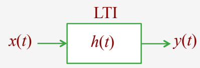
LTI Analog Systems in the Frequency Domain



- Recall

$$y(t)=\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

- For $x(t)=e^{j\Omega t}$, $-\infty < t < +\infty$

  we have

$$y(t)=\int_{-\infty}^{+\infty} h(\tau)e^{j\Omega(t-\tau)}d\tau$$

$$=\left[\int_{-\infty}^{+\infty} h(\tau)e^{-j\Omega\tau}d\tau\right]e^{j\Omega t}$$

Frequency Response

- Denote $\boxed{H(j\Omega)=\int_{-\infty}^{+\infty} h(\tau)e^{-j\Omega\tau}d\tau}$
- Then we have the input-output relation of the LTI analog system given by

$$y(t)=H(j\Omega)e^{j\Omega t}$$

Frequency Response

- Note: For a BIBO stable causal LTI analog system, $h(t)$ is absolutely integrable, and as a result its CTFT $H(j\Omega)$ exists

Frequency Response

- It follows from the input-output relation

$$y(t)=H(j\Omega)e^{j\Omega t}$$

  that the output signal $y(t)$ is also a complex exponential signal of the same angular frequency Ω multiplied by a complex constant $H(j\Omega)$

Frequency Response

- Consequence: For a stable LTI analog system, the steady-state output will not contain sinusoidal components of frequencies that do not appear in the input signal
- Otherwise, the analog system is not linear and time-invariant

Frequency Response

Example – Consider an ideal causal analog delay system for which $h(t)=\delta(t-t_o)$

- Its frequency response is thus

$$H(j\Omega)=\int_{-\infty}^{+\infty}\delta(\tau-t_o)e^{-j\Omega\tau}d\tau=e^{-j\Omega t_o}$$

Frequency Response

Example – Consider a causal LTI analog system with an impulse response

$$h(t)=e^{-\alpha t}\mu(t)\,,\ \mathcal{Re}\{\alpha\}>0$$

- Its frequency response is given by the CTFT of $h(t)$ and was computed earlier:

$$H(j\Omega)=\frac{1}{\alpha+j\Omega},\ \mathcal{Re}\{\alpha\}>0$$

Parts of Frequency Response

Real and Imaginary Parts

- For a real-valued $h(t)$

$$H(j\Omega) = H_{re}(j\Omega) + jH_{im}(j\Omega)$$

$$H_{re}(-j\Omega) = H_{re}(j\Omega)$$

$$H_{im}(-j\Omega) = -H_{im}(j\Omega)$$

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Parts of Frequency Response

Magnitude and Phase Functions

$$H(j\Omega) = |H(j\Omega)|e^{j\theta(\Omega)}$$

$|H(j\Omega)|$ - Magnitude function

$\theta(\Omega) = \arg\{H(j\Omega)\}$ - Phase function

- For a real-valued $h(t)$

$$|H(-j\Omega)| = |H(j\Omega)|$$

$$\theta(-\Omega) = -\theta(\Omega)$$

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Input-Output Relation in the Frequency Domain

- Applying the convolution integral property to $y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$ we arrive at

$$\boxed{Y(j\Omega) = H(j\Omega)X(j\Omega)}$$

↑ Frequency response

where $x(t) \overset{\text{CTFT}}{\leftrightarrow} X(j\Omega)$, $y(t) \overset{\text{CTFT}}{\leftrightarrow} Y(j\Omega)$

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Frequency Response

- Thus, the frequency response is given by

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

Example – The input and output signals of a causal stable analog system are

$$x(t) = 2e^{-1.5t}\mu(t)$$

$$y(t) = 4e^{-3t}\mu(t)$$

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Frequency Response

- From the example in Slide No. 6 we get

$$X(j\Omega) = \frac{2}{j\Omega + 1.5}$$

$$Y(j\Omega) = \frac{4}{j\Omega + 3}$$

- Hence,

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{2(j\Omega) + 3}{(j\Omega) + 3}$$

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Frequency Response

- The impulse response $h(t)$ of the analog system is thus given by the inverse CTFT of $H(j\Omega)$
- We first express $H(j\Omega)$ as

$$H(j\Omega) = K + \frac{\rho}{j\Omega + 3} = \frac{K(j\Omega) + (3k + \rho)}{j\Omega + 3}$$

- Equating the numerator of the above expression with that of $H(j\Omega)$ in the previous slide we get

