# Circuit Analysis Using Laplace Transform and Fourier Transform: 3-Element RC Circuit

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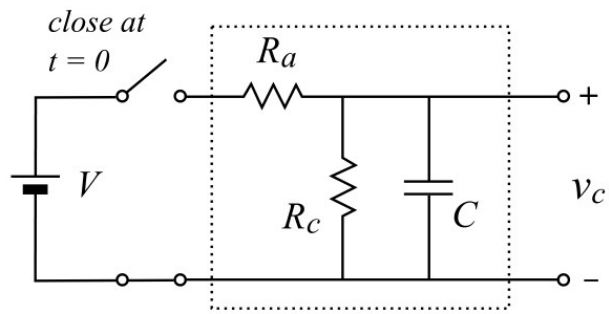
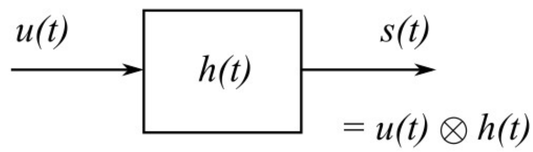
The schematic on the right shows a 3-element RC circuit. A constant voltage ($V$) is applied to the input of the circuit by closing the switch at $t = 0$. The output is the voltage across the capacitor *(C)*. The circuit can be represented as a linear time-invariant (LTI) system. We first normalize the input voltage: $V = 1$. Thus, the input is the unit step function *u(t),* and the output is the step response *s(t)*. The LTI system can be completely characterized by its impulse response *h(t)*. The step response is the convolution between the input step function and the impulse response: $s(t) = u(t) \otimes h(t)$.

**Circuit**

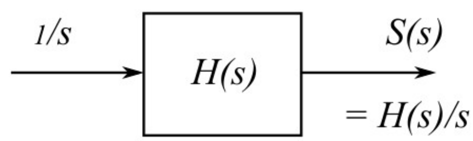



**Time-domain**




In the frequency domain, the transfer function *H(s)* is the Laplace transform (LT) of the impulse response *h(t)*. The LT of the unit step function is simply *s*. As convolution in time domain becomes multiplication in frequency domain, the Laplace transform of the step response is the product of *s* and *H(s)*.

**Frequency-domain**




**Circuit analysis using differential equations**

This is a first-order circuit with only one node, i.e. where the three elements join together. For $t \geq 0$, by evaluating Kirchhoff's current law at the node we have:

$$\frac{V - v_c}{R_a} = \frac{v_c}{R_c} + C\frac{dv_c}{dt} \text{ ------------------- (1)}$$

By rearranging terms, we have:

$$\frac{dv_c}{dt} = \frac{V}{R_a C} - \frac{R_a + R_c}{R_a R_c C} v_c \text{ -------------- (2)}$$

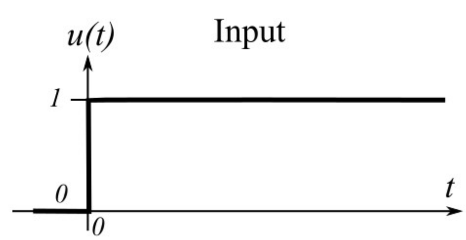
Assuming the capacitor is fully discharged at t = 0, the initial condition and the boundary condition, respectively, are:

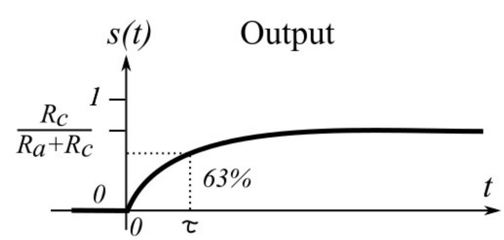
$$v_c = 0, \quad t = 0 \quad \text{and} \quad v_c = (R_c V)/(R_a + R_c), \quad t = \infty$$

The solution of the 1st-order differential equation is:

$$v_c = \frac{R_c V}{R_a + R_c} \left(1 - e^{-\frac{R_a + R_c}{R_a R_c C} t}\right) = \frac{R_c V}{R_a + R_c} \left(1 - e^{-t/\tau}\right)$$
------------------------------------------------ (3)

where $\tau = R_a R_c C / (R_a + R_c)$ is the time constant of the circuit.



