Cascade Connection

$$h(t) = h_1(t) \circledast h_2(t)$$

- In the frequency domain

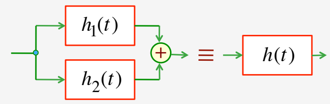
$$H(j\Omega) = H_1(j\Omega)H_2(j\Omega)$$

and in the s-domain

$$H(s) = H_1(s)H_2(s)$$

1

Parallel Connection



$$h(t) = h_1(t) + h_2(t)$$

- In the frequency domain

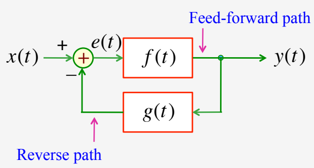
$$H(j\Omega) = H_1(j\Omega) + H_2(j\Omega)$$

and in the s-domain

$$H(s) = H_1(s) + H_2(s)$$

2

Feedback Connection



- Input-output relation

$$[\delta(t) + g(t) \circledast f(t)] \circledast y(t) = f(t) \circledast x(t)$$

3

Feedback Connection

- Taking the CTFT of both sides we get

$$[1 + G(j\Omega)F(j\Omega)]Y(j\Omega) = F(j\Omega)X(j\Omega)$$

- Therefore

$$Y(j\Omega) = \left(\frac{F(j\Omega)}{1 + F(j\Omega)G(j\Omega)}\right)X(j\Omega)$$

- Hence, in the frequency-domain

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{F(j\Omega)}{1 + F(j\Omega)G(j\Omega)}$$

4

Feedback Connection

and in the s-domain, $H(s) = \dfrac{F(s)}{1 + F(s)G(s)}$

Example – We determine the impulse response $h(t)$ of an analog feedback system for which

$$f(t) = 0.25\delta(t) + 0.25e^{-4t}\mu(t) \longleftarrow \text{Feed-forward path}$$

$$g(t) = -3\delta(t) + 9e^{-5t}\mu(t) \longleftarrow \text{Reverse path}$$

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Feedback Connection

- Corresponding frequency responses are given by

$$F(j\Omega) = 0.25 + \frac{0.25}{j\Omega + 4} = \frac{j\Omega + 5}{4(j\Omega + 4)}$$

$$G(j\Omega) = -3 + \frac{9}{j\Omega + 5} = \frac{3(j\Omega + 2)}{j\Omega + 5}$$

6

Feedback Connection

- The frequency response of the feedback system is thus given by

$$H(j\Omega) = \frac{F(j\Omega)}{1 - F(j\Omega)G(j\Omega)} = \frac{j\Omega + 5}{j\Omega + 10}$$

- Its transfer function is given by

$$H(s) = \frac{s+5}{s+10}$$

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Feedback Connection

- The partial-fraction expansion form of $H(s)$ is given by

$$H(s) = 1 - \frac{5}{s+10}$$

- The frequency response of the feedback system is thus

$$H(j\Omega) = 1 - \frac{5}{j\Omega + 10}$$

8

Feedback Connection

- The inverse CTFT of $H(j\Omega)$ yields the impulse response

$$h(t) = \delta(t) - 5e^{-10t}\mu(t)$$

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Inverse LTI Analog System

- Two causal LTI analog systems with impulse responses $g(t)$ and $h(t)$ are inverses of each other if

$$g(t) \circledast h(t) = \delta(t)$$

- Using the convolution integral property of CTFT we have

$$\boxed{G(j\Omega)H(j\Omega) = 1}$$

CTFT of $g(t)$ CTFT of $h(t)$

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Inverse LTI Analog System

- In terms of their transfer functions we have

$$\boxed{G(s)H(s) = 1}$$

- Hence, for a causal LTI analog system with a transfer function $H(s)$, the transfer function $H^{-1}(s)$ of its inverse causal LTI analog system is

$$\boxed{H^{-1}(s) = \frac{1}{H(s)}}$$

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Inverse LTI Analog System

- Note: If $H(s)$ is causal stable LTI analog system, its inverse system will be a stable system if it has the same number of zeros as poles with all zeros in the left-half s-plane
- Example: Consider the causal stable LTI analog system with a transfer function

$$H(s) = \frac{s+4}{s+2}$$

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Inverse LTI Analog System

- The transfer function of its inverse system is therefore given by

$$H^{-1}(s) = \frac{1}{H(s)} = \frac{s+2}{s+4}$$

- Since the zero of $H(s)$ is in the left-half s-plane, $H^{-1}(s)$ is stable

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Steady-State Response

- The output signal $y(t)$ of a stable LTI analog system characterized by a constant coefficient differential equation consists of two parts:

$$y(t) = y_c(t) + y_p(t)$$

complementary solution (output for $x(t) = 0$) ← $y_c(t)$

particular solution (output for given input $x(t)$) ← $y_p(t)$

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Sinusoidal Steady-State Response

