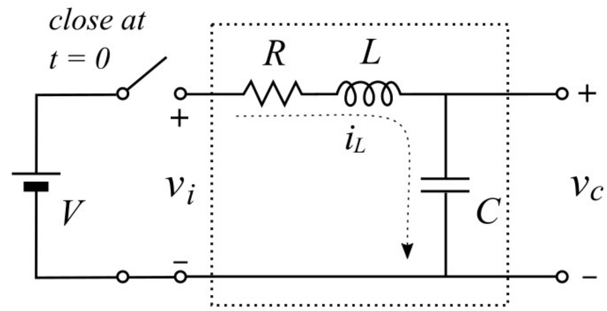
State Space Analysis *Ying Sun*

A state-space model represents a system by using a set of state variables. The number of the state variables defines the order of the system. The dynamics of the system is characterized by a set of first-order differential equations, one for each state variable. "State space" refers to the Euclidean space in which the state variables are on mutually orthogonal axes in multiple dimensions. The state of the system is represented as a vector within that space, which changes its length and direction over time.



The state-space analysis is demonstrated by use of the RLC circuit as shown. Refer to the handout "YS03_Circuit_2a". The circuit is characterized by the 2nd-order differential equation:

$$v_i = LC\frac{d^2 v_C}{dt^2} + RC\frac{d v_C}{dt} + v_C$$

Now, we take a different approach (state-space) to analyze this circuit. We choose the state variables as the current through the inductor (i_L) and the voltage across the capacitor (v_C). The circuit equations can be derived as follows:

$$\begin{cases} C\dfrac{d v_C(t)}{dt} = i_L \\ L\dfrac{d i_L(t)}{dt} = v_i(t) - R\,i_L(t) - v_C(t) \end{cases} \Rightarrow \begin{cases} \dfrac{d v_C(t)}{dt} = \dfrac{1}{C} i_L \\ \dfrac{d i_L(t)}{dt} = -\dfrac{1}{L} v_C(t) - \dfrac{R}{L} i_L(t) + \dfrac{1}{L} v_i(t) \end{cases}$$

Or in matrix form:

$$\begin{bmatrix} \dfrac{d v_C(t)}{dt} \\ \dfrac{d i_L(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_i(t)$$

The output is given by:

$$v_C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$$

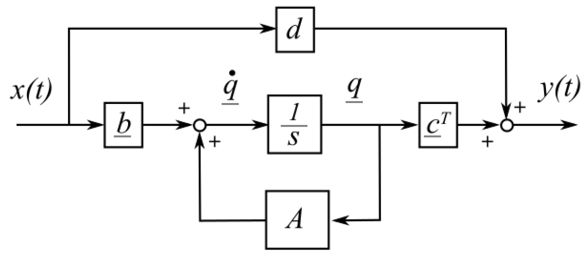
The state-space representation is given by

$$\dot{\underline{q}} = A\underline{q} + \underline{b} v_i$$

$$v_C = \underline{c}^T \underline{q}$$

where $\underline{q}$ is the state vector, A the plant matrix, $\underline{b}$ the input vector, and $\underline{c}$ the output vector.

To generalize, the state-space representation for a single-input single-output system is given below and shown on the right.



$$\dot{\underline{q}} = A\underline{q} + \underline{b}\,x(t)$$

$$y(t) = \underline{c}^T \underline{q}$$

Taking the LT, we have

$$sQ(s) = AQ(s) + \underline{b}\,X(s) \quad \Rightarrow \quad (sI - A)Q(s) = \underline{b}\,X(s) \quad \Rightarrow \quad Q(s) = (sI - A)^{-1}\underline{b}\,X(s)$$

$Y(s) = \underline{c}^T Q(s) \quad \Rightarrow \quad Y(s) = \underline{c}^T (sI - A)^{-1}\underline{b}\,X(s)$, where I is the identity matrix having 1's on the diagonal and 0's off diagonal.

Thus, the transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \underline{c}^T (sI - A)^{-1}\underline{b}$$

<u>Example</u> For the RLC circuit shown on previous page, let R = 1Ω, L = 1 H, and C = 1 F.

The transfer function based on the traditional method is derived below.

$$\omega_n = \frac{1}{\sqrt{LC}} = 1$$

$$\alpha = \frac{R}{2L} = \frac{1}{2}$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2} = \frac{1}{s^2 + s + 1}$$

Inversion of 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Now, we use the state-space method:

$$\begin{bmatrix} \dfrac{d v_C(t)}{dt} \\ \dfrac{d i_L(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_i(t)$$

$$v_C = \underline{c}^T \underline{q}$$

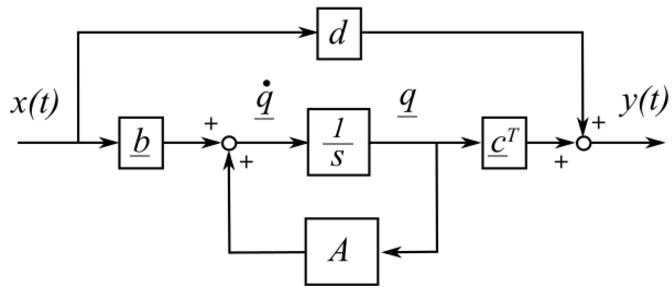
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The transfer function is:

$$H(s) = \underline{c}^T (sI - A)^{-1}\underline{b} = [1 \quad 0] \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\left(\frac{1}{s(s+1)+1}\right)[1 \quad 0] \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{1}{s^2+s+1}\right)[s+1 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+s+1} \quad \text{Check!}$$