**Circuit analysis using Laplace transform**

The transform function is simply the voltage divider in the Laplace domain, where the impedance of the capacitor is $1/sC$:

$$H(s) = \frac{R_c/(1+sR_cC)}{R_a+R_c/(1+sR_cC)} = \frac{R_c}{R_a+R_c+sR_aR_cC} \quad \text{------------------------------------------ (4)}$$

The LT of the step response is:

$$S(s) = H(s)/s = \frac{R_c/(R_a+R_c)}{s(1+sR_aR_cC/(R_a+R_c))} \quad \text{---------------------------------------------- (5)}$$

Applying the partial fraction technique and let $R_{ac} = R_aR_c/(R_a+R_c)$:

$$S(s) = \frac{A}{s} + \frac{B}{1+sR_{ac}C} = \frac{A(1+sR_{ac}C)+sB}{s(1+sR_{ac}C)} = \frac{A+s(AR_{ac}C+B)}{s(1+sR_{ac}C)} \quad \text{---------------- (6)}$$

Equating the numerators of eq (5) and (6), we have:

$$\frac{R_c}{R_a+R_c} = A + s(AR_{ac}C+B) \quad \text{---------------------------------------------------------------- (7)}$$

$$A = \frac{R_c}{R_a+R_c}, \text{ and } B = -AR_{ac}C \quad \text{------------------------------------------------------------ (8)}$$

Solving for *B*, we have:

$$B = -\left(\frac{R_c}{R_a+R_c}\right)R_{ac}C \quad \text{------------------------------------------------------------------------- (9)}$$

Substituting (8) and (9) into (6), the LT of the step response *S(t)* is given by:

$$S(s) = \left(\frac{R_c}{R_a+R_c}\right)\left(\frac{1}{s} - \frac{R_{ac}C}{1+sR_{ac}C}\right) = \left(\frac{R_c}{R_a+R_c}\right)\left(\frac{1}{s} - \frac{1}{s-(-1/\tau)}\right) \quad \text{------------------ (10)}$$

where $\tau = (R_aR_c)C/(R_a+R_c) = R_{ac}C$.

From the Table of Laplace Transforms: $\frac{1}{s} \rightarrow u(t)$ and $\frac{1}{s-a} \rightarrow e^{at}$ ---------------------- (11)

The step response is the inverse LT of (10):

$$s(t) = \frac{R_c}{R_a+R_c}\left(1 - e^{-t/\tau}\right) \quad \text{----------------------------------------------------------------- (12)}$$

which is the same as (3) with the input voltage *V* normalized to 1.

**Transfer function and impulse response**

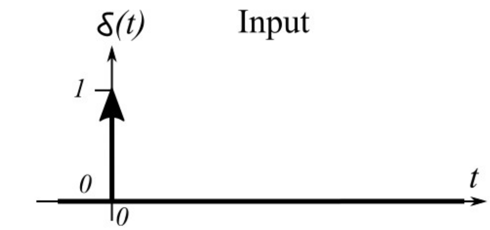
From (4), the transfer function is:

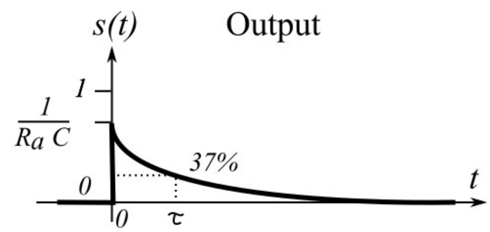
$$H(s) = \left(\frac{R_c}{R_a+R_c}\right)\frac{1}{1+s\,R_a R_c C/(R_a+R_c)} = \left(\frac{R_c}{\tau(R_a+R_c)}\right)\frac{1}{s-(-1/\tau)} = \left(\frac{1}{R_a C}\right)\frac{1}{s-(-1/\tau)} \text{ ---- (13)}$$