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Frequency Response

$$K(j\Omega) + (3K + \rho) = 2(j\Omega) + 3$$

- The solution of the above equation is given by

$$K = 2, \ \rho = -3$$

- Thus,

$$h(t) = 2\delta(t) - 3^{-3t}\mu(t)$$

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Transfer Function

- The transfer function $H(s)$ of an LTI analog system is given by the Laplace transform of its impulse response $h(t)$:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

where s is a complex variable

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Transfer Function

- The transfer function $H(s)$ can be seen to be a generalized version of the frequency response $H(j\Omega)$ with $j\Omega \rightarrow s$
- By setting $s = j\Omega$ in the expression for $H(s)$ we get

$$H(j\Omega) = H(s)\big|_{s=j\Omega} = \int_{-\infty}^{\infty} h(\tau)e^{-j\Omega\tau}d\tau$$

which is the CTFT of $h(t)$

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Transfer Function

- $H(j\Omega)$ converges uniformly if $h(t)$ is absolutely integrable
- Alternately, $H(s)$ of an LTI analog system with an absolutely integrable $h(t)$ can be derived from its frequency response $H(j\Omega)$ by replacing $j\Omega$ with s:

$$H(s) = H(j\Omega)\big|_{j\Omega=s} = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

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Transfer Function

- The impulse response of a stable LTI analog system is absolutely integrable and as a result its frequency response exists
- Hence, the $j\Omega$ axis is in the ROC of the transfer function of a stable LTI analog system

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General Form of Input-Output Relation

- The causal LTI analog systems of interest are characterized in the time-domain by an input-output relation given by a linear constant-coefficient differential equation:

$$\frac{d^N y(t)}{dt^N} + q_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + q_1\frac{dy(t)}{dt} + q_0 y(t$$

$$= p_M\frac{d^M x(t)}{dt^M} + p_{M-1}\frac{d^{M-1}x(t)}{dt^{N-1}} + \cdots + p_1\frac{dx(t)}{dt} + p_0 x(t)$$

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General Rational Form of Frequency Response

where $y(t)$ and $x(t)$ are, respectively, the output and input analog signals, and the coefficients $\{q_k\}$ and $\{p_k\}$ are constants

General Rational Form

- Applying the CTFT to both sides of the differential equation, we arrive at after some algebra

Rational Form of Frequency Response

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{P(j\Omega)}{Q(j\Omega)}$$

$$= \frac{p_M(j\Omega)^M + p_{M-1}(j\Omega)^{M-1} + \cdots + p_1(j\Omega) + p_0}{(j\Omega)^N + q_{N-1}(j\Omega)^{N-1} + \cdots + q_1(j\Omega) + q_0}$$

- $P(j\Omega)$ is the numerator polynomial of degree M and $Q(j\Omega)$ is the denominator polynomial of degree N in the variable $j\Omega$

Rational Form of Transfer Function

- By replacing $j\Omega$ with s we arrive at the rational form of the transfer function

$$H(s) = \frac{P(s)}{D(s)} = \frac{p_M s^M + p_{M-1}s^{M-1} + \cdots + p_1 s + p_0}{s^N + q_{N-1}s^{N-1} + \cdots + q_1 s + q_0}$$

- Note: Knowing the differential equation representation, we can determine by inspection the transfer function and vice-versa

Rational Form of Transfer Function

- **Example**: Let

$$h(t) = 4\delta(t) - 2e^{-3t}\mu(t) + 4e^{-2t}\mu(t)$$

- The frequency response of the system is thus given by

$$H(j\Omega) = 4 - \frac{2}{j\Omega + 3} + \frac{4}{j\Omega + 2}$$

Rational Form of Transfer Function

- Hence, the transfer function of the system is

$$H(s) = 4 - \frac{2}{s+3} + \frac{4}{s+2} = \frac{4s^2 + 22s + 32}{s^2 + +5s + 6}$$

- Comparing the above equation with the expression given in Slide No. 21 we have

$$p_2 = 4,\ p_1 = 22,\ p_0 = 32,\ q_1 = 5,\ q_0 = 5$$

Differential Equation from Rational Form

- Hence, the differential equation representation of the system is given by

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t)$$

$$= 4\frac{d^2x(t)}{dt^2} + 22\frac{dx(t)}{dt} + 32x(t)$$

Factored Form of Transfer Function

$$H(s) = \frac{P(s)}{Q(s)} = p_M \left(\frac{\prod_{k=1}^{M}(s+\xi_k)}{\prod_{k=1}^{N}(s+\lambda_k)} \right)$$

- The constant ξ_k is the zero and the constant λ_k is the pole of the transfer function $H(s)$

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Partial-Fraction Expansion Form

