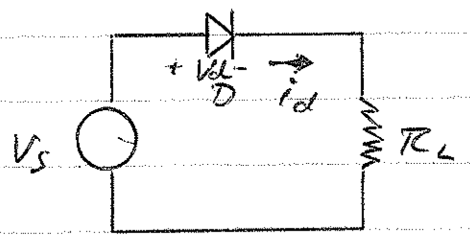
III Nonlinear Circuit Elements

We have seen in the previous chapter that op-amps can be utilized as linear as well as nonlinear circuit elements. Examples of linear circuits are amplifiers, filters, buffers, etc.
Nonlinear examples are limiters, comparators, and various Trigger circuits.
Nonlinear circuits are typically more difficult to analyze than their linear counterparts. Apart from the <u>direct mathematical method</u>, nonlinear circuits can be analyzed <u>graphically</u> (load line concept) or via an approximation method called <u>piecewise linear modeling</u>.
Note that some circuits containing nonlinear elements cannot be solved by direct mathematical calculation

<u>Example:</u> p-n junction diode with resistive load



KVL: $V_S = V_d + i_d R_L$ (1)

Diode: $i_d = I_S \left(e^{V_d/V_T} - 1 \right)$ (2)

(2) $V_d = V_T \ln\left[1 + \frac{i_d}{I_S}\right]$

$$\therefore \quad \boxed{V_S = V_T \ln\left[1 + \frac{i_d}{I_S}\right] + i_d R_L}$$

This is a transcendental equation which does not possess a closed-form solution

A) Numerical solution

$$i_d = \frac{V_S}{R_L} - \frac{V_T}{R_L} \ln\left[1 + \frac{i_d}{I_S}\right]$$

use iterative approach

$$\left\| \begin{aligned} i_{d_{n+1}} &= \frac{V_S}{R_L} - \frac{V_T}{R_L} \ln\left[1 + \frac{i_{d_n}}{I_S}\right] \\ i_{d_1} &= 0 \end{aligned} \right\| \qquad n = 1, 2, \ldots$$

