Methods for Evaluating the Inverse z-Transform

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ZT: $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

IZT: $$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$ where $z = r e^{j\omega}$, $\left|\sum_{-\infty}^{\infty} x[n] z^{-n}\right| < \infty$

There are three methods for evaluating the IZT: 1) partial fraction expansion, 2) power-series expansion, and 3) Residue theorem. They are demonstrated by use of the following examples.

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1. Partial-fraction expansion

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1}+z^{-2})}$$, ROC: $1<|z|<2$. Find $x[n]$.

Use the partial-fraction expansion to arrange X(z) such that the following know ZT's apply:

$-\alpha^n u[-n-1]$	$\Leftarrow z \Rightarrow$	$\frac{1}{1-\alpha z^{-1}}$,	for $\|z\|<\|\alpha\|$
$r^n \cos(\omega_0 n) u[n]$	$\Leftarrow z \Rightarrow$	$\frac{1-(r\cos\omega_0) z^{-1}}{1-(2r\cos\omega_0) z^{-1}+r^2 z^{-2}}$,	for $\|z\|>r$
$r^n \sin(\omega_0 n) u[n]$	$\Leftarrow z \Rightarrow$	$\frac{(r\sin\omega_0) z^{-1}}{1-(2r\cos\omega_0) z^{-1}+r^2 z^{-2}}$,	for $\|z\|>r$

$$X(z) = \frac{A}{(1-2z^{-1})} + \frac{B-Cz^{-1}}{1-z^{-1}+z^{-2}}$$

$$A = \left\{X(z)(1-2z^{-1})\right\}\Big|_{z=2} = \left\{\frac{1}{1-z^{-1}+2z^{-2}}\right\}\Big|_{z=2} = \frac{1}{1-\frac{1}{2}+\frac{1}{4}} = \frac{4}{3}$$

$$X(z) = \frac{\frac{4}{3}-\frac{4}{3}z^{-1}+\frac{4}{3}z^{-2}+B-2Bz^{-1}-Cz^{-1}+2Cz^{-2}}{(1-2z^{-1})(1-z^{-1}+z^{-2})} \Rightarrow$$

For the coefficient of z^0 : $\frac{4}{3}+B=1 \Rightarrow B=-\frac{1}{3}$

For the coefficient of z^{-2} : $\frac{4}{3}+2C=0 \Rightarrow C=-\frac{2}{3}$

For the coefficient of z^{-1} : $-\frac{4}{3}-2B-C=-\frac{4}{3}+\frac{2}{3}+\frac{2}{3}=0 \Rightarrow$ Checked.

$$X(z) = \frac{4}{3}\frac{1}{(1-2z^{-1})} + \frac{-\frac{1}{3}-(-\frac{2}{3})z^{-1}}{1-z^{-1}+z^{-2}} = \frac{4}{3}\frac{1}{(1-2z^{-1})} + (-\frac{1}{3})\frac{1-2z^{-1}}{1-z^{-1}+z^{-2}} =$$

$$\frac{4}{3}\frac{1}{(1-2z^{-1})} + (-\frac{1}{3})\left[\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+z^{-2}} - \sqrt{3}\frac{\frac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}}\right],$$

Compare to the known ZT's on the previous page, we have:

$$x[n] = -\frac{4}{3}2^{n}u[-n-1] - \frac{1}{3}\left[\cos(\frac{\pi}{3}n)-\sqrt{3}\sin(\frac{\pi}{3}n)\right]u[n]$$

note: $r=1$, $\omega_0=\frac{\pi}{3}$, $\cos\omega_0=\frac{1}{2}$, $\sin\omega_0=\frac{\sqrt{3}}{2}$

$$x[n] = -\frac{4}{3}2^{n}u[-n-1] - \frac{2}{3}\left[\frac{1}{2}\cos(\frac{\pi}{3}n)-\frac{\sqrt{3}}{2}\sin(\frac{\pi}{3}n)\right]u[n]$$

Applying the trigonometric identity:

$$\cos(\alpha+\beta) = \cos\alpha \ \cos\beta - \sin\alpha \ \sin\beta$$

We have:

$$x[n] = -\frac{4}{3}2^{n}u[-n-1] - \frac{2}{3}\cos(\frac{\pi}{3}n+\frac{\pi}{3})u[n]$$

2. Power-series expansion

$X(z)=e^{a z^{-1}}$, ROC: all z except $z = 0$. Find $x[n]$.

