Fourier Series for periodic functions

If $f(t)$ is periodic such that

$$\boxed{f(nT + t) = f(t)} \qquad n = -\infty, \ldots, 0, 1, 2, \ldots$$

Period

then

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn 2\pi \frac{1}{T} t}$$

$$a_n = \frac{1}{T} \int_{c}^{c+T} f(t) e^{-jn 2\pi \frac{1}{T} t} dt$$

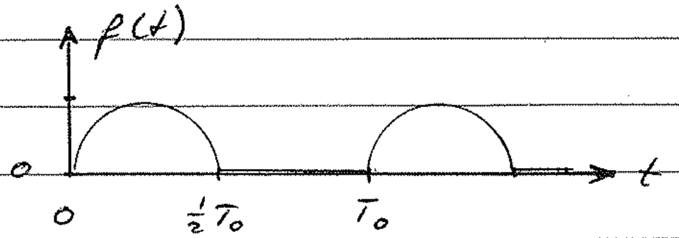
or

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} \left[\alpha_n \cos(n \tfrac{2\pi}{T} t) + \beta_n \sin(n \tfrac{2\pi}{T} t)\right]$$

$$\alpha_0 = a_0 \; ; \quad \alpha_n = \operatorname{Re}\{2a_n\} \quad n = 1, 2, \ldots$$

$$\beta_n = -\operatorname{Im}\{2a_n\} \quad n = 1, 2, \ldots$$

Example:



$$f_0(t) = \hat{V}_0 \sin(\omega_0 t) \qquad \text{where} \quad \omega_0 = \frac{2\pi}{T_0}$$

$$a_n = \frac{1}{T_0} \int_0^{\frac{1}{2}T_0} \hat{V}_0 \sin(\omega_0 t)\, e^{-jn\omega_0 t}\, dt$$

substitute, $\varphi = \omega_0 t \Rightarrow d\varphi = \omega_0\, dt$

$$\Rightarrow a_n = \frac{1}{2\pi} \int_0^{\pi} \hat{V}_0 \sin(\varphi)\, e^{-jn\varphi}\, d\varphi$$

$$= \frac{\hat{V}_0}{2\pi} \int_0^{\pi} \frac{[e^{j\varphi} - e^{-j\varphi}]}{2j}\, e^{-jn\varphi}\, d\varphi$$

$$= \frac{\hat{V}_0}{2\pi} \frac{[1 + e^{-jn\pi}]}{(1-n^2)}$$

$$\boxed{a_n = \begin{cases} \dfrac{\hat{V}_0}{\pi} \dfrac{1}{(1-n^2)} & n \text{ even} \\ -\hat{V}_0 \dfrac{j}{4} & n = 1 \\ 0 & \text{else} \end{cases}}$$

$$\left\| f(t) = \frac{\hat{V}_0}{\pi}\left[1 + \frac{\pi}{2}\sin(\omega_0 t) - \frac{2}{3}\cos(2\omega_0 t) - \frac{2}{15}\cos(4\omega_0 t) - \dots\right] \right\|$$

compare: full wave rectifier

$$\left| f(t) = \frac{\hat{V}_0}{\pi}\left[2 - \frac{4}{3}\cos(2\omega_0 t) - \frac{4}{15}\cos(4\omega_0 t) - \dots\right] \right|$$