e.g. $V_S = 10V$ $R_L = 3.3k\Omega$ $V_T = 30mV$ $I_S = 1pA$

Guess $i_{d_1} = I_S$

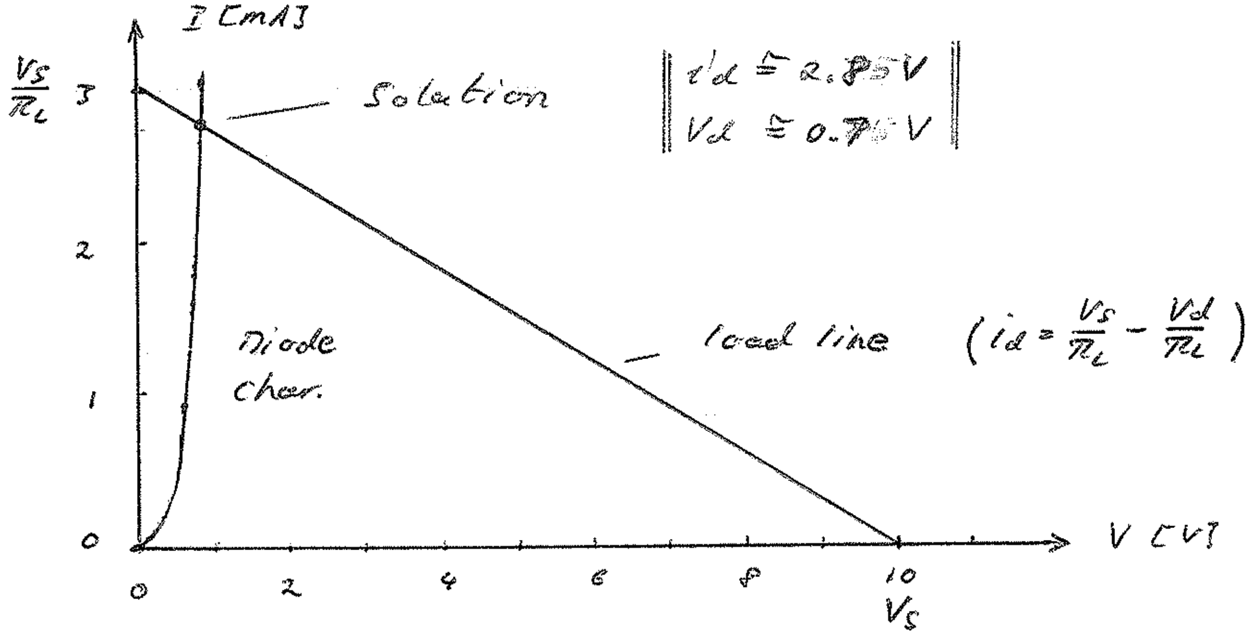
1^{st} it. $i_{d_2} = 3.03\,mA$

2^{nd} it. $i_{d_2} = 2.88\,mA$

3^{rd} it. $i_{d_3} = 2.88\,mA$

$\therefore$ $\left\| \begin{aligned} i_d &= 2.88\,mA \\ V_d &= 0.65\,V \end{aligned} \right\|$

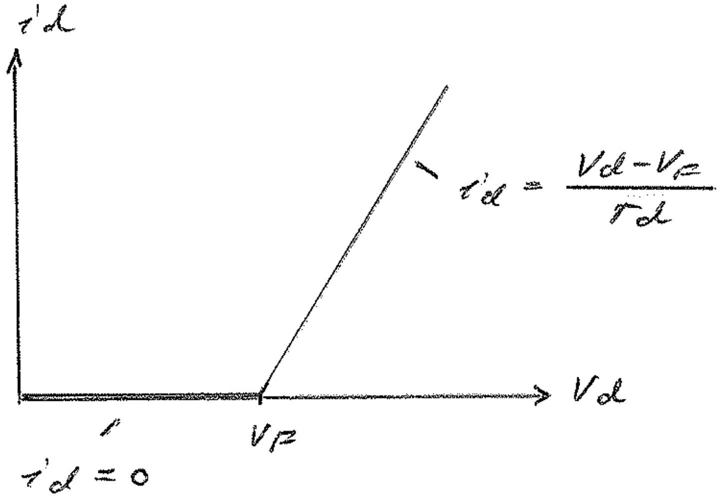
B) Graphical solution



C) Piecewise Linear Model

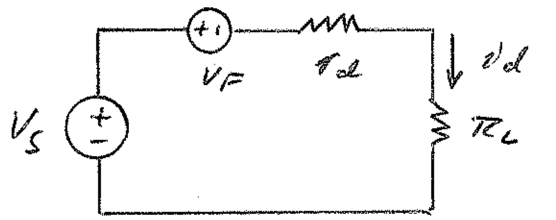
Example

$$i_d = \begin{cases} \dfrac{V_d - V_F}{r_d} & V_d \geq V_F \\ 0 & \text{else} \end{cases}$$



e.g. $V_F = 0.75V$ $\quad r_d = 10\,\Omega$ $\quad V_s = 10V$ $\quad R_L = 3.3k\Omega$

Linear equivalent circuit



condition for model validity:

$V_d \geq V_F$

$$V_s = V_F + i_d r_d + i_d R_L$$

$$\therefore \quad \boxed{\begin{aligned} i_d &= \frac{V_s - V_F}{r_d + R_L} = 2.79\,mA \\ V_d &= V_F + i_d r_d = 0.78\,V \end{aligned}}$$

Iterative approach to circuit analysis (Newton Method)

Example:
$$\left| \begin{aligned} V_s &= V_d + I_d R_L \\ I_d &\cong I_s\, e^{V_d/V_T} \end{aligned} \right| \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

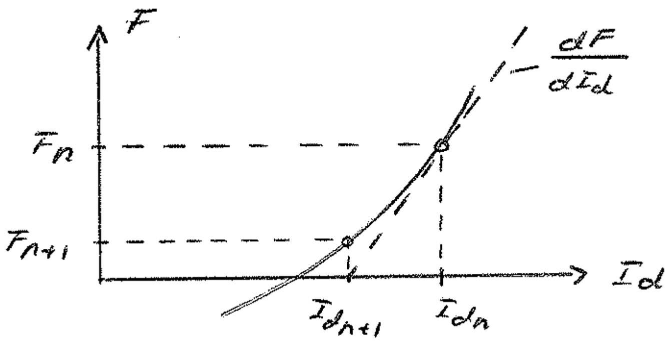
(2) $\therefore\ V_d = V_T \ln\left[\frac{I_d}{I_s}\right]$

$$\therefore \left| V_s = V_T \ln\left[\frac{I_d}{I_s}\right] + I_d R_L \right| \qquad (3)$$

or
$$I_d - \frac{V_s}{R_L} + \frac{V_T}{R_L} \ln\left[\frac{I_d}{I_s}\right] = 0$$