From the power-series expansion:

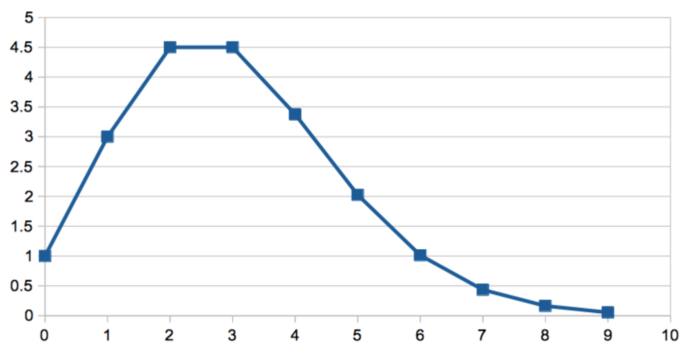
$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^{az^{-1}} = \sum_{n=0}^{\infty}\frac{(az^{-1})^n}{n!} = \sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n} = \sum_{n=-\infty}^{\infty}\left(\frac{a^n}{n!}u[n]\right)z^{-n}$$

From the definition of ZT, we see

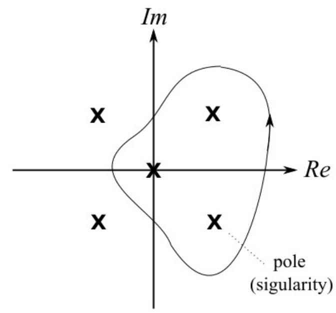
$$x[n] = \frac{a^n}{n!}u[n]$$

Here is an numerical example with $a = 3$:



3. Residue Theorem

The closed-loop integration of a function on a complex plane equals to $2\pi j$ times the sum of the residues for all poles inside the loop.



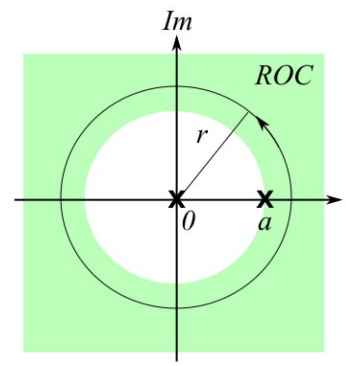
$$\oint_C f(z)\,dz = 2\pi j \sum_{\text{enclosed poles}} Residues$$

where, for an nth-order pole z_i,

$$Residue_i = \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-z_i)^n f(z) \right]_{z=z_i}$$

Example: Show $x[n]=a^n u[n] \quad \Leftarrow z \Rightarrow \quad \dfrac{1}{1-az^{-1}}$, for $|z|>|\alpha|$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz = \frac{1}{2\pi j} \left[2\pi j \left(Residue_{z=a} + Residue_{z=0}, \text{if } n \le 0 \right) \right]$$



For n > 0, $\left[(z-a) \dfrac{z^{n-1}}{1-az^{-1}} \right]_{z=a} = \left[(z-a) \dfrac{z^n}{z-a} \right]_{z=a} = a^n$

For n = 0, $Residue_{z=a} = a^n$

$Residue_{z=0} \Rightarrow$

$$\left[z \times \frac{z^{-1}}{1-az^{-1}} \right]_{z=0} = \left[z \times \frac{1}{z-a} \right]_{z=0} = 0$$