Taking the inverse LT results in the impulse response:

$$h(t) = \frac{1}{R_a C}\, e^{-t/\tau} \text{ ------------------------------- (14)}$$

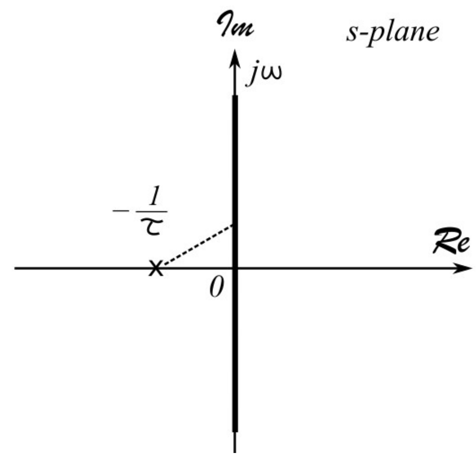
which completely characterizes the dynamics of the linear time-invariant system. The time constant τ, given by $R_a R_c C/(R_a+R_c)$, is when the output drops to 37% (1/e) of the initial value. Notice that the initial value $1/R_a C$ is independent of $R_c$.




**Fourier transform**

Zeros are points on the *s*-place where the transfer function *H(s)* is 0. Poles are points on the *s*-place where *H(s)* approaches to ∞. From (13), we see that this first-order system has one zero at $s = \infty$ and one pole at $s = -1/\tau$. As shown in the figure, the pole is marked by "X". Because the Fourier transform (FT) is evaluated along the *j*ω axis of the *s*-plane, the denominator of *H(s)*, $s-(-1/\tau)$, is represented by the dashed line. As the frequency (ω) increases from 0 (DC) to infinity, the magnitude of the line increases and the magnitude of *H(jω)* decreases. Thus, the system is a low-pass filter.




The FT is obtained by substituting $s = j\omega$ in (13), we have:

$$H(j\omega) = \left(\frac{1}{R_a C}\right)\frac{1}{j\omega+1/\tau} = \left(\frac{1}{R_a C}\right)\frac{1/\tau - j\omega}{1/\tau^2+\omega^2} \text{ ---- (15)}$$

Taking the magnitude of (15), we have:

$$|H(j\omega)| = \left(\frac{1}{R_a C}\right)\frac{1}{\sqrt{1/\tau^2+\omega^2}} \text{ ---------------------- (16)}$$

The angle is given by:

$$\measuredangle H(j\omega) = -\tan^{-1}\tau\omega \text{ -------------------------------- (17)}$$

**Frequency response**

Let's assign values to the circuit components so that we can plot the frequency response as an example. Let's assume:

$$R_a = R_c = 10\,K\Omega\,; \quad C = 0.1\,\mu F \text{ ------------------------------------------- (18)}$$

Based on these values, we have:

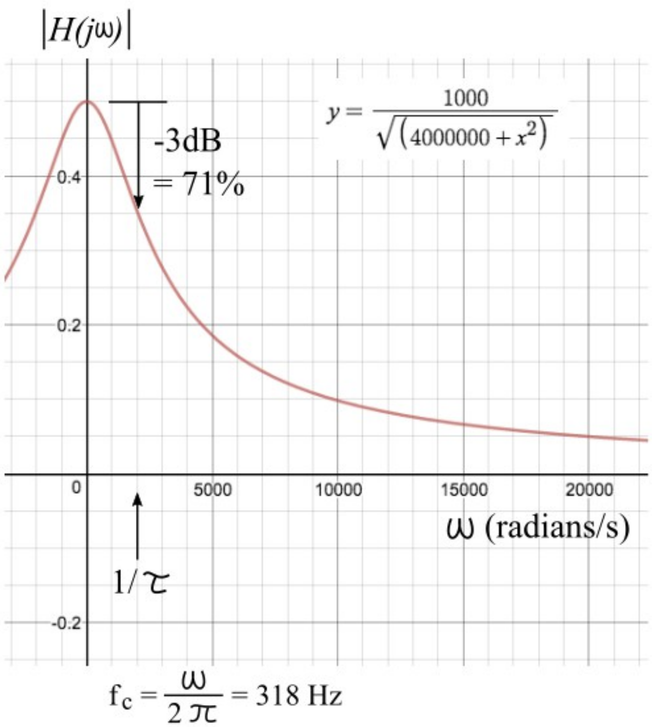
$$\frac{R_c}{R_a+R_c} = 0.5\,; \text{ and } \tau = \frac{R_a R_c C}{R_a+R_c} = 0.0005 \text{ ----------------------------------- (19)}$$

