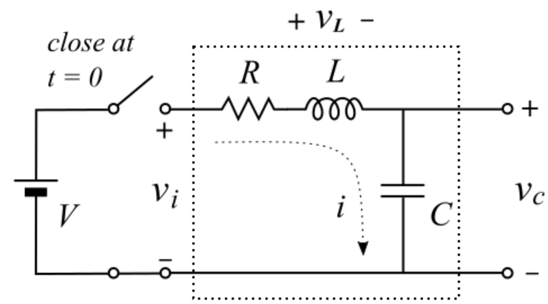
Circuit Analysis Using Laplace Transform and Fourier Transform: RLC Low-Pass Filter

Ying Sun



The schematic on the right shows a 2nd-order RLC circuit. A constant voltage (*V*) is applied to the input of the circuit by closing the switch at $t = 0$. The output is the voltage across the capacitor *(C)*. The circuit can be represented as a linear time-invariant (LTI) system. The input is v_i . If a constant voltage is applied at $t = 0$, it is a step input. We further normalize the input voltage $V = 1$ such that it's unit step function. Thus, the input is the unit step function *u(t),* and the output is the step response *s(t)*. The LTI system can be completely characterized by its impulse response *h(t)*. The step response is the convolution between the input step function and the impulse response: $s(t) = u(t) \otimes h(t)$.

Circuit analysis using Laplace transform

The circuit analysis can be done by use of the Kirchhoff's voltage law and the properties of capacitor and inductor:

$$i = C\frac{d\,v_c}{dt} \text{ , and } v_L = L\frac{d\,i}{dt} \quad \text{--- (1)}$$

$$v_i = Ri + v_L + v_c = R\,i + L\frac{d\,i}{dt} + v_c \quad \text{--- (2)}$$

By substituting (1) into (2), we have:

$$v_i = LC\frac{d^2 v_c}{d\,t^2} + RC\frac{d\,v_c}{dt} + v_c \quad \text{-- (3)}$$

Although a closed form solution can be obtained by solving the above 2nd-order differential equation, we will take the frequency-domain approach. Taking LT on both side, we have:

$$V_i(s) = LC\,s^2\,V_c(s) + RC\,s\,V_c(s) + V_c(s) \quad \text{-- (4)}$$

The transfer function is given by:

$$H(s) = \frac{V_c(s)}{V_i(s)} = \frac{1}{LC\,s^2 + RC\,s + 1} \quad \text{--- (5)}$$

We now use a different set of parameters:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\,\alpha\,s + \omega_n^2} \quad \text{--- (6)}$$

where $\omega_n = \dfrac{1}{\sqrt{LC}}$, natural frequency

$\alpha = \dfrac{R}{2\,L}$, damping factor

We further define damped frequency ω_d :

$$\omega_d = \sqrt{\omega_n^2-\alpha^2}\,; \quad \text{or} \quad \omega_n^2 = \omega_d^2+\alpha^2\,; \quad \text{or} \quad \omega_n = \sqrt{\omega_d^2+\alpha^2}$$

To obtain the impulse response, the transfer function is further extended to:

$$H(s) = \frac{\omega_n^2}{s^2+2\alpha s+\omega_n^2} = \frac{\omega_n^2}{(s+\alpha+\sqrt{\alpha^2-\omega_n^2})(s+\alpha-\sqrt{\alpha^2-\omega_n^2})} = \frac{\omega_n^2}{(s+\alpha+j\omega_d)(s+\alpha-j\omega_d)} =$$

$$\frac{\omega_d^2+\alpha^2}{s^2+2\alpha s+\alpha^2+\omega_d^2} = (\omega_d+\frac{\alpha^2}{\omega_d})(\frac{\omega_d}{(s+\alpha)^2+\omega_d^2}) \text{ -- (7)}$$

The system has two poles at: $-\alpha+j\omega_d$ and $-\alpha-j\omega_d$.