- Consider a causal stable LTI system with a real valued impulse response $h(t)$
- Let $\tilde{x}(t) = A\cos(\Omega_o t + \phi)$, $-\infty < t < +\infty$
- Using the trigonometric identity we have $\tilde{x}(t) = g(t) + g^*(t)$ where $g(t) = \frac{1}{2}Ae^{j\phi}e^{j\Omega_o t}$

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Sinusoidal Steady-State Response

- Now for an input $e^{j\Omega_o t}$, the output is given by $H(j\Omega_o)e^{j\Omega_o t}$
- Hence, linearity property of the system, output for an input $g(t) = \frac{1}{2}Ae^{j\phi}e^{j\Omega_o t}$ is

$$v(t) = \tfrac{1}{2}Ae^{j\phi}H(j\Omega_o)e^{j\Omega_o t}$$

$$= \tfrac{1}{2}Ae^{j\phi}|H(j\Omega_o)|e^{j\theta(\Omega_o)}e^{j\Omega_o t}$$

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Sinusoidal Steady-State Response

- Output for an input $g^*(t) = \frac{1}{2}Ae^{-j\phi}e^{-j\Omega_o t}$ is

$$v^*(t) = \tfrac{1}{2}Ae^{-j\phi}H^*(j\Omega_o)e^{-j\Omega_o t}$$

$$= \tfrac{1}{2}Ae^{-j\phi}|H(-j\Omega_o)|e^{-j\theta(\Omega_o)}e^{-j\Omega_o t}$$

$$= \tfrac{1}{2}Ae^{-j\phi}|H(j\Omega_o)|e^{-j\theta(\Omega_o)}e^{-j\Omega_o t}$$

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Sinusoidal Steady-State Response

- Hence, for an input $x(t) = A\cos(\Omega_o t + \phi)$, the output is given by

$$y(t) = v(t) + v^*(t)$$

$$= \tfrac{1}{2}Ae^{j\phi}|H(j\Omega_o)|e^{j\theta(\Omega_o)}e^{j\Omega_o t}$$

$$+ \tfrac{1}{2}Ae^{-j\phi}|H(j\Omega_o)|e^{-j\theta(\Omega_o)}e^{-j\Omega_o t}$$

$$= A|H(j\Omega_o)|\left(e^{j\phi}e^{j\theta(\Omega_o)}e^{j\Omega_o t} + e^{-j\phi}e^{0j\theta(\Omega_o)}e^{-j\Omega_o t}\right)$$

$$\boxed{\tilde{y}(t) = A|H(j\Omega_o)|\cos(\Omega_o t + \theta(\Omega_o) + \phi)}$$

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Sinusoidal Steady-State Response

- Therefore, the output is a sinusoidal signal of the same frequency as the input except for an amplitude scaled by $|H(j\Omega_o)|$ and a phase lag of $\theta(\Omega_o)$ radians
- The derivation of the expression for the output assumes that the input signal has been present for all values of time in the range $-\infty < t < +\infty$

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Sinusoidal Steady-State Response

- **Example** – The frequency response of a causal stable analog system is given by

$$H(j\Omega) = \frac{2(j\Omega)+3}{(j\Omega)+6}$$

- We determine its steady-state response $\tilde{y}(t)$ for an input given by

$$\tilde{x}(t) = 5\cos(20t + 0.3)$$

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Sinusoidal Steady-State Response

- The frequency response for $\Omega = 20$ is given by

$$H(j20) = \frac{3 + j40}{6 + j20} = 1.8761 + j0.4128$$

- Its magnitude and phase at $\Omega = 20$ are

$$|H(j20)| = \sqrt{1.8761^2 + 0.4128^2} = 1.9210$$

$$\arg\{H(j20)\} = \tan^{-1}(0.4128/1.8761) = 0.2166$$

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Sinusoidal Steady-State Response

