Root Locus Analysis

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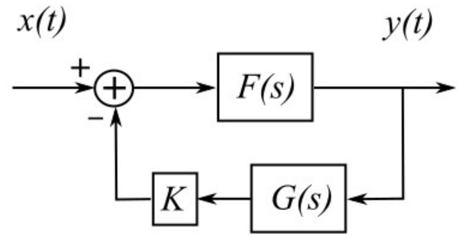
A gain parameter (K) is added to the feedback loop as shown. We have negative feedback for $K > 0$, positive feedback for $K < 0$. In order to explore the stability of the feedback system, K is adjusted over a wide range. For a given K, the roots of the system are plotted. As K is varied, the roots follow certain trajectories called root locus. The overall transfer function is:



$$H(s) = \frac{F(s)}{1+K\,G(s)\,F(s)}$$

For those K's where the root locus gets on the $j\omega$ axis or into the right side of the s-plane, the system becomes unstable. The roots of the system are given by:

$$1+K\,G(s)\,F(s) = 0\,,\ \text{or}\ \ G(s)\,F(s) = -\frac{1}{K}$$ ($G(s)F(s)$ is called the loop gain.)

Example

Plot the root locus for $G(s)\,F(s) = \dfrac{s-1}{(s+1)(s+2)}$.

$$G(s)\,F(s) = \frac{s-1}{(s+1)(s+2)} = -\frac{1}{K} \quad\Rightarrow\quad -K\,s+K = s^2+3\text{s}+2 \quad\Rightarrow$$

$$s^2+(3+K)s+(2-K) = 0$$

The roots are at: $s = \dfrac{-K-3\pm\sqrt{(K+3)^2+4\text{K}-8}}{2} = \dfrac{-K-3\pm\sqrt{K^2+10\text{K}+1}}{2}$.

For $K = 0$ (no feedback):

roots at $s = \dfrac{-3\pm1}{2} = -1,\ -2;$ zeros at $1, \infty$.

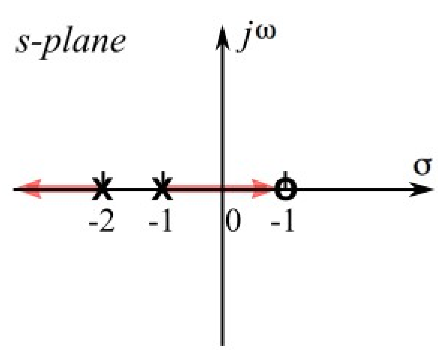
For $K > 0$ (negative feedback):

Now we find the root locus for positive K starting from 0. Because the zeros are at 1 and ∞, the root locus must enter either 1 or ∞, as $K \to \infty$.

$K = 2 \Rightarrow$ s = 0, -5. This is where one of the roots moves cross to the right side.

$K = 5 \Rightarrow$ s = 0.4, -8.4.

$K \to \infty \Rightarrow s \to 1.0, -\infty$. $G(s)F(s) \to$ - 0 (approaching 0 from the negative side).



The root locus plot is shown on the right.

For $K < 0$ (positive feedback):

Solve for $K^2 + 10K + 1 = 0$, we have K = -0.1, -9.9, corresponding to s = -1.45, 3.45, where the root locus breaks away or re-enters the real axis.

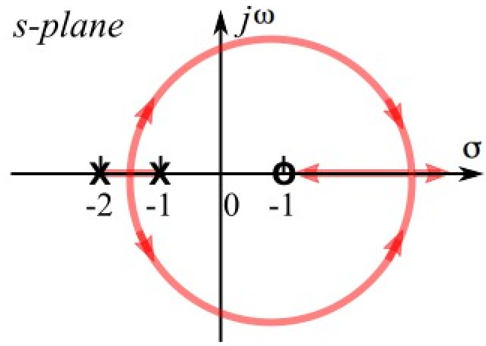
K = -0.1 $\Rightarrow$ s = -1.45. This is where the two roots move together and then break away from the real axis in the opposite directions.

K = -3 $\Rightarrow$ s = 2.24j, -2.24j. These locations are where the two roots cross the $j\omega$ axis and enter the right side plane.

K = -9.9 $\Rightarrow$ s = 3.45. This is where the two roots move together on the real axis and then move away in the opposite directions on the real axis.

$K \rightarrow -\infty \Rightarrow s \rightarrow 1.0, \infty$. $G(s)F(s) \rightarrow +0$ (approaching 0 from the positive side).

The root locus plot is shown on the right.

