

# Diode Models

ELE 338

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## 1 Large Signal Model: The Shockley Diode Equation

In this section we are considering large signal diode models. When we study the large signal behavior of the diode, we replace the schematic symbol with a non-linear model of the diode. The large signal diode model is a controlled current source. This is shown in figure 1. The diamond shape for the current source is my notation for a controlled current source (vs. an independent source for which I use a circle).

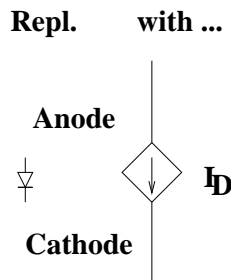


Figure 1. Non-linear diode model.

The current,  $I_D$ , is given by the *Shockley Diode Equation*:

$$I_D = I_S \left[ e^{\left( \frac{V_D}{V_T} \right)} - 1 \right] \quad (1)$$

where  $I_S$  is the saturation current,  $V_D$  is the voltage across the diode terminals and  $V_T = \frac{k_B T}{q_e}$ , is the thermal voltage,  $k_B$  is Boltzman's Constant,  $T$  is the temperature in Kelvin and  $q_e$  is the charge of an electron. At room temperature (e.g.  $T = 300k$ ),  $V_T = 26mV$ .

Equation (1) crosses the zero axis (e.g.  $I_D = 0$  for  $V_D = 0$ ). Notice, that when  $V_D < 0$ , that  $I_D \approx -I_s$ . The current,  $I_S$ , is leakage current; in Silicon, this

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current is on the order of 0.1 to 100 fA at room temperature. If  $V_D > 0$ , then  $I_D$  increases exponentially with  $V_D$ . Equation (1) can be solved for  $V_D$  giving:

$$V_D = V_T \ln \left( \frac{I_D - I_S}{I_S} \right) \quad (2)$$

This form of the equation is also useful since it allows one to compute a terminal voltage of the diode for a given diode current.

When the diode is forward biased and  $e^{\left(\frac{V_D}{V_T}\right)} \gg 1$  the second term in equation (1) is often negligible (even for relatively small positive values of  $V_D$ ). If this assumption holds, then equations (1) and (2) can be approximated to:

$$I_D \approx I_S e^{\left(\frac{V_D}{V_T}\right)}; \text{ when } e^{\left(\frac{V_D}{V_T}\right)} \gg 1 \quad (3)$$

and

$$V_D \approx V_T \ln \left( \frac{I_D}{I_S} \right); \text{ when } I_D \gg I_S \quad (4)$$

If, however,  $e^{\left(\frac{V_D}{V_T}\right)}$  becomes close to 1 in value then these approximations are no longer valid.

In this course, we are interested in analyzing and designing circuits which contain non-linear elements starting with the diode. The circuit in figure 2, for example, has a resistor,  $R_1$ , connected in series with the diode  $D_1$ . The voltage source is  $V_x$ .

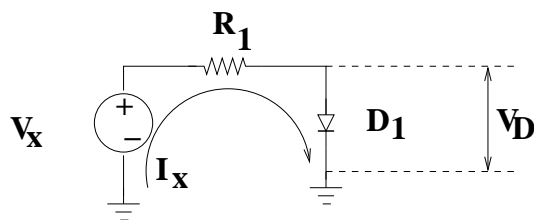


Figure 2. Schematic diagram of a voltage source in series with a resistor and a diode.

Notice that the current through the resistor is equal to that of the current flowing through the diode ( $I_x = I_D$ ). Although the diode is a non-linear element, KVL and KCL will work the same as they would if  $D_1$  were a linear circuit element. By applying KCL, we get:

$$f(V_D) = I_x - I_D = \frac{V_x - V_D}{R_1} - I_S \left[ e^{\left(\frac{V_D}{V_T}\right)} - 1 \right] \quad (5)$$

where  $f(V_D)$  is the sum of *ALL* currents at the output node. This means that  $f(V_D)$  *must* be equal to zero according to KCL; however, this will only be true for a single, specific value of  $V_D$  (which is unique).