- Therefore

$$\tilde{y}(t) = 5 \times 1.921\cos(20t + 0.2166 + 0.3)$$
$$= 9.605\cos(20t + 0.5166)$$

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Steady-State Response to a Causal Input

- Thus, the LTI system is in the steady state throughout this range of time

Response to a Causal Input

- In practical applications, the input is a causal signal applied at a finite instant of time
- It is of interest to develop the expression for the output signal for a causal input

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Steady-State Response to a Causal Input

- We assume that the input is a causal exponential signal applied at time $t = 0$:

$$x(t) = e^{j\Omega t}\mu(t)$$

- As $x(t) = 0$ for $t < 0$, for a causal system the output $y(t) = 0$, for $t < 0$
- Thus we have for $t \geq 0$

$$y(t) = \int_0^\infty h(\tau)e^{-j\Omega(t-\tau)}\mu(t-\tau)d\tau$$

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Steady-State Response to a Causal Input

which reduces to

$$y(t) = \left(\int_0^t h(\tau) e^{-j\Omega\tau} d\tau \right) e^{-j\Omega t}$$

as $\mu(t - \tau) = 0$ for $t > \tau$

- We rewrite the above equation as

$$y(t) = \underbrace{\left(\int_0^\infty h(\tau) e^{-j\Omega\tau} d\tau \right)}_{H(j\Omega)} e^{j\Omega t} - \left(\int_t^\infty h(\tau) e^{-j\Omega\tau} d\tau \right) e^{j\Omega t}$$

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Steady-State Response to a Causal Input

- First term in the expression for $y(t)$ is the steady-state response

$$y_{ss}(t) = \left(\int_0^\infty h(\tau) e^{-j\Omega\tau} d\tau \right) e^{j\Omega t}$$

and the second term is the transient response

$$y_{tr}(t) = -\left(\int_t^\infty h(\tau) e^{-j\Omega\tau} d\tau \right) e^{j\Omega t}$$

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Steady-State Response to a Causal Input

- Note

$$|y_{tr}(t)| = \left| \int_t^\infty h(\tau) e^{-j\Omega\tau} d\tau \right| \leq \int_t^\infty |h(\tau)| d\tau \leq \int_0^\infty |h(\tau)| d\tau$$

- For a causal and stable LTI analog system, the impulse response is absolutely integrable, and hence the transient response $y_{tr}(t)$ is a bounded signal

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Steady-State Response to a Causal Input

- As $t \to \infty$, $\int_t^\infty h(\tau) e^{-j\Omega t} d\tau \to 0$

indicating that the transient response approaches zero value as t gets very large

- In most practical situations, the system is assumed to have reached the steady-state as the transient response becomes extremely small after a finite amount of time

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Phase and Group Delays

- Recall that for a sinusoidal input signal $\tilde{x}(t) = A\cos(\Omega_o t + \phi)$, $-\infty < t < \infty$, the output signal of a causal stable LTI analog system with a frequency response $H(j\Omega)$ is given by

$$\tilde{y}(t) = A|H(j\Omega_o)|\cos(\Omega_o t + \theta(\Omega_o) + \phi)$$

and thus has a phase lead of

$$\theta(\Omega_o) = \arg\{H(j\Omega_o)\}$$

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Phase and Group Delays

- A measure of the time delay of the phase is called the phase delay and is given by

$$\tau_p(\Omega_o) = -\frac{\theta(\Omega_o)}{\Omega_o}$$

where the minus sign indicates phase lag

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Phase and Group Delays

- Usually the input signal to a frequency-selective LTI analog system is composed of an weighted combination of many sinusoidal signals with different angular frequencies that are not harmonically related
- Each sinusoidal signal present in the input undergo different phase delays

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Phase and Group Delays

- In such cases, the time delay between the output and input signals is determined using an alternate parameter given by

$$\tau_g(\Omega) = -\frac{d\theta(\Omega)}{d\Omega}$$

- The parameter $\tau_g(\Omega)$ is known as the group delay

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Phase and Group Delays

- For the computation of the group delay, the phase function $\theta(\Omega)$ is assumed to have been unwrapped
- The physical difference between the phase delay and the group delay is illustrated in the next slide

