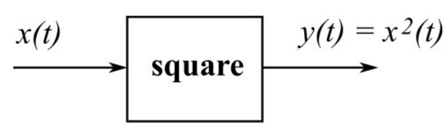
From the Laplace transforms table, we see that $e^{at} \rightarrow \frac{1}{s-a}$. Thus, the transfer function in Example 1 is: $H(s) = \frac{1}{s+1}$. Later on, we will see that this is a first-order low-pass filter.

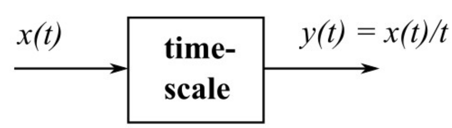
Example 2: For a LTI, show that $y(t) = h(t) \otimes x(t)$ results in $Y(s) = H(s)X(s)$.

$$Y(s) = \int_0^\infty y(t)e^{-st}\,dt = \int_0^\infty \int_{-\infty}^\infty x(\tau)h(t-\tau)e^{-st}\,d\tau\,dt = \int_{-\infty}^\infty x(\tau)[\int_0^\infty h(t-\tau)e^{-st}\,dt]\,d\tau =$$

$$\int_{-\infty}^\infty x(\tau)e^{-s\tau}[\int_{-\tau}^\infty h(t-\tau)e^{-s(t-\tau)}d(t-\tau)]\,d\tau = \int_{-\infty}^\infty x(\tau)e^{-s\tau}H(s)\,d\tau = H(s)X(s) \text{ ------ QED}$$

Example 3: The "square" function shown on the right (top) is a nonlinear time-invariance system. The "time-scale" function shown on the right (bottom) is a linear time-varying system.





Square function

Linearity: $y_1(t) = x_1^2(t)$; $y_2(t) = x_2^2(t)$

$[ax_1(t)+bx_2(t)]^2$ does not equal to

$ax_1^2(t)+bx_2^2(t)$ in general. Thus, linearity is not satisfied.

Time-invariance: Delay the input, the output is $x^2(t+T)$.

Delay the output, the output is also $x^2(t+T)$. Thus, time-invariance is satisfied.

Time-scale function

Linearity: $y_1(t) = x_1(t)/t$; $y_2(t) = x_2(t)/t$

$[ax_1(t)+bx_2(t)]/t$ is the same as $ax_1(t)/t+bx_2(t)/t$. Thus, linearity is satisfied.

Time-invariance: Delay the input, the output is $x(t+T)/t$. Delay the output, the output is also $x(t+T)/(t+T)$, which is different. Thus, time-invariance is not satisfied.