- In practice, the degree M of the numerator polynomial of $H(s)$ is less than or equal to the degree N of the denominator polynomial
- In addition, the poles λ_k, $1 \le k \le N$, of $H(s)$ are distinct
- Such a transfer function can be expressed in a partial-fraction expansion form as shown in the next slide

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Partial-Fraction Expansion Form

$$H(s) = K + \sum_{k=1}^{N} \frac{\rho_k}{s+\lambda_k}$$

where

$$K = \begin{cases} 0, & M < N \\ p_N, & M = N \end{cases}$$

$$\rho_k = H(s)(s+\lambda_k)\big|_{s=-\lambda_k}, \quad 1 \le k \le N$$

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Partial-Fraction Expansion Form

- **Example** – Determine the transfer function $H(s)$ and the impulse response $h(t)$ of an LTI analog system with an input-output relation given by

$$\frac{d^2 y(t)}{dt^2} + 3.5\frac{dy(t)}{dt} + 3y(t) = 4\frac{dx(t)}{dt} + 2.5x(t)$$

- Here $q_1 = 3.5$, $q_0 = 3$, $p_1 = 4$, $p_0 = 2.5$

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Partial-Fraction Expansion Form

- The transfer function is thus given by

$$H(s) = \frac{4s+2.5}{s^2+3.5s+3} = \frac{4s+2.5}{(s+2)(s+1.5)}$$

- Its partial-expansion is of the form

$$H(s) = K + \frac{\rho_1}{s+2} + \frac{\rho_2}{s+1.5}$$

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Partial-Fraction Expansion Form

- Since the numerator polynomial degree is less than the denominator polynomial degree, we have $K = 0$
- The residues ρ_1 and ρ_2 are given by

$$\rho_1 = H(s)(s+2)\big|_{s=-2} = 11$$

$$\rho_2 = H(s)(s+1.5)\big|_{s=-1.5} = -7$$

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Partial-Fraction Expansion Form

- Hence the transfer function is given by,

$$H(s) = \frac{11}{s+2} - \frac{7}{s+1.5}$$

The frequency response is obtained by replacing s with $j\Omega$:

$$H(j\Omega) = \frac{11}{j\Omega+2} - \frac{7}{j\Omega+1.5}$$

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Partial-Fraction Expansion Form

- The inverse CTFT of $H(j\Omega)$ yields the impulse response

$$h(t) = 11e^{-2t}\mu(t) - 7e^{-1.5t}\mu(t)$$

- **Example** – Consider a causal LTI analog system with an input-output relation given by

$$\frac{dy(t)}{dt} + 4y(t) = 2x(t)$$

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Partial-Fraction Expansion Form

- We determine its output response $y(t)$ for an input signal $x(t) = 5e^{-2t}\mu(t)$
- Its transfer function is given by

$$H(s) = \frac{2}{s+3}$$

- Hence, its frequency response is

$$H(j\Omega) = \frac{2}{j\Omega+3}$$

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Partial-Fraction Expansion Form

- The CTFT of the input signal is

$$X(j\Omega) = \frac{5}{j\Omega+2}$$

- Hence, the CTFT of the output signal is given by

$$Y(j\Omega) = H(j\Omega)X(j\Omega) = \frac{10}{(j\Omega+3)(j\Omega+2)}$$

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Partial-Fraction Expansion Form

- The output signal $y(t)$ is then obtained by taking the inverse CTFT of $Y(j\Omega)$
- To this end, we develop using MATLAB the partial-fraction expansion of $Y(j\Omega)$ which is of the form

$$Y(j\Omega) = \frac{\rho_1}{j\Omega+3} + \frac{\rho_2}{j\Omega+2}$$

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Partial-Fraction Expansion Form

- Code fragments used are:

```
num = 10;
den = [1  5  6];
[r,p,k] = residue(num,den)
```

which yield

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Partial-Fraction Expansion Form

```
r =
   -10.0000
    10.0000
p =
   -3.0000
   -2.0000
k =
```

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Partial-Fraction Expansion Form

- Therefore

$$Y(j\Omega) = -\frac{10}{j\Omega+3} + \frac{10}{j\Omega+2}$$

whose inverse CTFT yields

$$y(t) = -10e^{-3t}\mu(t) + 10e^{-2t}\mu(t)$$

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DC Gain

- An important parameter characterizing a causal LTI analog system
- It is the value of its frequency response $H(j\Omega)$ at $\Omega = 0$
- For a real rational frequency response it is given by

$$H(j0) = H(s)\big|_{s=0} = \frac{p_0}{q_0}$$

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DC Gain

- Usually given in decibels
- For $$H(s) = \frac{4s^2+22s+32}{s^2+5s+6}$$

the dc gain in dB is

$$\mathcal{G}(0) = 20\log_{10}\left(\frac{p_0}{q_0}\right) = 20\log_{10}\left(\frac{32}{6}\right) = 14.54 \ \ dB$$