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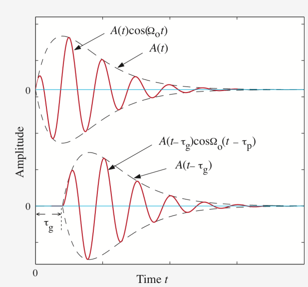
Phase and Group Delays



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Ideal LTI Analog Systems

- In practice, the analog signal as a linear combination of many, often infinite, number of sinusoidal analog signals
- Some applications require analog filters that passes components with frequencies in a specified range without distortion and block components outside this range

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Ideal Analog Filters

- The key to the design of analog filters is

$$y(t) = A|H(j\Omega_o)|\cos(\Omega_o t + \theta(\Omega_o) + \phi)$$

from which it can be seen that the value of the magnitude function at a given angular frequency Ω_o determines the value of the output response of the filter at that frequency Ω_o

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Ideal Analog Filters

- To pass sinusoidal components of the input in a specified frequency range, ideally the magnitude function should have a value of 1 in that frequency range
- To block signal components outside the specified frequency range, the magnitude function should have a value 0 in that frequency range

Ideal Analog Delay

- The system implementing the time shifting operation $y(t) = x(t - t_o)$ for a positive value of t_o
- Its impulse response is $h(t) = \delta(t - t_o)$
- Its frequency response is thus

$$H(j\Omega) = e^{-j\Omega t_o}$$

- Note: $|H(j\Omega)| = 1$ and $\theta(\Omega) = -\Omega t_o$

Ideal LTI Analog Systems

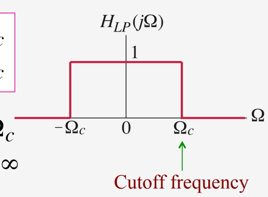
- Since $|H(j\Omega)| = 1$, any input analog signal appears at the output of the delay without any distortion of its amplitude
- The group delay of an ideal analog delay system is

$$\tau_g(\Omega) = -\frac{d\theta(\Omega)}{d\Omega} = t_o$$

Basic Analog Filters

Ideal Lowpass Analog Filter

$$H_{LP}(j\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

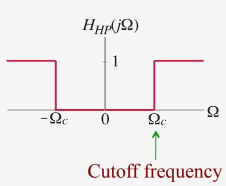


- Passband: $0 \le \Omega \le \Omega_c$
- Stopband: $\Omega_c < \Omega < \infty$

Basic Analog Filters

Ideal Highpass Analog Filter

$$H_{HP}(j\Omega) = \begin{cases} 0, & |\Omega| < \Omega_c \\ 1, & |\Omega| \ge \Omega_c \end{cases}$$



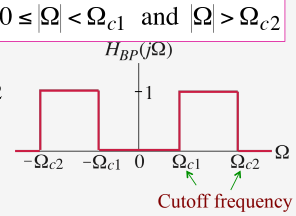
- Passband: $\Omega_c < \Omega < \infty$
- Stopband: $0 \le \Omega \le \Omega_c$

Basic Analog Filters

Ideal Bandpass Analog Filter

$$H_{BP}(j\Omega) = \begin{cases} 1, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0, & 0 \le |\Omega| < \Omega_{c1} \text{ and } |\Omega| > \Omega_{c2} \end{cases}$$

- Passband: $\Omega_{c1} \le |\Omega| \le \Omega_{c2}$
- Stopbands: $0 \le \Omega < \Omega_{c1}$, $\Omega_{c2} < \Omega < \infty$

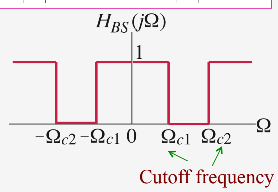


Basic Analog Filters

Ideal Bandstop Analog Filter

$$H_{BS}(j\Omega) = \begin{cases} 0, & \Omega_{c1} < |\Omega| < \Omega_{c2} \\ 1, & 0 \le |\Omega| \le \Omega_{c1} \text{ and } |\Omega| \le \Omega_{c2} \end{cases}$$

- Stopband:
  $\Omega_{c1} \le |\Omega| \le \Omega_{c2}$
- Passbands:
  $0 \le \Omega < \Omega_{c1}$
  $\Omega_{c2} < \Omega < \infty$