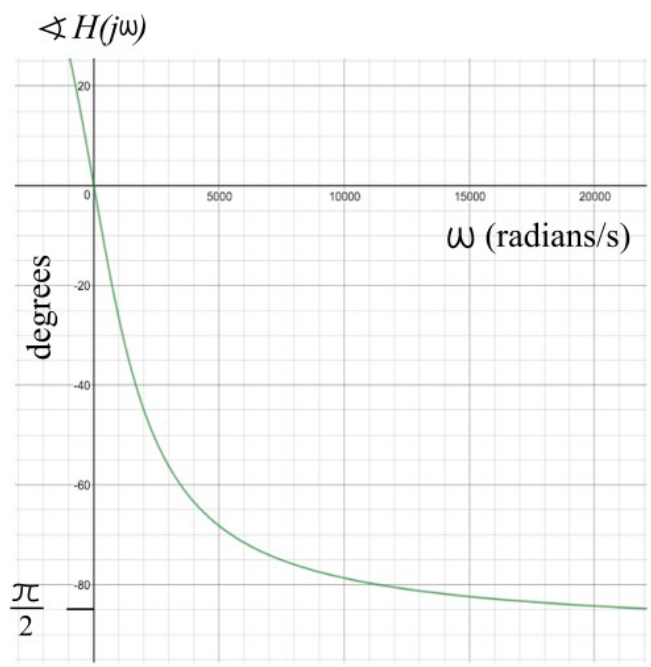
Inserting these values into (16), we have:

$$|\,H(j\omega)\,| = (\frac{1}{R_a C})\frac{1}{\sqrt{1/\tau^2+\omega^2}} = \frac{1000}{\sqrt{4000000+\omega^2}} \text{ --------------------------- (20)}$$

$$\sphericalangle H(j\omega) = -\tan^{-1} 0.0005\,\omega \text{ ----------------------------------------------------- (21)}$$