The objective is to find a value for  $V_D$  (and  $I_D$ ),  $(V_D^*, I_D^*)$ , which will force equation (5) to be exactly zero. Since  $V_D$  is the independent variable, this can be solved by finding  $V_D^*$ . The difficult part is finding the exact value of  $V_D$  which solves KCL. If the diode were a linear device we could use linear equations to solve it; however, equation (5) does not have a direct analytical solution for which simple algebra can be used. In the case where  $V_x \leq V_{ON}$  for the circuit in figure 2 (and  $V_{ON}$  is known exactly), then the initial guess,  $V_D = V_x$ , will be very close to a reasonable, exact solution. If this is not true then a different guess for  $V_D$  will need to be plugged into equation (5). An evaluation will need to be made for a new choice of  $V_D$  based upon how close  $f(V_D)$  is to zero and the resulting sign (e.g. was  $f(V_D)$  greater than zero or less than zero for the initial estimate of  $V_D$ ). Since the  $I_D$  can change by a large amount for a very small change in  $V_D$  (due to the large derivative), this can be quite tedious.

There are two methods which can be used to find  $V_D^*$ : a graphical method or a numerical method. The graphical method requires that we plot equation (1) and then the equation of the resistor, (the  $I - V$  curve for  $R_1$ ). The plots of the diode and resistor  $I - V$  curves are shown in figure 3. In order to plot the

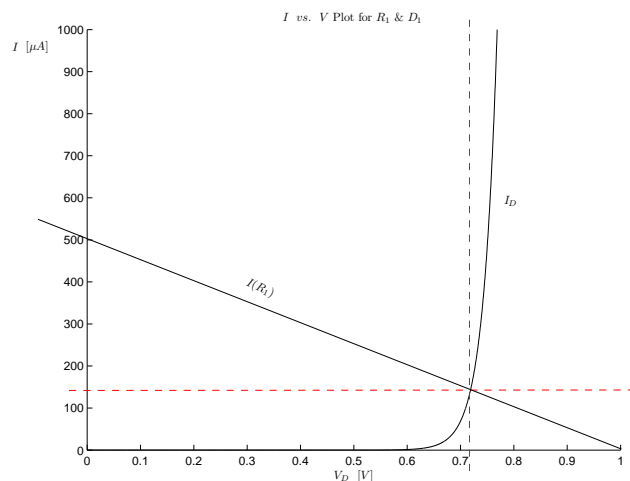


Figure 3.  $IV$  plot of  $R_1$  and  $D_1$ .

diode  $I - V$  curve we will need  $I_S$ ; then we will select an appropriate range of values for  $V_D$ . The equation for the resistor can be found by examining the Thevenin equivalent circuit at the diode terminals. If we remove the diode,  $V_{oc} = V_x$  and  $I_{sc} = \frac{V_x}{R_1}$ . Since  $I_x = I_D = 0$  then the value  $V_{oc}$  is the voltage source and it is placed on the horizontal axis.  $I_{sc}$  is plotted on the vertical axis since  $V_D = 0$  when  $I_D = I_{sc}$ ; then we draw a straight line between these

two points. For our example, the circuit in figure 2, we will assume  $R_1 = 2k\Omega$ ,  $I_S = 0.1fA$  and  $V_x = 1V$ .

Notice that the value of  $(V_D, I_D)$  for equation (5) which satisfies KCL, e.g.  $f(V_D^*) = 0$ , solves this problem can be found at the intersection of the two curves.

The graphical method will work; however, it is cumbersome to use. If there are more than one non-linear device this method can be rendered impractical. Obtaining very good accuracy, down to the micro-Amp range for  $f(V_D)$  requires a great deal of effort. It is also not easily implemented on a computer. The second approach is a numerical method known as the *Newton-Raphson root-finding* algorithm (it is also referred to as *Newton's Method*).

Newton's method is a numerical algorithm which allows one to find the roots (e.g. zero crossings) of a nonlinear equation. Newton's method works by taking an initial estimate of the independent variable, and then iteratively improving upon the estimate. This can be shown graphically. We can then extract the basic algorithm from the graphical picture. For example, let's assume the same values as we did for the previous example. Our goal is identical, that is to find the value of  $V_D$  in equation (5) for which  $f(V_D) = 0$ . In this case, since  $V_{ON} \approx 0.6V-0.8V$ , let's take  $V_{D,0} = 0.8V$  as our initial estimate. Next, we find  $f(V_{D,0}) = 2,650.4\mu A$ . We then draw a line tangent to the point  $V_{D,0}, f(V_{D,0})$ ; the point  $V_{D,1}$  is found from the intersection of the tangent line and the  $f(V_{D,0})$  zero axis. This is shown in figure 4.