Basic Analog Filters

Impulse Response

- Expressions for the impulse responses of the above four ideal filters can be computed by applying the inverse CTFT to their frequency responses
- For example, the impulse response of the ideal lowpass filter is
- $$h_{LP}(t) = \frac{\sin(\Omega_c t)}{\pi t}, -\infty < t < \infty$$

Basic Analog Filters

- Note: $h_{LP}(t)$ is not absolutely integrable
- Hence, the ideal lowpass analog filter is not BIBO stable
- Also, $h_{LP}(t)$ is of doubly infinite length from $-\infty < t < \infty$, implying that the ideal lowpass analog filter is not a causal system

Basic Analog Filters

- The remaining three ideal filters also have a noncausal impulse response of doubly infinite length which are not absolutely summable making these filters unstable
- Filters with an ideal "brick wall" frequency response cannot be realized in practice

Basic Analog Filters

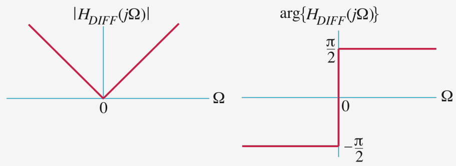
- To develop a stable and realizable filter, a transition band between the passband and stopband is included in the frequency response specification of the filter
- Also the magnitude response characteristic in the passband and the stopband are allowed to vary within a very small range called "ripples"

Ideal Analog Differentiator

- This system implements the differentiation operation
  $$y(t) = \frac{dx(t)}{dt}$$
- Taking the CTFT of both sides we arrive at its frequency response
  $$H_{DIFF}(j\Omega) = j\Omega$$

Ideal Analog Differentiator

- Plots of the magnitude and phase functions are shown below



Ideal Analog Differentiator

- From the magnitude function plot, it can be seen that high-frequency components of the input signal of a differentiator are amplified more compared to the low-frequency components
- Hence, a differentiator can be used to amplify portions of the input signal exhibiting sharp transitions

Concept of Filtering

- Consider the ideal highpass filter of Slide No. 41 with an input signal given by
  $\tilde{x}(t) = A\cos(\Omega_1 t) + B\cos(\Omega_2 t)$ where
  $0 < \Omega_1 < \Omega_c < \Omega_2 < \infty$
- Note: $A\cos(\Omega_1 t)$ is in the stopband of the filter and $B\cos(\Omega_2 t)$ is in the passband of the filter

Concept of Filtering

- From Slide No. 29 we note that the output of the highpass filter is

$$\tilde{y}(t) = A|H(j\Omega_1)|\cos(\Omega_1 t + \theta(\Omega_1)) + B|H(j\Omega_2)|\cos(\Omega_2 t + \theta(\Omega_2))$$
$$= B\cos(\Omega_2 t + \theta(\Omega_2))$$

as $|H(j\Omega_1)| = 0$ and $|H(j\Omega_2)| = 1$

A Simple Design Example

- **Example**: Design a LTI analog lowpass filter with a dc gain of 0 dB and a gain of -1 dB at 100 Hz
- Assume
$$H(j\Omega) = \frac{a}{j\Omega + b}$$
  with a and b real and positive numbers

A Simple Design Example

- Square-magnitude function is given by
$$|H(j\Omega)|^2 = H(j\Omega)H(-j\Omega) = \frac{a^2}{b^2 + \Omega^2}$$
- The gain function is given by
$$\mathcal{G}(\Omega) = 10\log_{10}|H(j\Omega)|^2 \text{ dB}$$

A Simple Design Example

- Design specifications:

$$\mathcal{G}(\Omega) = \begin{cases} 0, & \Omega = 0 \\ -20, & \Omega = 2\pi(100) \end{cases}$$

or, equivalently,

$$|H(j\Omega)|^2 = \begin{cases} 1, & \Omega = 0 \\ 0.01, & \Omega = 200\pi \end{cases}$$

A Simple Design Example

- Now $|H(j\Omega)|^2 = \frac{a^2}{b^2} = 1 \Longrightarrow a = b$
- Next, we set

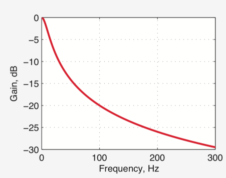
$$|H(j200\pi)|^2 = \frac{a^2}{a^2 + (200\pi)^2} = 0.01$$

which yields $a = 63.1484$

A Simple Design Example

- Thus, the frequency response of the filter is

$$H(j\Omega) = \frac{63.1484}{j\Omega + 63.1484}$$



Realizable Analog Filters

- We describe next a few low-order causal LTI analog filters with frequency responses approximating the ideal frequency responses presented earlier
- These low-order filters often provide satisfactory performances in many applications