<u>Define</u> $\left| F = I_d - \frac{V_s}{R_L} + \frac{V_T}{R_L} \ln\left[\frac{I_d}{I_s}\right] \right|$

The problem is now reduced to finding the value of I_d where F becomes zero



$$\frac{dF}{dI_d} = \frac{F_n}{(I_{dn} - I_{dn+1})}$$

$$\therefore \left\| I_{dn+1} = I_{dn} - \frac{F_n}{dF/dI_d} \right\|$$

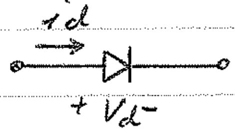
$$\frac{dF}{dI_d} = 1 + \frac{V_T}{R_L}\frac{1}{I_d}$$

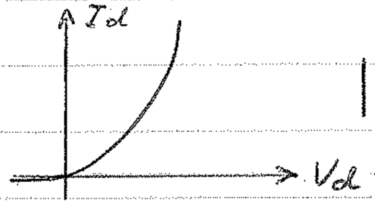
$$\therefore\ I_{dn+1} = \frac{\cancel{I_{dn}} + V_T/R_L - \cancel{I_{dn}} + V_s/R_L - V_T/R_L \ln\left[\frac{I_{dn}}{I_s}\right]}{1 + \frac{V_T}{R_L}\frac{1}{I_{dn}}}$$

$$\left\| I_{dn+1} = \frac{V_s/R_L - V_T/R_L\left(\ln\left[\frac{I_{dn}}{I_s}\right] - 1\right)}{1 + V_T/R_L\ {}^1/I_{dn}} \cong \frac{V_s}{R_L} - \frac{V_T}{R_L} \ln\left[\frac{I_{dn}}{I_s}\right] \right\|$$

Examples of Nonlinear Elements

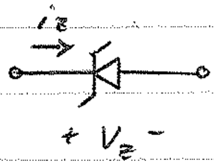
A) PN Junction Diode

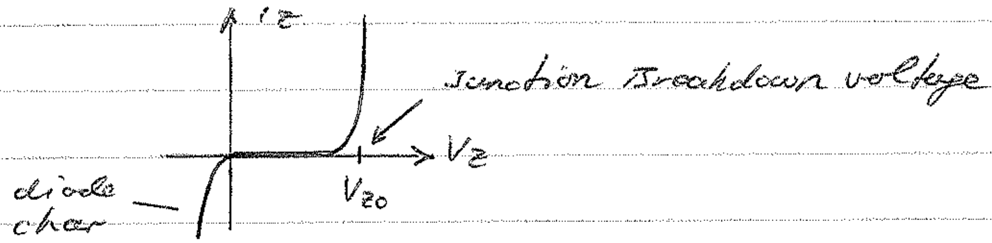




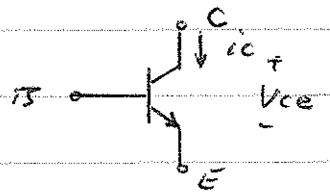
$$\boxed{I_d \cong I_s \left[e^{\frac{V_d}{nV_T}} - 1 \right]}$$

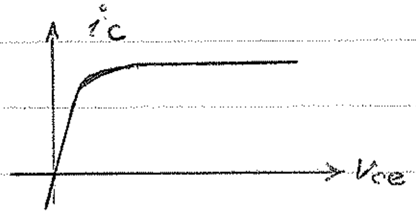
B) Zener Diode (Reverse biased Junction Breakdown)



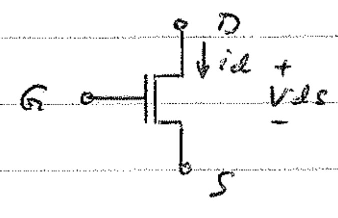


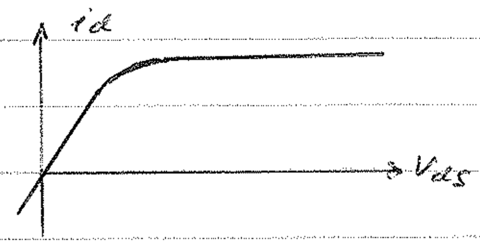
c) Bipolar Junction Transistor (BJT)