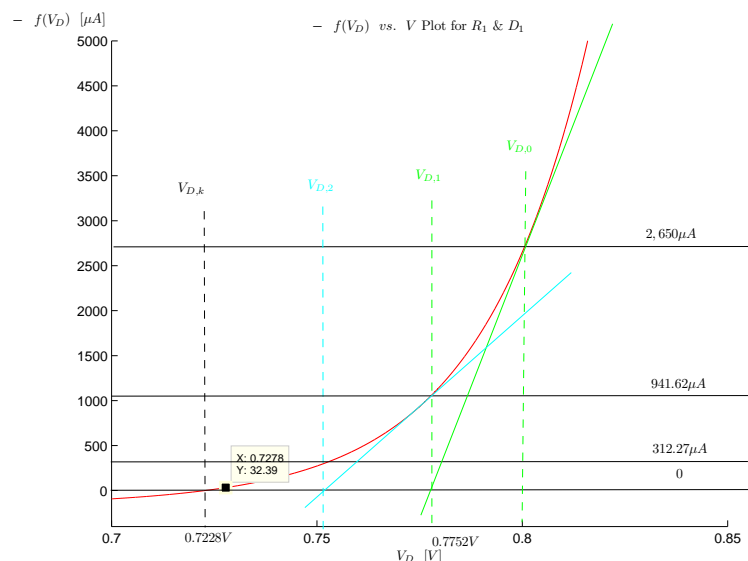


Figure 4. Graphical description of the Newton-Raphson Method.

$f(V_D)$  is plotted in red; note that I have actually plotted  $-f(V_D)$  vs  $V_D$ . The

reason is that it becomes easier for me to illustrate this technique. The points  $V_{D,0}$  and  $V_{D,1}$  are indicated with vertical green dashed lines which intersect  $f(V_D) = 0$  and  $f(V_{D,0})$  and  $f(V_{D,1})$ , respectively ; the tangent line is indicated with a green solid line. The value of  $f(V_{D,1}) = 941.62\mu A$  and  $V_{D,1} = 0.7752V$ . Notice that  $|f[V_{D,1}]| < |f[V_{D,0}]|$ . The slope of this line is given by:

$$Slope = \frac{f(V_D) - f[V_D(0)]}{V_D(1) - V_D(0)} \quad (6)$$

where  $V_{D,0}, f[V_{D,0}]$  is the first point and  $V_{D,1}, f(V_D) = 0$  is the second point.

Since the slope of this line is the derivative with respect to  $V_D$  and the tangent line intersects  $f(V_D) = 0$  (by definition), then the equation (6) can be rewritten as

$$Slope = -\frac{f[V_{D,0}]}{V_{D,1} - V_{D,0}} = \left. \frac{df[(V_D)]}{dV_D} \right|_{V_D=V_{D,0}} \quad (7)$$

where  $\left. \frac{df[(V_D)]}{dV_D} \right|_{V_D=V_{D,0}}$  is the derivative of  $f(V_D)$  evaluated at  $V_{D,0}$ .

This means that once an estimate for  $V_{D,0}$  is found, it can be used to graphically be used to find a better estimate of  $V_D$ ,  $V_{D,1}$ . In fact, we have repeated this for a second iteration, where we use the same method to find  $f(V_{D,2}) = 312.27\mu A$  and  $V_{D,2} = .7524V$ . In general, we can improve our solution with each iterative estimate.