Taking the ILT, the impulse response is:

$$h(t) = (\omega_d+\frac{\alpha^2}{\omega_d})e^{-\alpha t}\sin\omega_d t \text{ -- (8)}$$

Next, we want to get a closed form solution for the step response. This will be accomplished by extending *H(s)* to *H(s)/s*, or *S(s)*, which is the LT of the step response.

$$S(s) = \frac{H(s)}{s} = (\omega_d+\frac{\alpha^2}{\omega_d})(\frac{1}{s})(\frac{\omega_d}{(s+\alpha)^2+\omega_d^2}) = \frac{a}{s}+\frac{bs+c}{(s+\alpha)^2+\omega_d^2}$$

$$\omega_d^2+\alpha^2 = a(s+\alpha)^2+a\omega_d^2+bs^2+cs = (a+b)s^2+(2a\alpha+c)s+a(\omega_d^2+\alpha^2)$$

We have $a+b=0$, $2a\alpha+c=0$, and $\omega_d^2+\alpha^2=a(\omega_d^2+\alpha^2) \Rightarrow a = 1, b = -1$, and $c = -2\alpha$.

$$S(s) = \frac{1}{s}-\frac{s+2\alpha}{(s+\alpha)^2+\omega_d^2} = \frac{1}{s}-\frac{s+\alpha}{(s+\alpha)^2+\omega_d^2}-(\frac{\alpha}{\omega})\frac{\omega_d}{(s+\alpha)^2+\omega_d^2} \text{ ------------------------- (9)}$$

Using the LT Table, we obtain the step response *s(t)*:

$$s(t) = 1-e^{-\alpha t}[\cos\omega_d t+(\frac{\alpha}{\omega_d})\sin\omega_d t] \text{ --- (10)}$$

Next, we want to combine cosine and sine into one term with a phase angle. We further define the damping ratio η, $\eta=\frac{\alpha}{\omega_n}$.

$$\frac{\alpha}{\omega_d} = \frac{\alpha}{\sqrt{\omega_n^2-\alpha^2}} = \frac{1}{\sqrt{\omega_n^2/\alpha^2-1}} = \frac{1}{\sqrt{1/\eta^2-1}} = \frac{\eta}{\sqrt{1-\eta^2}} \text{ --------------------------------- (11)}$$

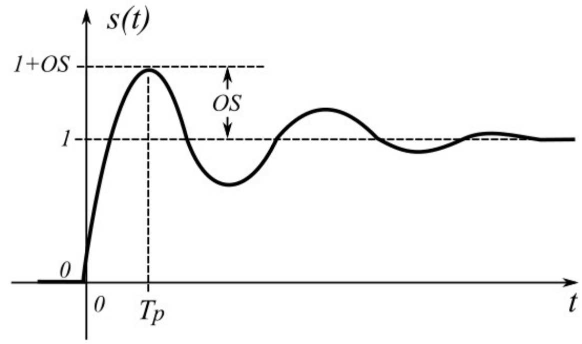
$$s(t) = 1-e^{-\alpha t}[\cos\omega_d t+(\frac{\eta}{\sqrt{1-\eta^2}})\sin\omega_d t] = 1-\frac{e^{-\alpha t}}{\sqrt{1-\eta^2}}[\sqrt{1-\eta^2}\cos\omega_d t+\eta\sin\omega_d t] =$$

$$1-\frac{e^{-\alpha t}}{\sqrt{1-\eta^2}}[\sin\phi\cos\omega_d t+\cos\phi\sin\omega_d t] = 1-\frac{e^{-\alpha t}}{\sqrt{1-\eta^2}}\sin(\omega_d t+\phi) \text{ --------------------- (12)}$$

where $\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sqrt{1-\eta^2}}{\eta}$, $\phi = \tan^{-1}\frac{\sqrt{1-\eta^2}}{\eta}$.