Analog Lowpass Filters

- Transfer function of a stable causal first-order analog lowpass filter is

$$H_{LP}(s) = \frac{\Omega_c}{s + \Omega_c}, \ \Omega_c > 0$$

- Its gain response is

$$\mathcal{G}_{LP}(\Omega) = 10\log_{10}|H_{LP}(j\Omega)|^2 = 10\log_{10}\left(\frac{\Omega_c^2}{\Omega^2 + \Omega_c^2}\right)$$

Analog Lowpass Filters

- dc gain $\mathcal{G}_{LP}(0) = 10\ \log_{10}(1) = 0$ dB
- Gain at $\Omega = \Omega_c$ is

$$\mathcal{G}(\Omega_c) = 10\ \log_{10}(1/2) = -3.0103 \text{ dB}$$

- As the gain of the filter at $\Omega = \Omega_c$ is about 3 dB below the dc gain, Ω_c is called the 3-dB cutoff frequency of the lowpass filter

Analog Lowpass Filters

- **Example**: Design a first-order analog lowpass filter with a 3-dB cutoff frequency of 400 Hz
- The 3-dB cutoff angular frequency is thus

$$\Omega_c = 2\pi(400) = 2513.3$$

- Hence,

$$H_{LP}(s) = \frac{2513.3}{s + 2513.3}$$

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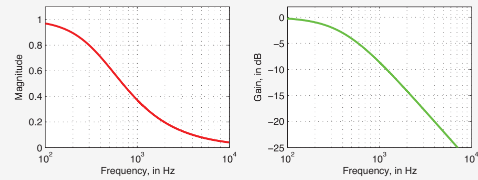
Analog Lowpass Filters

- Plots of the magnitude and gain functions of the lowpass filter are shown below



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Analog Highpass Filters

- Transfer function of a stable causal first-order analog highpass filter is

$$H_{HP}(s) = \frac{s}{s + \Omega_c}, \; \Omega_c > 0$$

- Its gain response is

$$\mathcal{G}_{HP}(\Omega) = 10 \log_{10} |H_{HP}(j\Omega)|^2 = 10 \log_{10}\left(\frac{\Omega^2}{\Omega^2 + \Omega_c^2}\right)$$

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Analog Highpass Filters

- The value of the gain response as $\Omega \to \infty$ is

$$\mathcal{G}_{HP}(\infty) = 10 \;\; \log_{10}(1) = 0 \text{ dB}$$

- Gain at $\Omega = \Omega_c$ is

$$\mathcal{G}(\Omega_c) = 10 \;\; \log_{10}(1/2) = -3.0103 \text{ dB}$$

- The gain of the filter at $\Omega = \Omega_c$ is about 3 dB below the gain at ∞, Ω_c is called the 3-dB cutoff frequency of the highpass filter

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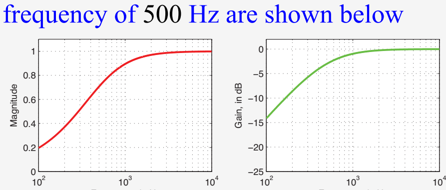
Analog Highpass Filters

- Plots of the magnitude and gain functions of a highpass filter with a 3-dB cutoff frequency of 500 Hz are shown below



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Analog Bandpass Filters

- The transfer function of a stable causal second-order analog bandpass filter with a frequency response approximating the frequency response of an ideal bandpass filter is

$$H_{BP}(s) = \frac{Bs}{s^2 + Bs + \Omega_o^2}, \;\; \Omega_o > 0, B > 0$$

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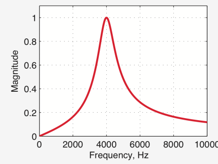
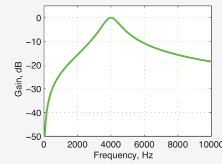
Analog Bandpass Filters

- The magnitude function has a maximum value of 1 at $\Omega = \Omega_o$, called the center frequency
- Let Ω_1 and Ω_2, with $\Omega_1 < \Omega_o < \Omega_2$, where the gain is -3 dB
- Parameter $B = \Omega_2 - \Omega_1$ is known as the 3-dB bandwidth of the passband