In iterative methods values are selected for the independent variable; in this case  $V_D$ . These values are indexed. In our case we will use  $k$  as the index;  $k = 0, 1, 2, \dots$ . The idea is that as  $k$  is incremented, that the  $(k+1)^{th}$  value for  $V_D$  can be found from the  $k^{th}$  value, or  $V_D(k) \rightarrow V_D(k+1) \rightarrow V_D(k+2)$ , etc.; as  $k$  is incremented, our estimate of  $V_D^*$ , the value that leads to the root, e.g.  $f(V_D^*) = 0$  will be a *better* estimate of  $V_D^*$ . Another way of saying this is by examining the error:

$$|V_{D,(k+1)} - V_D^*| < |V_{D,k} - V_D^*| \quad (8)$$

or

$$f[(V_d^*)] = 0 < |f[(V_{D,(k+1)})]| < |f[(V_{D,k})]| \dots \quad (9)$$

For the  $k^{th}$  estimate, we can re-write equation (7) as

$$Slope = -\frac{f[V_{D,k}]}{V_{D,(k+1)} - V_{D,k}} = \left. \frac{df[(V_D)]}{dV_D} \right|_{V_D=V_{D,k}} \quad (10)$$

where  $\left. \frac{df[(V_D)]}{dV_D} \right|_{V_D=V_{D,k}}$  is the derivative of  $f(V_D)$  evaluated at  $V_{D,k}$ .

This can be carried out graphically; however, it is much more convenient to solve equation (10) for  $V_{D,(k+1)}$ . This gives the following expression:

$$\boxed{V_{D,(k+1)} = V_{D,k} - \frac{f(V_D)}{\left[\frac{df(V_D)}{dV_D}\right]} \Big|_{V_D=V_{D,k}}} \quad (11)$$

which is the Newton-Raphson algorithm.

The absolute value of equation (5),  $|f(V_D)|$ , approaches zero for each successive approximation. It is possible that we never quite find  $V_D^*$  precisely. This means that we will also need an *exit condition*. When the error is less than a value, say  $\epsilon$ , then we will decide that we are *close enough* to the zero crossing.

There are additional details. First, equation (11) needs a starting point. Thus, the first point,  $V_{D,0}$  ( $k = 0$ ) must be found in order to start the algorithm. The initial estimate for  $V_D$  will be  $V_{D,0}$ ; the guidelines for selecting the initial estimate will be discussed at the end of this section.

Second, we must find an expression for the derivative. In this example, we must differentiate equation (5), which leads to the following expression:

$$\frac{df[V_D]}{dV_D} = -\frac{1}{R_1} - \left(\frac{I_s}{V_T}\right) e^{\left(\frac{V_D}{V_T}\right)} \quad (12)$$

Notice that the units of equation (12) is  $\Omega^{-1}$ . The reciprocal of resistance is conductance ( $\Omega^{-1}$ ); the units are *mhos*, which uses the symbol,  $\mathcal{U}$ , or the *SI-unit*, *Siemens*, which uses the symbol  $S$  (upper case S).

Finally, we must decide on the maximum error tolerable; in other words, how close must  $|f[V_{D,k}]|$  be to zero. We will decide that  $|f[V_{D,k}]| < \epsilon$ ,  $\epsilon = 10^{-6}A$ , or  $\epsilon = 1\mu A$ . Obtaining this level of accuracy via a graphical solution will be quite challenging; however, with a good selection of  $V_{D,0}$ , finding  $|f[V_{D,k}]|$  to within this error will typically take 3 – 6 iterations (assuming a good choice for  $V_{D,0}$ ) using Newton's Method. Newton's method has been carried out for this example and the results are given in Table 1<sup>2</sup> for an initial estimate of  $V_D = 0.8V$ . The first three estimates (e.g.  $k = 0, 1 \& 2$ ) were also plotted in figure 4. The value for  $I_{D,k}$  is computed using the current estimate,  $V_{D,k}$  and plugging it into equation (1). The current voltage drop across  $R_1$  is given by  $V_{R_1,k} = I_{D,k}R_1$ , and the current estimate of the supply voltage,  $\hat{V}_{x,k}$ , is given by the sum of the current estimates for the voltage drop across  $R_1$ ,  $V_{R_1,k}$ , and the voltage drop across the diode,  $V_{D,k}$ . Notice that KVL and KCL cannot be correctly solved unless the estimates for  $I_{D,k}$ ,  $V_{D,k}$  are *reasonably close* to  $I_D^*$ ,  $V_D^*$ . My definition of *reasonably close* is four significant digits for  $I_{D,k}$ ,  $V_{D,k}$

<sup>2</sup>  $V_T = 25.85mV$

and  $\hat{V}_{x,k}$ . Since  $V_x$  is known, and both KVL and KCL must hold,  $|V_x - \hat{V}_{x,k}|$  is easily found and can be used as a check to insure that the correct results have been obtained.