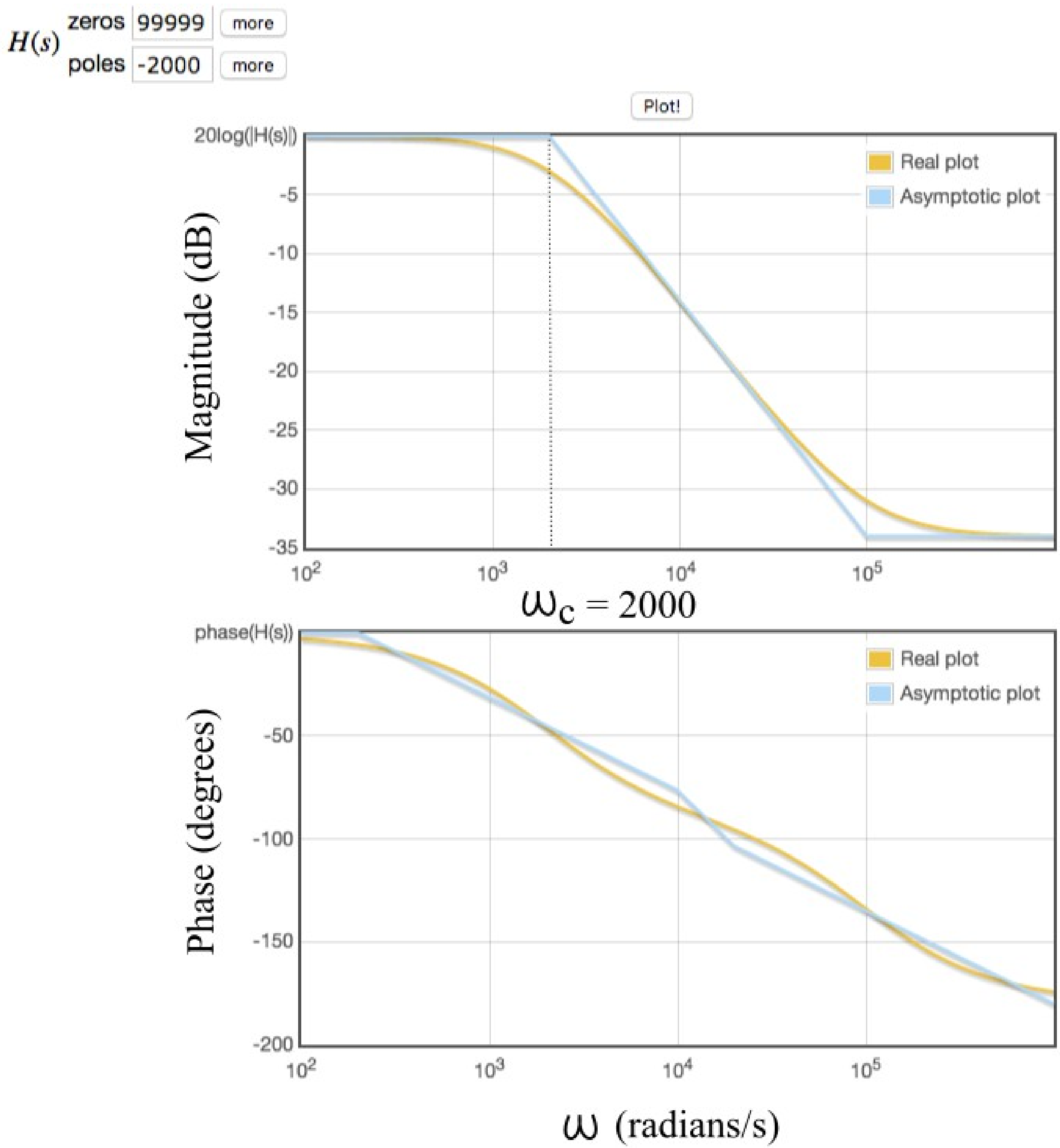
Using an online graphing calculator at <www.desmos.com/calculator>, the magnitude of the FT can be plotted as shown in the figure. The low-pass filter has a gain (magnitude) of 0.5 at DC, which decreases as frequency increases. The cutoff frequency *fc* is at $\omega_c = 1/\tau = 1/0.0005$ = 2000 radians/s, where the gain decreases to 71% ( $1/\sqrt{2}$ ) of the DC gain. This is called the 3 dB point, because $20\log_{10}(1/\sqrt{2}) = -3$ dB. The angular frequency is related to the regular frequency by $\omega = 2\pi f$ . Thus, the cutoff frequency is $f_c = \omega_c/2\pi = 318$ Hz. The phase of *H(s)* given by (21) is also plotted.



**Bode plot**

Bode plot is a graph of the frequency response of a system on the log-log scales. The transfer function *H(s)* in (13) shows one zero at $s = \infty$ and one pole at $s = -1/\tau = -2000$. Using an online Bode plot generator at <http://http://www.onmyphd.com/?p=bode.plot>. We obtain the following:




Please note that the above Bode plot assumes that the DC gain is 1 or 0 dB, which should be 0.5 or -6 dB. So the actual magnitude Bode plot should be shifted down by 6 dB. For a first-order low-pass filter, the slope of the declining portion of the Bode plot is -20 dB per decade. For example, the magnitude drops from 0 dB to -20 dB for frequency from 2000 radians/s to 20000 radians/s.