From the previous circuit course, such as ELE 212, you might have learned what the damping ratio means: a) underdamping $(\eta<1)$; b) critical damping $(\eta=1)$; and c) overdamping $(\eta>1)$.

Assume the underdamping situation, *s(t)* is shown on the right. The 2nd-order system requires two parameters to define, such as the damped frequency ω_d and the damping factor α . These two parameters can be obtained from the *s(t)* curve by making two measurements. A prominent feature point is the first peak after the onset. We measure the time to peak T_p and the amount of overshoot (OS). This point occurs when the derivative of the curve is 0.



$$\frac{d\,s(t)}{dt} = 0$$

$$\frac{d\,s(t)}{dt} = h(t) = (\omega_d + \frac{\alpha^2}{\omega_d}) e^{-\alpha t} \sin \omega_d t = 0 \text{ . --- (13)}$$

The peaks and valleys occur when $\omega_d t$ = 0, π, 2π, 3 π, 4π, …. The first peak occurs when $\omega_d t = \pi$.

Thus, $T_p = \frac{\pi}{\omega_d}$, or $\omega_d = \frac{\pi}{T_p}$ -- (14)

At the first peak T_p ,

$$s(T_p) = 1 - \frac{e^{-\alpha T_p}}{\sqrt{1-\eta^2}} \sin(\omega_d T_p + \tan^{-1}\frac{\sqrt{1-\eta^2}}{\eta}) = 1 - \frac{e^{-\alpha T_p}}{\sqrt{1-\eta^2}} \sin(\pi + \tan^{-1}\frac{\sqrt{1-\eta^2}}{\eta}) =$$

$$1 + \frac{e^{-\alpha T_p}}{\sqrt{1-\eta^2}} \sin(\tan^{-1}\frac{\sqrt{1-\eta^2}}{\eta}) = 1 + \frac{e^{-\alpha T_p}}{\sqrt{1-\eta^2}} \sqrt{1-\eta^2} = 1 + e^{-\alpha T_p} = 1 + OS \text{ . ------------ (15)}$$

Thus, $e^{-\alpha T_p} = OS$, or $\alpha = -\frac{\ln OS}{T_p}$. --- (16)

Frequency response

Let's assign values to the circuit components so that we can plot the frequency response as an example. Let's assume:

$$R = 40\,\Omega \text{ ; } L = 1 \; mH \text{ ; } C = 0.1\,\mu F \text{ --- (17)}$$

Based on these values, we have:

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\,mH \times 0.1\,\mu F}} = 100000 \text{ radians/s; and}$$

$$\alpha = \frac{R}{2\mathrm{L}} = \frac{40\,\Omega}{2 \times 1\,mH} = 20000$$

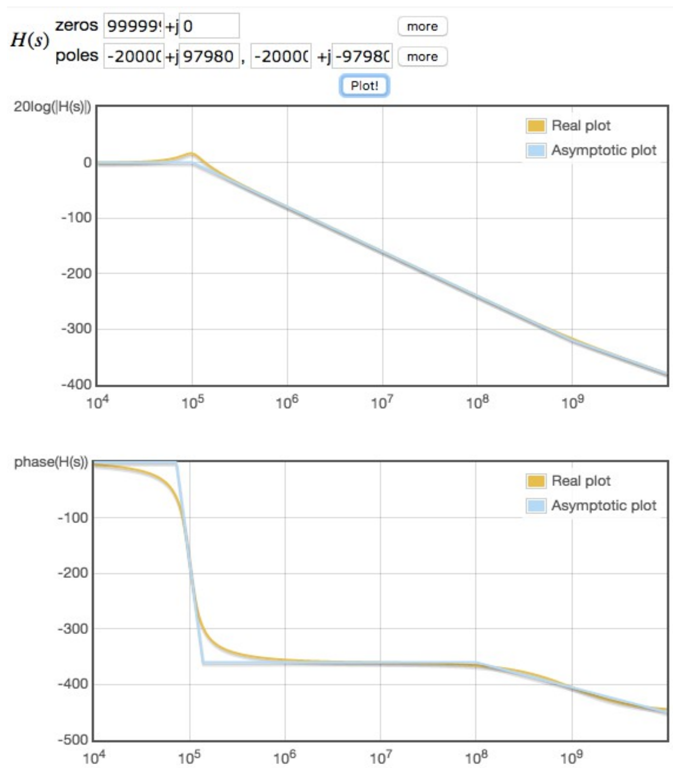
$$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = \sqrt{100000^2 - 20000^2} = 97980 \text{ radians/s.}$$