For n < 0, we do a time reversal: $x[-n] \quad \Leftarrow z \Rightarrow \quad X(z^{-1})$ ROC: R_X^{-1}

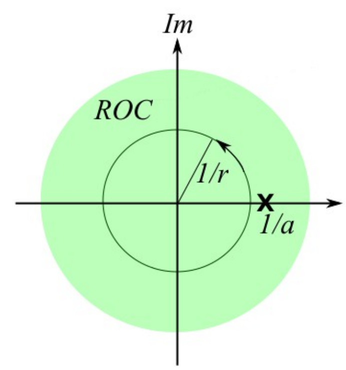
Let $\rho = z^{-1}$

$$x[n] = \frac{1}{2\pi j} \oint_r X(z) z^{n-1} dz =$$

$$\frac{1}{2\pi j} \oint_{1/r} X\left(\frac{1}{\rho}\right) \rho^{-n-1} d\rho =$$

$$\frac{1}{2\pi j} \oint_{1/r} \frac{\rho^{-n-1}}{1-a\rho} d\rho = 0,$$

because there is no pole inside the loop.



In summary, $x[n]=a^n u[n]$.

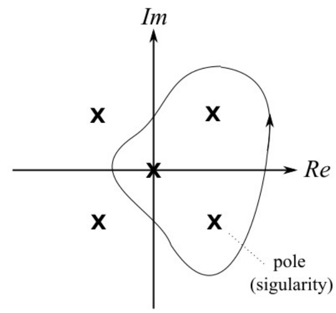
Sketch proof of the Residue Theorem

The closed-loop integration of a function on a complex plane equals to $2\pi j$ times the sum of the residues for all poles inside the loop.

$$\oint_C f(z)\,dz = 2\pi j \sum_{enclosed\ poles} Residues$$

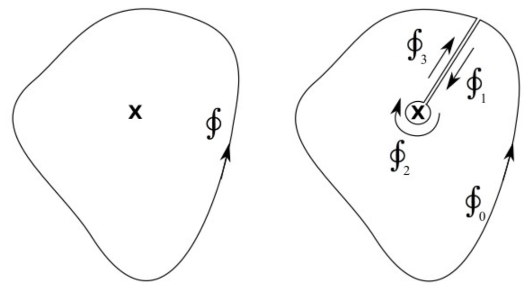
where, for an nth-order pole z_i,

$$Residue_i = \left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_i)^n f(z)\right]_{z=z_i}$$



Without losing generality, we will show how it works for a system with only one single pole inside the loop.

We create a new integration path $\oint_0$ that circle around the pole. $\oint_0$ Should be 0 because it does not enclose any pole.



$$\oint_0 = \oint + \oint_1 + \oint_2 + \oint_3 = 0$$

$\oint_1$ and $\oint_3$ cancel out with each other because their paths are opposite and very close. Thus,

$$\oint + \oint_2 = 0, \text{ or } \oint = -\oint_2$$

Let $f(z) = \dfrac{g(z)}{z-z_0}$, where z_0 is the pole.

Let z_0. Take derivative, we have $dz = j\rho e^{j\theta}d\theta$.

$\oint_2 = \int_{2\pi}^{0} f(z)dz = \int_{2\pi}^{0} \dfrac{g(z)}{z-z_0}dz = g(z)\int_{2\pi}^{0} \dfrac{dz}{z-z_0}$, assuming the integration loop is small and *g(z)* is constant around the pole.

$$\oint_2 = g(z)\int_{2\pi}^{0}\frac{dz}{z-z_0} = g(z)\int_{2\pi}^{0}\frac{j\rho e^{j\theta}}{\rho e^{j\theta}}d\theta = g(z)\int_{2\pi}^{0} j\,d\theta = -2\pi j\,g(z)$$

Thus,

$$\oint = -\oint_2 = 2\pi j\left[(z-z_0)f(z)\right]_{z=z_0}$$