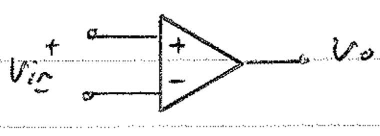


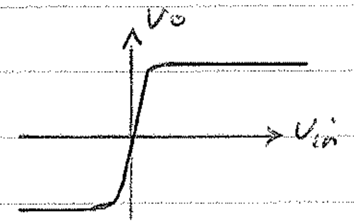
d) Metal-Oxide-Semicond. Field Effect Transistor (MOSFET)



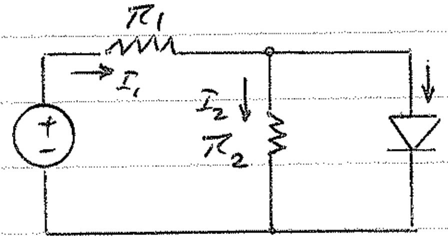


e) Op-amp





Example



$V_S = 10\,V$

$R_1 = 1.5\,k\Omega$

$R_2 = 1.0\,k\Omega$

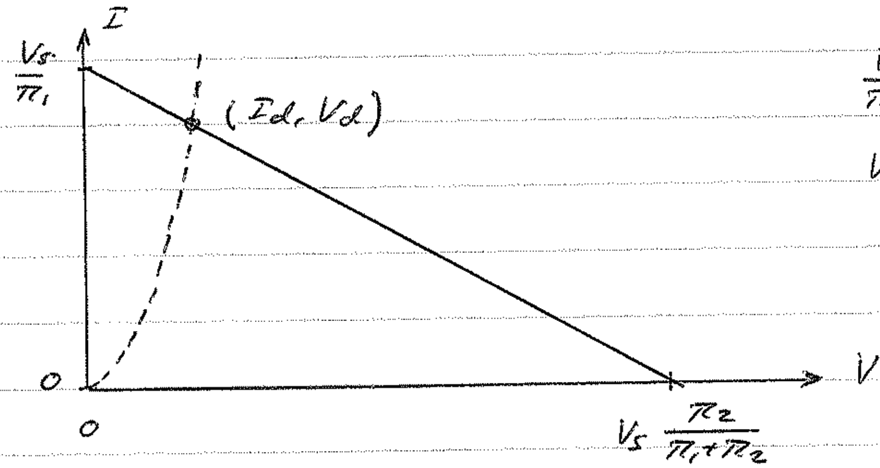
Determine I_d and V_d by

a) The graphical method

b) Replacing the diode with a piecewise linear model

Solution

a) KCL: $\left| I_d = \dfrac{V_S - V_d}{R_1} - \dfrac{V_d}{R_2} = \dfrac{V_S}{R_1} - \dfrac{V_d}{R_1 \| R_2} \right|$ load line eq.

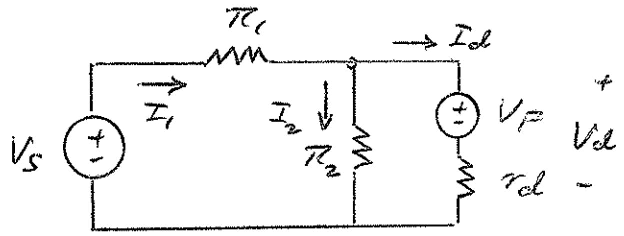


$\dfrac{V_S}{R_1} = 6\tfrac{2}{3}\,mA$

$V_S \dfrac{R_2}{R_1+R_2} = 4V$

$$\left\| \begin{array}{l} I_d \cong 5\tfrac{1}{3}\,mA \\ V_d \cong 0.8\,V \end{array} \right\| \quad \therefore \quad \left\| \begin{array}{l} I_2 \cong 0.8\,mA \\ I_1 \cong 6.13\,mA \end{array} \right\|$$

b) Piecewise linear model



$V_F \cong 0.7V$

$r_d \cong 10\Omega$

Network equations

$$\left|\begin{array}{ll} KVL1: & V_S = I_1 R_1 + I_2 R_2 \\ KVL2: & I_2 R_2 = V_F + I_d r_d \\ KCL: & I_1 = I_2 + I_d \end{array}\right| \quad \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