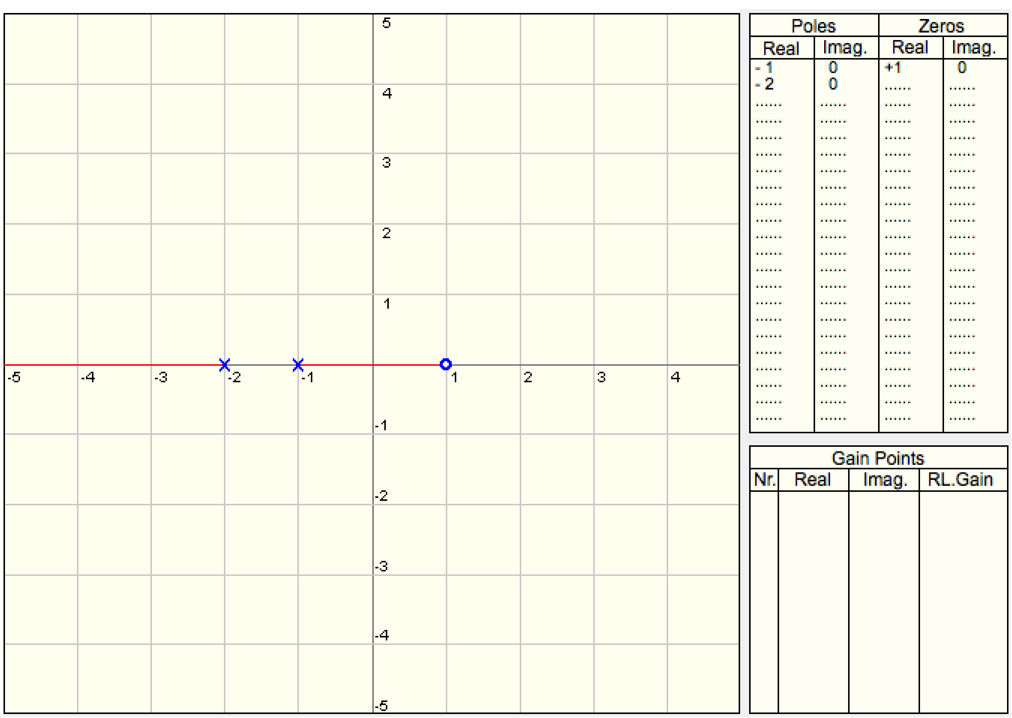


Because $s = \dfrac{-K - 3 \pm \sqrt{K^2 + 10K + 1}}{2}$, the root locus touches the $j\omega$ axis when K = -3,

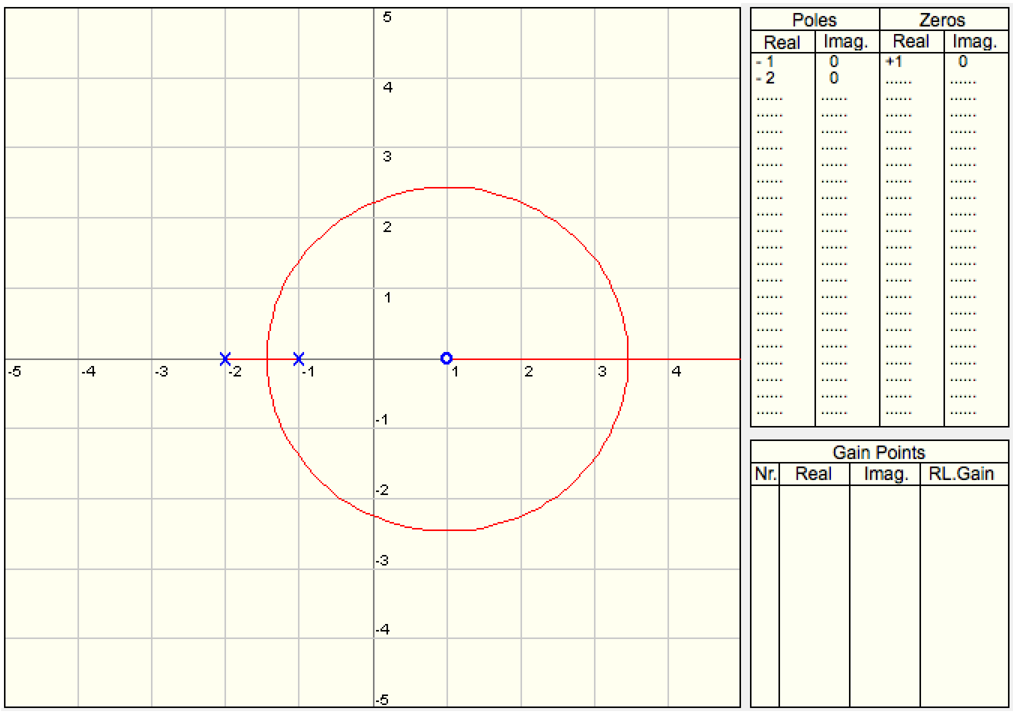
where $s = \dfrac{3 - 3 \pm \sqrt{3^2 + 10(-3) + 1}}{2} = \pm\sqrt{-5} = \pm 2.24\, j$.

