# Feedback Implementation

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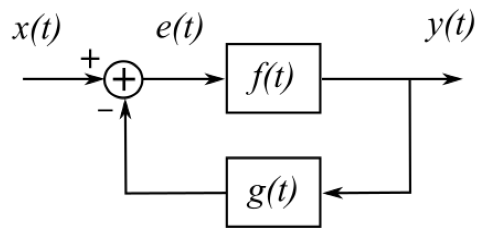
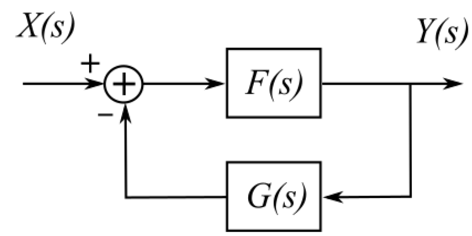
## LTI System Implemented with a Feedback Connection




The system shown on the right has a feedforward unit *f(t)* and a feedback unit *g(t)*. What is it's overall transfer function *H(s)*?

The output of the feedback unit is $y(t) \otimes g(t)$.

The error function *e(t)* is given by

$$e(t) = x(t) - y(t) \otimes g(t)$$

The output of the allover system is

$$y(t) = [x(t) - y(t) \otimes g(t)] \otimes f(t)$$

$$y(t) = x(t) \otimes f(t) - y(t) \otimes g(t) \otimes f(t)$$

$$y(t) + y(t) \otimes g(t) \otimes f(t) = x(t) \otimes f(t)$$

$$[\delta(t) + g(t) \otimes f(t)] \otimes y(t) = x(t) \otimes f(t)$$

Take LT on both sides

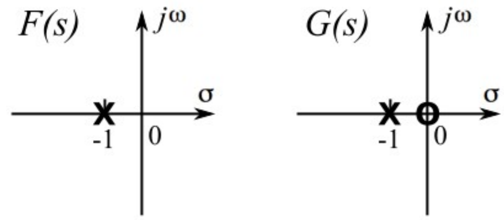
$$[1 + G(s)F(s)]Y(s) = X(s)F(s)$$

The transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(S)}{1 + G(s)F(s)}$$

Example 1: Let *F(s)* be a low-pass filter and *G(s)* be a high-pass filter. Thus, the low-pass filter is further enhanced by subtracting the high-pass result from the output. The individual transfer functions are given below. Find the overall transfer function, pole-zero plot, Bode plot, and the filter differential equation.

$$F(s) = \frac{1}{s+1}\,; \qquad G(s) = \frac{s}{s+1}\,.$$

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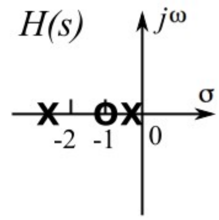
$$H(s) = \frac{F(s)}{1+F(s)G(s)} = \frac{\frac{1}{s+1}}{1+(\frac{1}{s+1})(\frac{s}{s+1})} =$$

$$= \frac{s+1}{(s+1)^2+s} = \frac{s+1}{(s+1)^2+s} =$$

$$\frac{s+1}{s^2+3s+1} = \frac{s+1}{(s+0.382)(s+2.618)}\,.$$




Note: Solving quadratic equation $s^2+3s+1$, the two roots are $s=(-3\pm\sqrt{3^2-4})/2$.
The pole-zero plots of the original systems and the overall system are shown. The poles and zeros of the systems are: Poles at -0.382 and -2.618; Zeros at -1; ∞.
The Bode plot obtained from <http://www.onmyphd.com/?p=bode.plot> is shown.