(1) $\quad I_1 = \dfrac{V_S}{R_1} - I_2 \dfrac{R_2}{R_1}$

(2) $\quad I_2 = \dfrac{V_F}{R_2} + I_d \dfrac{r_d}{R_2}$

(3) $\quad I_d = I_1 - I_2 = \dfrac{V_S}{R_1} - \left(1 + \dfrac{R_2}{R_1}\right)\left(\dfrac{V_F}{R_2} + I_d \dfrac{r_d}{R_2}\right)$

$$\therefore \left\| I_d = \frac{V_S/R_1 - V_F/R_2 - V_F/R_1}{1 + \frac{r_d}{R_1} + \frac{r_d}{R_2}} = \frac{V_S R_2 - V_F (R_1 + R_2)}{R_1 R_2 + r_d (R_1 + R_2)} \right\|$$

$$\left\| \begin{array}{l} I_d = 5.41\,mA \\ I_2 = 0.71\,mA \\ I_1 = 6.12\,mA \end{array} \right\| \qquad \therefore \left\| V_d = 0.75V \right\|$$

Temperature Dependence of PN Junction Diode

Diode equation: $\boxed{I_d \cong I_S \, e^{\frac{V_d}{V_T}}}$

or

$$\boxed{V_d \cong V_T \ln\left(\frac{I_d}{I_S}\right)}$$

$$\boxed{\begin{aligned} V_T &= \frac{kT}{q} \\ I_S &\cong I_{S0}\, e^{-\frac{E_G}{kT}} \end{aligned}}$$

E_G: Band Gap Energy of Semicond.

We now determine the temp. dependence of V_d if we keep the value of I_d constant

$$\frac{dV_d}{dT} = \frac{dV_T}{dT} \ln\left(\frac{I_d}{I_S}\right) - V_T \frac{\cancel{I_S}}{\cancel{I_d}} \frac{\cancel{I_d}}{I_S^{\cancel{2}}} \frac{dI_S}{dT}$$

$$\frac{dV_T}{dT} = \frac{k}{q} = \frac{V_T}{T}$$

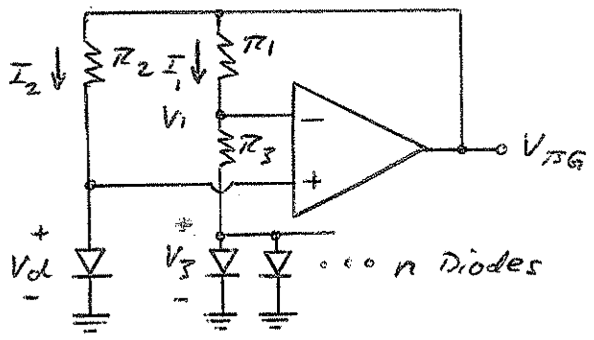
$$\frac{dI_S}{dT} = I_{S0}\, e^{-\frac{E_G}{kT}} \left(\frac{E_G}{kT^2}\right) = I_S \frac{E_G}{kT^2}$$

$$\therefore \frac{dV_d}{dT} = \frac{k}{q} \ln\left(\frac{I_d}{I_S}\right) - V_T \frac{E_G}{kT^2}$$

$$= \frac{V_d}{T} - \frac{E_G}{q \cdot T}$$

$$\boxed{\left.\frac{dV_d}{dT}\right|_{I_d = \text{const}} = -\frac{1}{T}\left[\frac{E_G}{q} - V_d\right]}$$

Band-Gap Reference Circuit with Junction Diodes



Motivation

In many circuit applications, it is important to possess a very stable reference voltage (e.g. threshold voltages in data converters).
The output voltage of the above circuit is supposed to be a (first-order) temperature compensated voltage.
Let us proof that by analyzing the circuit under the assumption of employing an ideal op-amp.
$\therefore$ $V_1 = V_d$ Furthermore, for the sake of simplicity, we will assume that $R_1 = R_2$ $\therefore$ $I_1 = I_2$
We can now express the two diode voltage drops as

$$V_d = V_T \ln\left(\frac{I_2}{I_S}\right) = V_T \ln\left(\frac{I_1}{I_S}\right) \tag{1}$$

$$V_3 = V_T \ln\left(\frac{I_1}{n I_S}\right) \tag{2}$$

In addition

$$V_1 - V_3 = V_d - V_3 = V_T \left[\ln\left(\frac{I_1}{I_S}\right) - \ln\left(\frac{I_1}{n I_S}\right)\right] = V_T \ln(n) \tag{3}$$