A freeware called RootLocs can be downloaded from <http://www.coppice.myzen.co.uk/RootLocs_Site/RootLocs.html>. The results of using RootLocs for this problem are shown below.

Negative Feedback: Enter the two roots (-1 and -2) and the zero (+1). Select negative feedback and click START.



<u>Positive Feedback:</u> Enter the two roots (-1 and -2) and the zero (+1). Select positive feedback and click START.

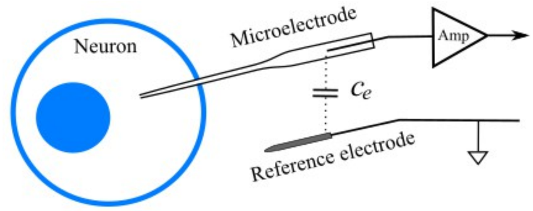


Applications

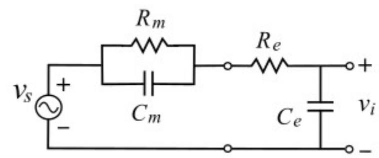
For designing amplifier circuits, we almost always use negative feedback. When designing oscillator circuits, we put poles right on the $j\omega$ axis to create oscillations. Poles on the right side of the s-plane are not desirable in general, because they represent unbounded (exponentially-growing) responses. From the above example, notice that there is a range of K for the positive feedback ($0>K>-3$) where the poles are still on the left side of the s-plane. In very rare situations, this positive feedback can be exploited in the design of special-purpose amplifiers. An example can be found in the design of a "negative capacitance" amplifier for neuroscience instrumentation, as detailed on next page.

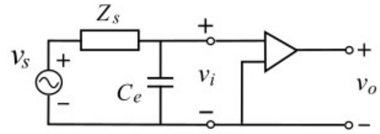
Negative Capacitance: Compensation for Microelectrode Capacitance

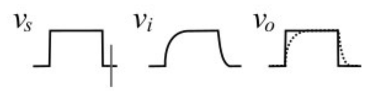
The standard technique for recording action potentials from a neuron is to insert a microelectrode, which is made from a glass pipette by using an electrode puller. The micro-electrode tapers and becomes very small at the tip (~1 μm). The microelectrode has a large resistance R_e as well as a large stray capacitance C_e with respect to the surrounding bath solution.



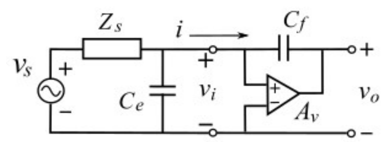
The equivalent circuit is shown on the right, where R_m = membrane resistance, C_m = membrane capacitance, v_s = voltage signal to be measured, and v_i = input voltage to the amplifier. The equivalent circuit can be further reduced, where the source impedance Z_s includes R_e, R_m and C_m. If v_s is a square pulse, v_i would show a low-pass filtered waveform because of the presence of C_e. To recover the lost higher frequency components, an amplifier is designed such that the output voltage v_o resembles v_s, as illustrated.







A technique typically used for this situation is based on a positive feedback circuit that creates a "negative capacitance" to cancel out C_e. As shown in the equivalent circuit, the gain of the amplifier is A_v, where $v_o = A_v \times v_i$. A positive feedback is created by connecting C_f from the output to the positive input terminal of the amplifier.

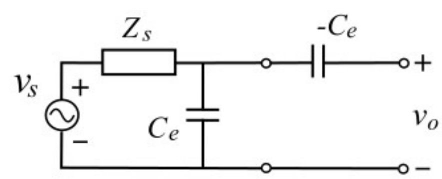


Let i be the current through C_f, we have

$$v_i = v_o + \frac{1}{C_f}\int i\,di = A_v v_i + \frac{1}{C_f}\int i\,di$$

$$(1-A_v)v_i = \frac{1}{C_f}\int i\,di \;\Rightarrow\; v_i = \frac{1}{(1-A_v)C_f}\int i\,di$$

Choose $A_v = 1 + \frac{C_e}{C_f} \;\Rightarrow\; \frac{1}{(1-A_v)C_f} = \frac{1}{-C_e}$.



The final equivalent circuit is shown on the right, where the negative capacitance $-C_e$ cancels out the electrode capacitance C_e, if the gain of the positive feedback A_v is properly set.

The gain should be set at $A_v = 1 + C_e / C_f$ in order to reproduce the square wave. Too low or too high a gain would affect the wave shape as shown on the right.