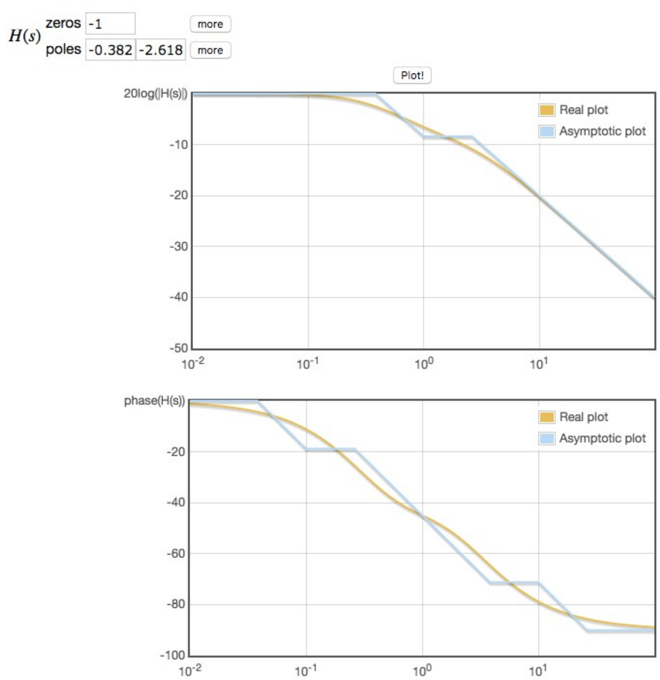
The filter differential equation can be obtained from the transfer function.

$$H(s) = \frac{s+1}{s^2+3s+1}\,.$$

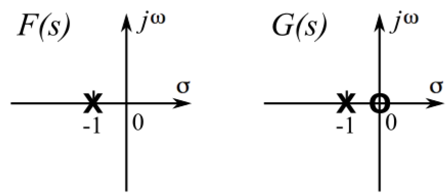
$$Y(s)(s^2+3s+1) = X(s)(s+1)$$

Take the ILT, we have

$$\frac{d\,y^2(t)}{dt}+3\frac{d\,y(t)}{dt}+y(t) = \frac{d\,x(t)}{dt}+x(t)\,.$$



Example 2: Using the same *F(s)* and *G(s)*, what is the overall transfer function H(s) if the two are connected in parallel?




$$F(s) = \frac{1}{s+1}; \qquad G(s) = \frac{s}{s+1}.$$

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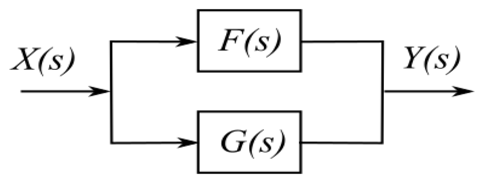
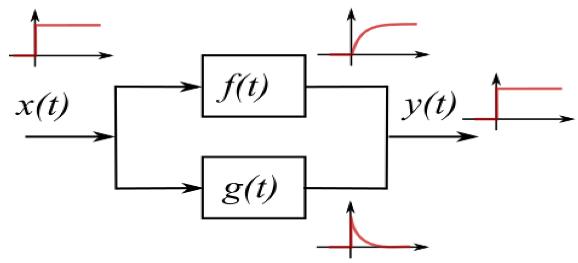
$$H(s) = F(s)+G(s) = \frac{1}{s+1} + \frac{s}{s+1} =$$

$$\frac{1+s}{s+1} = 1$$

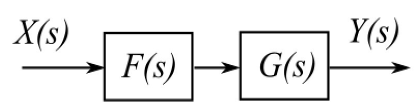



The resulting system is an identity system. The low-pass filter *F(s)* and the low-pass filter *G(s)* exactly complement each other, as demonstrated by the step input shown on the right.




Example 3: Using the same *F(s)* and *G(s)*, what is the overall transfer function H(s) if the two are connected in series?

$$F(s) = \frac{1}{s+1}; \qquad G(s) = \frac{s}{s+1}.$$

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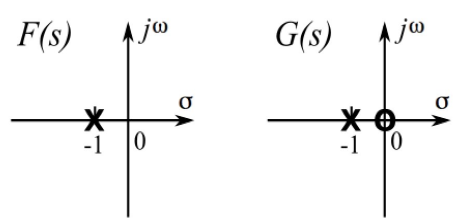
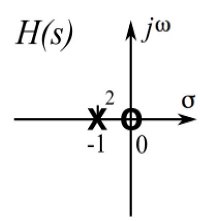
$$H(s) = F(s)G(s) = \left(\frac{1}{s+1}\right)\left(\frac{s}{s+1}\right) =$$

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$




The resulting *H(s)* has a zero at 0 and double poles at -1, with the pole-zero plot shown on the right.



The Bode plot is given on the right.

The impulse response is:

$$h(t).=(1+t)e^{-t}u(t)$$

as plotted below.