and

$$V_1 - V_3 = I_1 R_3 \tag{4}$$

Finally, we can express the output voltage as

$$V_{BG} = V_d + I_2 R_2 = V_d + I_1 R_2 \qquad (5)$$

Combining (3) (4) and (5) yields

$$\left\| V_{BG} = V_d + V_T \frac{R_2}{R_3} \ln(n) \right\| \qquad (6)$$

In order to keep V_{BG} thermally stable, we have to meet the following condition:

$$\frac{d}{dT} V_d + \frac{d}{dT} \left[V_T \frac{R_2}{R_3} \ln(n) \right] = 0 \qquad (7)$$

If the two resistors R_2 and R_3 are realized by the same material, their temperature coefficients will track very well. Therefore

$$\frac{d}{dT} V_d + \frac{R_2}{R_3} \ln(n) \frac{d}{dT} V_T = 0 \qquad (7a)$$

$$-\frac{1}{T} \left[\frac{E_G}{q} - V_d \right] + \frac{R_2}{R_3} \ln(n) \frac{V_T}{T} = 0 \qquad (7b)$$

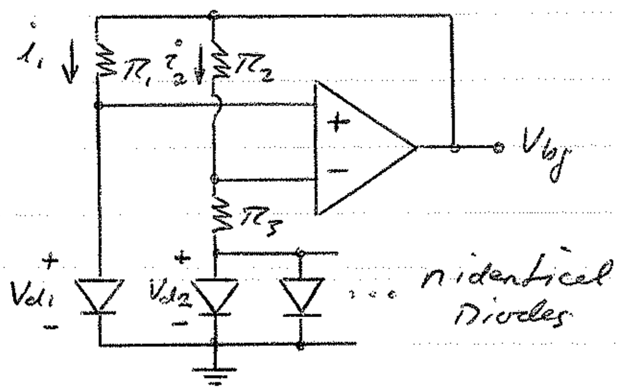
This finally yields

$$\left\| \frac{R_2}{R_3} \ln(n) = \frac{E_G}{kT} - \frac{V_d}{V_T} \right\| \qquad (8)$$

Inserting (8) into (6) yields

$$\left\| V_{BG} = \frac{E_G}{q} \right\| \qquad (9)$$

Band-Gap Reference Circuit



Op amp is assumed ideal

Diode eq. (forw. bias)

$$\left\| \, i_d = I_S \, e^{\frac{V_d}{V_T}} \, \right\|$$

where $V_T = \frac{kT}{q}$

$$\left| \, I_S = C_0 T^m e^{-\frac{E_G}{kT}} \, \right| \quad m \approx 3$$

NW Equations:

(1) $i_1 R_1 = i_2 R_2$ — Op amp

(2) $V_{d1} - V_{d2} = \Delta V_d = i_2 R_3$ — KVL

(3) $\Delta V_d = V_T \left[\underbrace{\ln\left(\frac{i_1}{I_S}\right) - \ln\left(\frac{i_2}{n I_S}\right)}_{\ln\left(\frac{i_1}{I_S} \frac{I_S}{i_2} n\right) = \ln\left(n \frac{R_2}{R_1}\right)} \right]$ — Diode

(4) $\left| V_{bg} = V_{d1} + i_1 R_1 \right|$ — KVL

(3) into (2) $\therefore \left| V_T \ln\left(n \frac{R_2}{R_1}\right) = i_2 R_3 \right|$ (5)

(5) into (4)

$$\left\| \, V_{bg} = V_{d1} + V_T \frac{R_2}{R_3} \ln\left(n \frac{R_2}{R_1}\right) \, \right\| \quad (6)$$

bg 2

In order to keep V_{bg} temperature independent

$$\frac{\partial V_{bg}}{\partial T} = 0 \qquad \therefore \quad \left| \frac{\partial V_{d1}}{\partial T} + \frac{\partial V_T}{\partial T} \frac{R_2}{R_3} \ln\left(n \frac{R_2}{R_1}\right) = 0 \right| \qquad (7)$$

assumption:
current i_1 varies little over operating temp. range

$$\left.\frac{\partial V_d}{\partial T}\right|_{i_d = const} = \frac{\partial V_T}{\partial T} \ln\left(\frac{i_d}{I_s}\right) - V_T \frac{\cancel{I_s}}{\cancel{i_d}} \frac{\cancel{i_d}}{I_s^{\cancel{2}}} \frac{\partial I_s}{\partial T}$$