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BIBO Stability Condition

- Recall, the inverse CTFT of $\rho_k/(j\Omega+\lambda_k)$ exists if $\mathcal{R}e\{\lambda_k\} > 0$ and is given by

$$h_k(t) = \rho_k e^{-\lambda_k t}\mu(t)$$

- For a causal LTI analog system with a transfer function $H(s)$ in the form

$$H(s) = K + \sum_{k=1}^{N} \frac{\rho_k}{s+\lambda_k}$$

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BIBO Stability Condition

the frequency response is given by

$$H(j\Omega) = K + \sum_{k=1}^{N} \frac{\rho_k}{j\Omega+\lambda_k}$$

- Its impulse response is given by

$$h(t) = K\delta(t) + \sum_{k=1}^{N} h_k(t) = K\delta(t) + \sum_{k=1}^{N} \rho_k e^{-\lambda_k t}\mu(t)$$

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BIBO Stability Condition

- Since $h_k(t)$ is absolutely integrable for $\mathcal{Re}\{\lambda_k\} > 0$, $1 \le k \le N$, the causal LTI analog system is BIBO stable for $\mathcal{Re}\{\lambda_k\} > 0$, $1 \le k \le N$
- As the N poles of $H(s)$ are at $s = -\lambda_k$, the causal LTI analog system is BIBO stable if all its poles are in the left-half s-plane

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BIBO Stability Condition

First-Order Causal LTI Analog System

- Here

$$H(s) = \frac{p_1 s + p_o}{s + d_o}$$

- The system is BIBO stable if the pole of $H(s)$ is in the left-half s-plane

$$\Rightarrow d_0 > 0$$

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BIBO Stability Condition

Second-Order Causal LTI Analog System

- Here

$$H(s) = \frac{p_2 s^2 + p_1 s + p_o}{s^2 + q_1 s + q_0} = \frac{p_2 s^2 + p_1 s + p_o}{(s+\lambda_1)(s+\lambda_2)}$$

$$= \frac{p_2 s^2 + p_1 s + p_o}{s^2 + (\lambda_1 + \lambda_2)s + \lambda_1 \lambda_2}$$

- Note: $q_1 = \lambda_1 + \lambda_2$, $q_0 = \lambda_1 \lambda_2$

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BIBO Stability Condition

- The system is BIBO stable if the poles of $H(s)$ are in the left-half s-plane, that is

$$\mathcal{Re}\{\lambda_1\} > 0 \quad \mathcal{Re}\{\lambda_2\} > 0$$

- If λ_1 and λ_2 are complex numbers, then $\lambda_2 = \lambda_1^*$
- Thus, $q_1 = 2\mathcal{Re}\{\lambda_1\}$ and $q_0 = |\lambda_1|^2$

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BIBO Stability Condition

- Hence, system is BIBO stable if

$$\Rightarrow q_1 > 0, \; q_0 > 0$$

- If λ_1 and λ_2 are real numbers, then the poles of $H(s)$ are in the left-half s-plane if

$$q_1 = \lambda_1 + \lambda_2 > 0$$
$$q_0 = \lambda_1 \lambda_2 > 0$$

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BIBO Stability Condition

Higher Order Causal LTI Analog System

- Now, the denominator polynomial $Q(s)$ of $H(s)$ can be expressed as a product of first-order and/or second-order polynomials
- Hence, a necessary condition for BIBO stability is that all coefficients of $Q(s)$ of the transfer function must be positive and greater than zero

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BIBO Testing Using MATLAB

- If one or more coefficients are either missing or are negative, then the system is unstable

Stability Testing Using MATLAB

- Stability of $H(s)$ can be tested by determining the factors of $Q(s)$ using the MATLAB function `roots` for $M \leq N$

BIBO Testing Using MATLAB

- **Example**: Consider

$$H(s) = \frac{5s^2 - 4s + 3}{s^2 + 0.5s^2 + 0.2025s + 0.7025}$$

- Code fragments used are

```
den = [1 0.5 0.2025 0.7025];
lambda = roots(den)
```

which yield

BIBO Testing Using MATLAB

```
lambda =
  -1.0000
   0.2500 + 0.8000i
   0.2500 – 0.8000i
```

- As the two poles are in the right-half s-plane, the analog system is unstable