Analog Bandpass Filters

- Plots of the magnitude and gain functions of a bandpass filter with $B = 1000$ Hz and $\Omega_o = 4000$ Hz are shown below





Analog Bandstop Filters

- The transfer function of a stable causal second-order analog bandstop filter with a frequency response approximating the frequency response of an ideal bandstop filter is

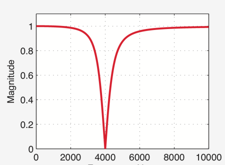
$$H_{BS}(s) = \frac{s^2 + \Omega_o^2}{s^2 + Bs + \Omega_o^2}, \quad \Omega_o > 0\,, B > 0$$

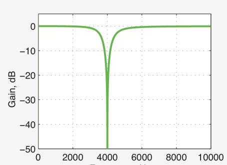
Analog Bandstop Filters

- The magnitude function has a maximum value of 0 at $\Omega = \Omega_o$, called the notch frequency
- Parameter B is known as the 3-dB notch bandwidth of the stopband
- The analog bandstop filter is also called an analog notch filter

Analog Bandstop Filters

- Plots of the magnitude and gain functions of a bandpass filter with $B = 1000$ Hz and $\Omega_o = 4000$ Hz are shown below





Analog Allpass Filters

- The analog allpass filter is a causal stable analog system with a transfer function $\mathcal{A}(s)$ whose magnitude square function is

$$|\mathcal{A}(j\Omega)|^2 = 1, \ -\infty < \Omega < +\infty$$

- For a real coefficient $\mathcal{A}(s)$, the above condition is equivalent to the condition

$$\mathcal{A}(s)\mathcal{A}(-s) = 1$$

Analog Allpass Filters

- If $Y(j\Omega)$ and $X(j\Omega)$ denote the CTFTs of the output and input analog signals, $y(t)$ and $x(t)$, respectively, then the condition

$$|\mathcal{A}(j\Omega)|^2 = 1, \; -\infty < \Omega < +\infty$$

implies

$$|Y(j\Omega)|^2 = |X(j\Omega)|^2$$

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Analog Allpass Filters

- Using the Parseval's relation we have

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

implying that a stable analog allpass filter is a lossless analog system

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Analog Allpass Filters

- We review next the transform-domain and time-domain representations of causal stable first-order and second-order analog allpass filters
- A higher-order allpass analog filter can be realized as a cascade of first- and/or second-order analog allpass filter sections

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Analog Allpass Filters

First-Order Analog Allpass Filter

- The transfer function of a stable causal first-order analog allpass filter with real coefficients is given by

$$\mathcal{A}_1(s) = \frac{s - q_0}{s + q_0}, \; q_0 > 0$$

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Analog Allpass Filters

Second-Order Analog Allpass Filter

- The transfer function of a stable causal second-order analog allpass filter with real coefficients is given by

$$\mathcal{A}_2(s) = \frac{s^2 - q_1 s + q_0}{s^2 + q_1 s + q_0}, \; q_1 > 0, \; q_0 > 0$$

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Analog Allpass Filters

A Simple Application

- One common application is in correcting the phase response of an analog frequency-selective filter designed to meet a specific magnitude response by placing in cascade with it an analog allpass filter

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Analog Allpass Filters

- Let $H(s)$ denote the transfer function of a frequency-selective causal stable analog filter
- Consider the cascade of $H(s)$ and a stable allpass filter with a transfer function $\mathcal{A}(s)$
- The overall transfer function of the cascade is
$$G(s) = H(s)\mathcal{A}(s)$$

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Analog Allpass Filters

- The magnitude response of the cascade is
$$|G(j\Omega)| = |H(j\Omega)\mathcal{A}(j\Omega)| = |H(j\Omega)||\mathcal{A}(j\Omega)|$$
$$= |H(j\Omega)|$$
- The phase response of the cascade is
$$\arg\{G(j\Omega)\} = \arg\{H(j\Omega)\} + \arg\{\mathcal{A}(j\Omega)\}$$

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Analog Allpass Filters

- Thus, the allpass filter can be designed so that $\arg\{G(j\Omega)\}$ is approximately linear in the frequency range of interest without changing the desired magnitude response $|H(j\Omega)|$

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