$k$	$V_{D,(k+1)}$ [Volts]	$V_{D,k}$ [Volts]	$f[V_{D,k}]$ [ $\mu\text{Amps}$ ]	$f'[V_{D,k}]$ [ $m\Omega^{-1}$ ]	$I_{D,k}$ [ $\mu\text{Amps}$ ]	$V_{R,k}$ [Volts]	$\hat{V}_{x,k}$ [Volts]
0	0.7752	0.8000	-2,650.4	-106.9	2,750.4	5.5008	6.3008
1	0.7524	0.7752	-941.62	-41.2712	1,054.0	2.1080	2.8832
2	0.7344	0.7524	-312.27	-17.3683	436.1	0.8722	1.6245
3	0.7249	0.7344	-84.737	-8.9145	217.5	0.4351	1.1695
4	0.7228	0.7249	-13.057	-6.3256	150.6	0.3012	1.0261
5	0.7228	0.7228	-0.4675	-5.8786	139.0	0.2781	1.0009
6	0.7228	0.7228	$-657.2 \times 10^{-6}$	-5.8621	138.6205	0.2772	1.0000
7	0.7228	0.7228	$-1.3 \times 10^{-9}$	-5.8621	138.6199	0.2772	1.0000
8	0.7228	0.7228	$135.5 \times 10^{-15}$	-5.8621	138.6199	0.2772	1.0000

Table 1

Results from Newton's Method for  $V_x = 1V$ ,  $R_1 = 2k\Omega$  &  $I_S = 0.1fA$ .

This table was implemented in the spreadsheet application *gnumeric*. The same solution can be readily be found in any other spreadsheet application, or in Matlab by writing the algorithm as an m-file or with a calculator by hand. It typically takes no more than 5 – 6 iterations to find a very accurate solution. I have carried out many more iterations to illustrate how quickly the absolute error in estimating  $I_D^*$ ,  $V_D^*$  approaches zero, e.g.  $|f[V_{D,k}]| \rightarrow 0$ .

The convergence to the correct answer is not necessarily guaranteed; this is the reason behind the famous non-convergence errors in SPICE simulators. Newton's Method can fail to converge for several reasons: 1) the function,  $f(x)$ , might be more complicated than anticipated; this could happen with 2-3 different nonlinear devices; 2) a poor initial estimate for  $x_0$  (*most common*); or 3) something else that is *ill conditioned* with respect to the evaluation of  $f(x)$  or  $f'(x)$ . In most cases, Newton's method is quite useful.

Obtaining the initial estimate for  $V_D$  is important. In cases where the power supply voltage in figure 2 is much less than  $V_{ON}$ , then  $V_D = V_X$  is both the initial guess and will also be very close to the final answer. Examples of initial estimates vs. the number of iterations required for a reliable estimate are given in Table 2.<sup>3</sup>

It is important to recognize that the expressions obtained in equations (5)

<sup>3</sup>  $V_T = 25.85mV$

$V_x$ [Volts]	$V_{D,0}$ [Volts]	$I_{D,0}$ [μAmps]	$f[V_{D,0}]$ [μAmps]	$V_{D,k}$ [Volts]	$I_{D,k}$ [μAmps]	$f[V_{D,k}]$ [μAmps]	$k$
0.5	0.5		$25.1 \times 10^{-3}$	0.4999	$25.1 \times 10^{-3}$	$47.1 \times 10^{-9}$	1
0.6	0.6	1.2010	-1.2010	0.5978	1.1031	$4.22 \times 10^{-3}$	1
0.8	0.8	2,750	-2,750	0.6971	51.4350	$56.3 \times 10^{-6}$	7
1.0	0.8	2,750	-2,650	0.7228	138.6205	$657 \times 10^{-6}$	6
1.2	0.8	2,750	-2,550	0.7361	232.0142	$52.1 \times 10^{-3}$	5
5.0	0.8	2,750	-650.39	0.7931	2,103.5	$294 \times 10^{-6}$	3

Table 2

Results obtained over multiple supply voltages, for  $R_1 = 2k\Omega$  &  $I_S = 0.1fA$ .

and (12) will be entirely dependent upon the circuit topology and the specific non-linear elements involved. A second example has been given as a practice problem. The schematic is shown in figure 5.