where $$\frac{\partial I_s}{\partial T} = c_0 \left[m\, T^{(m-1)} e^{-\frac{E_G}{kT}} + T^m \frac{E_G}{kT^2} e^{-\frac{E_G}{kT}} \right] = \frac{I_s}{T}\left[m + \frac{E_G}{kT}\right]$$

$$\therefore \left\| \left.\frac{\partial V_d}{\partial T}\right|_{i_d = const} = \frac{V_T}{T} \ln\left(\frac{i_d}{I_s}\right) - V_T \frac{1}{T}\left[m + \frac{E_G}{kT}\right] = \frac{1}{T}\left[V_d - V_T\left(m + \frac{E_G}{kT}\right)\right] \right\| (8)$$

plug (8) into (7)

$$\frac{1}{T}\left[V_{d1} - V_T\left(m + \frac{E_G}{kT}\right)\right] + \frac{1}{T} V_T \frac{R_2}{R_3} \ln\left(n \frac{R_2}{R_1}\right) = 0$$

$$\therefore \left\| V_T \frac{R_2}{R_3} \ln\left(n \frac{R_2}{R_1}\right) = V_T\left(m + \frac{E_G}{kT}\right) - V_{d1} \right\| \quad (9)$$

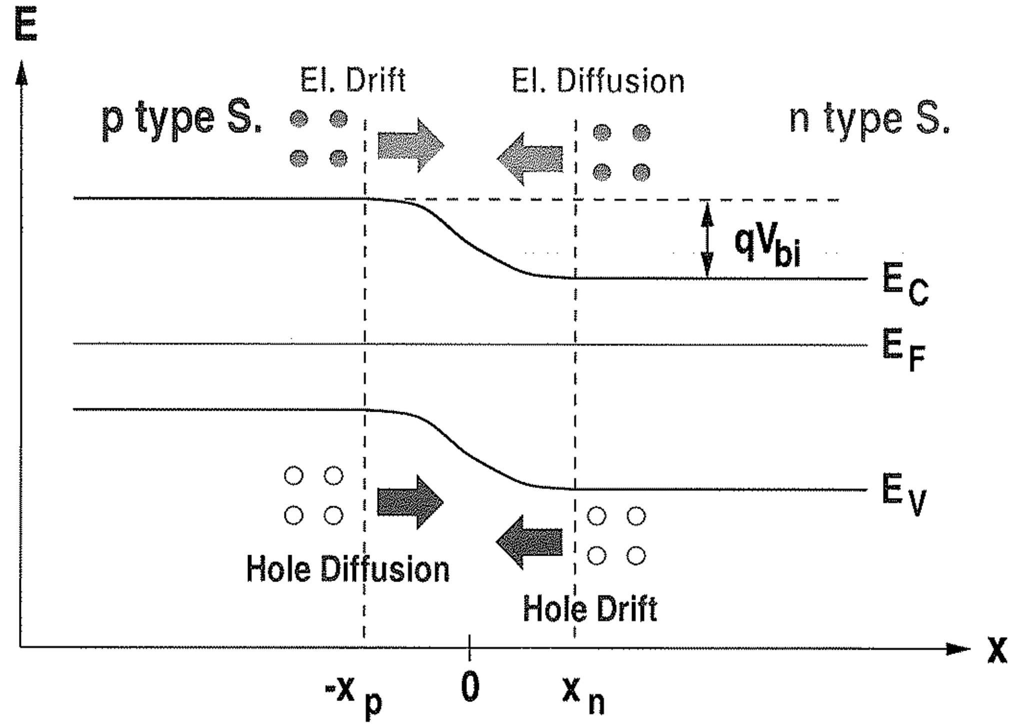
Note: Eq. (9) keeps V_{bg} temperature independent

Finally, insert (9) into (6)

$$\left\| V_{bg} = V_{d1} + V_T\left(m + \frac{E_G}{kT}\right) - V_{d1} = m V_T + \frac{E_G}{q} \right\| \qquad (10)$$

$$\left\| V_{bg} = \frac{E_G}{q} + 3 V_T \right\|$$

PN Junction at Thermal Equilibrium



Current Densities under forward Bias