$$\eta = \frac{\alpha}{\omega_n} = \frac{20000}{100000} = 0.2 \text{ ; } \phi = \tan^{-1}\frac{\sqrt{1-\eta^2}}{\eta} = \tan^{-1}\frac{\sqrt{1-0.2}}{0.2} = 1.35 \text{ radians.}$$

The transfer function is $H(s) = (\omega_d + \frac{\alpha^2}{\omega_d})(\frac{\omega_d}{(s+\alpha)^2+\omega_d^2})$. The system has the poles at $-\alpha \pm j\omega_d$, or $-20000 \pm 97980\,j$. The zeros are at infinity.

Bode plot

Enter the poles of $-20000 \pm 97980\,j$ at the online Bode plot generator <http://http://www.onmyphd.com/?p=bode.plot>. We obtain the following:



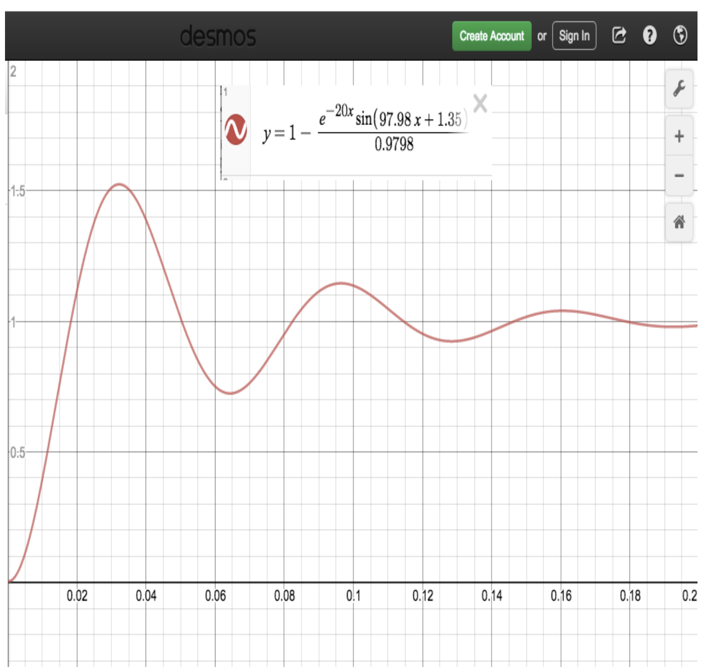
Step response

$$s(t) = 1 - \frac{e^{-\alpha t}}{\sqrt{1-\eta^2}} \sin(\omega_d t + \phi) =$$

$$1 - \frac{e^{-20000\,t}}{0.9798} \sin(97980\,t + 1.35) =$$

Due to a range issue with the graphing calculator, we change the time unit from s to ms:

$$s(t) = 1 - \frac{e^{-20\,t}}{0.9798} \sin(97.98\,t + 1.35)$$



From the graph, we see that the peak occurs roughly at: $T_p = 0.0323$ ms, and OS = 0.52.

Using the formula derived previously:

$$\omega_d = \frac{\pi}{T_p} = \frac{3.14159}{0.0000323} = 97263 \,,$$

as compare to $\omega_d = 97980$.

$$\alpha = -\frac{\ln OS}{T_p} = -\frac{\ln 0.52}{0.0000323} = 20245 \,,$$

as compare to $\alpha = 20000$.