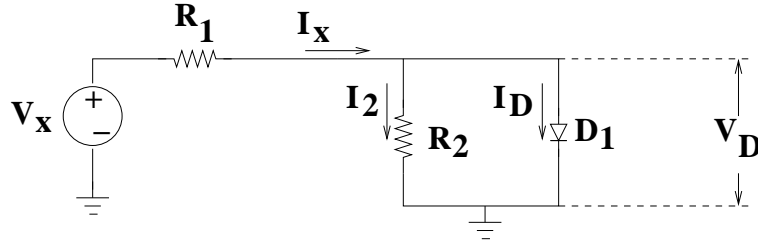


Figure 5. Schematic of practice problem.

Find expressions  $f(V_D)$  and  $f'(V_D)$ . Find the nodal equation at the diode terminal; this can be found using KCL at this node,  $I_x - I_2 - I_D = 0$ . Substitute expressions for  $I_x$ ,  $I_2$  and  $I_D$  and re-write this equation; then take the derivative with respect to  $V_D$ , the voltage at the diode terminals.

I will summarize (assuming a single non-linear element; a diode).

- 1) Find expressions  $f(V_D)$  and  $f'(V_D)$ .
- 2) Check the Thevenin equivalent circuit at the diode terminals. What is the relative difference between  $V_{oc}$  and  $V_{D,ON}$  ?
  - i) If  $V_{oc} < V_{D,ON}$ , then  $V_{D,0} = V_D = V_{oc}$  is the starting point. This approximation should be sufficient, but it can obviously be checked using KCL or KVL and plugging in the estimate of  $I_D$ , or by merely observing the value:  $|f(V_D)|$ .
  - ii) If  $V_{oc} > V_{D,ON}$ , then  $V_{D,0} = V_{D,ON}$  is a good starting point. This will probably require at least several iterations to achieve reasonable accu-



racy.

- iii) If  $V_{oc} \approx V_{D,ON}$ , for example,  $V_{D,ON} = 0.7V$  and  $V_{oc} \approx 0.65V - 0.75V$ , then the starting point  $V_{D,0} = V_{D,ON}$ . 1 – 4 iterations are needed depending upon the degree of accuracy required.
- 3) If iterations are required, then equation (11) will be employed to find the next estimate,  $V_{D,(k+1)}$ . Obviously, the value of  $|f(V_D)|$  and a KVL and/or KCL check will determine if the approximate operating point is sufficient.

## 2 DC Models

DC Models of Diodes are implemented with *piecewise linear* solutions to equation (1). This means that a series of lines are fitted to the diode's  $I - V$  characteristic. The equations of the lines are found by inserting linear circuit elements. The three common models are shown in figure 6. One can use: (a) an ideal diode in series with a single resistor, or (b) an ideal diode in series with an independent voltage source, or (c) a series connection of an ideal diode, an independent voltage source and a resistor.

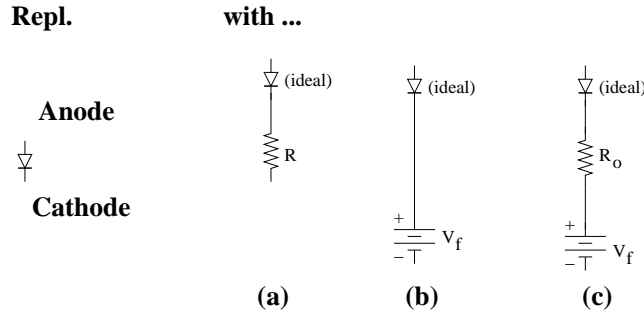


Figure 6. Piece-wise, linear DC diode Models.

If the diode is known to be forward biased then the ideal diode can be replaced by a short circuit. If the diode is reverse biased, then the ideal diode is replaced with an open circuit and each of the 3 models will be replaced with open circuits. The  $I - V$  plots for each of the 3 piecewise linear models are overlayed on top of a diode  $I - V$  characteristic (plotted in red). The model from figure 6 (a) is plotted with a black solid line; the model from figure 6 (b) is plotted with a green dashed line; and the model from figure 6 (c) is plotted with a light blue solid line.