For a 2nd-order system, we summarize the derivation of the state-space representation from the differential equation. We begin with a 2nd-order differential equation, where



$\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2 y}{dt^2}$, etc.

$$\ddot{y} + a_1 \dot{y} + a_2 y = b_1 \dot{x} + b_0 x$$

Notice that, for simplicity, there is no $\ddot{x}$. This means that there is no direct link from the input to the output, or $d = 0$ in the signal flow diagram. This also means that the transfer function H(s) has a 2nd-order denominator and a 1st-order numerator.

Taking Laplace Transform: $Y(s)(s^2 + a_1 s + a_2) = X(s)(b_1 s + b_0)$. The transfer function is given by:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_2}$$

Now, we choose the state variables $q_1 = y$ and $q_2 = \dot{y} - b_1 x$. Take derivative of q_1, we have

$\dot{q}_1 = \dot{y} = q_2 + b_1 x$. Take derivative of q_2, we have

$$\dot{q}_2 = \ddot{y} - b_1 \dot{x} = -a_2 y - a_1 \dot{y} + b_0 x = -a_2 y - a_1(\dot{y} - b_1 x) + (b_0 - a_1 b_1) x =$$

$$-a_2 q_1 - a_1 q_2 + (b_0 - a_1 b_1) x$$

$$\underline{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_0 - a_1 b_1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad d = 0.$$

The state-space representation and the output equation are given below;

$$\dot{\underline{q}} = A\underline{q} + \underline{b}x = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_0 - a_1 b_1 \end{bmatrix} x; \qquad y = \underline{c}^T \underline{q} + 0 \cdot x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The transfer function is

$$H(s) = \underline{c}^T (sI - A)^{-1} \underline{b} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ a_2 & s + a_1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_0 - a_1 b_1 \end{bmatrix} =$$

$$\frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + a_1 & 1 \\ -a_2 & s \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 - a_1 b_1 \end{bmatrix} = \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s + a_1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 - a_1 b_1 \end{bmatrix} =$$

$$\frac{b_1 s + a_1 b_1 + b_0 - a_1 b_1}{s^2 + a_1 s + a_2} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_2}$$ ---------------- Check!

Example

The differential equation is given by $\ddot{y}+3\dot{y}+2y = \dot{x}$.

Take the Laplace transform, we have $Y(s^2+3s+2) = sX$.

The transfer function is given by $H = \dfrac{Y}{X} = \dfrac{s}{s^2+3s+2}$.

Choose the state variables: $q_1=y$, $q_2=\dot{y}-x$. We have

$$\dot{q}_1 = \dot{y}+x \;;\quad \text{and} \quad \dot{q}_2 = \ddot{y}-\dot{x} = -2y-3\dot{y} = -2y-3(\dot{y}-x)-3\text{x} = -2q_1-3q_2-3x$$

The state-space representation is given by

$$\dot{\underline{q}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}+\begin{bmatrix} 1 \\ -3 \end{bmatrix}x$$

The output equation is given by

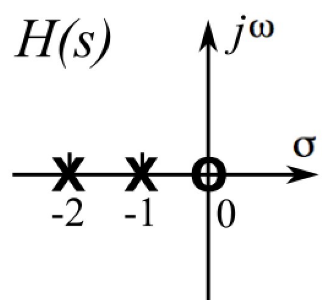
$$y = [1 \quad 0]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The transfer function is given by

$$H(s) = \underline{c}^T(sI-A)^{-1}\underline{b} = [1 \quad 0]\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -3 \end{bmatrix} =$$

$$\frac{1}{s(s+3)+2}[1 \quad 0]\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{s^2+3s+2}[s+3 \quad 1]\begin{bmatrix} 1 \\ -3 \end{bmatrix} =$$

$$\frac{s+3-3}{s^2+3\text{s}+2} = \frac{s}{s^2+3\text{s}+2}$$ ---------------- Check!



The pole-zero plot of the system is shown on the right.

$$H(s) = \frac{s}{s^2+3\text{s}+2} = \frac{s}{(s+1)(s+2)}$$

The magnitude of the Fourier transform is

$$H(j\omega) = \frac{j\omega}{(j\omega)^2+3j\omega+2} =$$

$$\frac{j\omega}{-\omega^2+3j\omega+2} = \frac{j\omega}{(2-\omega^2)+3j\omega}$$

The magnitude of the Fourier transform is

$$|H(j\omega)| = \frac{\omega}{\sqrt{(2-\omega^2)^2+9\omega^2}}$$

It's a band-pass filter.