Piecewise linear approximations to the diode  $I - V$  characteristic are used to find DC current biasing solutions; hence, the name DC model. The advantage

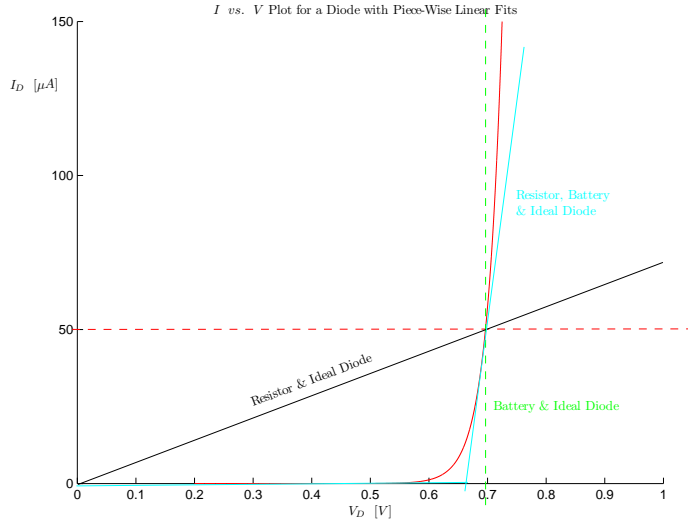


Figure 7. Graphical illustration of each piecewise linear diode model.

to using one of the DC models is that the circuit analysis is reduced to that of solving a set (or a single) of linear equations. The disadvantage is that these solutions are only accurate at a single point on the diode  $I - V$  curve; thus, KVL and KCL will not hold if the bias point changes significantly from the assumed value for  $(I_D, V_D)$ .

Typically, the most common DC models are either the ideal diode in series with an independent voltage source or the series connection of an ideal diode, a resistor and an independent voltage source. In either case, the operating point,  $(I_D, V_D)$ , set by either of these two DC models is only valid for a single point on the actual diode  $I - V$  curve. For example, if we are using the series connection of an ideal diode and an independent voltage source as our DC model, then the value,  $V_f = V_D$  specifies a unique value for  $I_D$ . There will be a unique value of  $I_D$  in equation (1) for the diode terminal voltage,  $V_{D,ON}$ ,  $I_{D,ON}$ . If the value of  $V_f$  is set to  $V_{D,ON}$  and used for  $I_D$  values other than  $I_{D,ON}$ , for example, then both KVL and KCL might not be satisfied the actual value of  $I_D$  found by equation (1) is substituted. One could employ Newton's Method to refine the estimate of  $V_f$ ; alternatively, this could be refined in a SPICE DC simulation.

### 3 AC *Small-Signal* Models

The diode also has a linear ac model. In this case,  $v_D(t) = V_D + v_d(t)$ , where  $v_D(t)$  is an ac voltage,  $v_d(t)$ , with a DC offset  $V_D$ . The DC offset,  $V_D$ , sets the DC operating point,  $(I_D, V_D)$ . The ac voltage superimposed upon  $V_D$  will be linear as long as  $v_d(t) \ll V_D$ ; hence the name, *Small-Signal* model. If this

relation holds then the diode can be replaced by its ac model with the  $v_d(t)$  as the voltage source. In order to find the ac model of the diode we must find the derivative of  $I_D$  with respect to  $v_D(t)$ . This is given by

$$\frac{d}{dv_D} I_D = \left( \frac{I_S}{V_T} \right) e^{\left( \frac{v_D}{V_T} \right)} = \frac{(I_D - I_S)}{V_T} \approx \frac{I_D}{V_T} \quad (13)$$

The dynamic resistance,  $r_d$ , can be found from equation (13); it is the reciprocal of the derivative and is given by

$$r_d = \frac{1}{\frac{dI_D}{dv_D}} = \frac{V_T}{I_D} \quad (14)$$

This is the simplest example of the small signal diode model; it is a resistor with the value,  $r_d$ . Notice that the dynamic resistance,  $r_d$ , required by the small signal model, is set by the DC operating point of the diode. It depends upon  $I_D$ ; thus,  $I_D$ , the DC operating point, is expected to be maintained within a very small deviation. As long as this holds, the linear circuit used in the small signal model will provide reliable results.