ELE/MCE 504 **Homework #5** **Spring '09**

1. Recall that the estimate of a random vector, $\mathbf{x}$, from two measured random vectors, $\mathbf{m}_1$ and $\mathbf{m}_2$, may be obtained using the "two-measurement" formulas given below:

$$\begin{aligned} \hat{\mathbf{m}}_2 &= \mathbf{G}\mathbf{m}_1, \text{ where } \mathbf{G} = \mathbf{C}_{m_2 m_1}\mathbf{C}_{m_1 m_1}^{-1} \\ \mathbf{e}_2 &= \mathbf{m}_2 - \hat{\mathbf{m}}_2 \\ \hat{\mathbf{x}} &= \mathbf{F}\mathbf{m}_1 + \mathbf{F}_2\mathbf{e}_2, \text{ where } \mathbf{F}_1 = \mathbf{C}_{xm_1}\mathbf{C}_{m_1 m_1}^{-1} \text{ and } \mathbf{F}_2 = \mathbf{C}_{xe_2}\mathbf{C}_{e_2 e_2}^{-1}. \end{aligned}$$

Show that $\hat{\mathbf{x}}$ is the optimal MMSE estimator of $\mathbf{x}$ because it satisfies the orthogonality principle. That is, show that the first and second terms identified below are both equal to zero.

$$E\left\{(\mathbf{x} - \hat{\mathbf{x}})\left[\, \mathbf{m}_1^T \quad \mathbf{m}_2^T \,\right]\right\} = E\left\{(\mathbf{x} - \mathbf{F}_1\mathbf{m}_1 - \mathbf{F}_2\mathbf{e}_2)\left[\, \mathbf{m}_1^T \quad \mathbf{m}_2^T \,\right]\right\}$$

The first term is $E\{(\mathbf{x}-\mathbf{F}_1\mathbf{m}_1-\mathbf{F}_2\mathbf{e}_2)\mathbf{m}_1^T\}$ and the second term is $E\{(\mathbf{x}-\mathbf{F}_1\mathbf{m}_1-\mathbf{F}_2\mathbf{e}_2)\mathbf{m}_2^T\}$.

2. The purpose of this problem is to construct an MMSE estimator for a random vector given a vector of two measurements and to compare this estimator with the "two-measurement" formula using each element of the measurement vector individually.

Consider a two-element zero-mean, random measurement vector

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

with covariance matrix $\mathbf{R}$. Note that $\mathbf{m}$ can be simulated as $\mathbf{m} = \mathbf{L}\mathbf{w}$, where $\mathbf{w}$ is a two-element zero-mean random variable with identity covariance matrix, with $\mathbf{R} = \mathbf{L}\mathbf{L}^T$. Note also that $\mathbf{R}$ and $\mathbf{L}$ are 2×2 matrices. Let a two-element random vector $\mathbf{x}$ be given by

$$\mathbf{x} = \mathbf{A}\mathbf{m} + \mathbf{v}$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{v}$ is a zero-mean random vector with identity covariance matrix, and $\mathbf{v}$ and $\mathbf{m}$ are uncorrelated. From the information given above, compute the following covariance matrices:

(a) $\mathbf{C}_{xm} = E(\mathbf{x}\mathbf{m}^T)$

(b) $\mathbf{C}_{mm}$

(c) $\mathbf{C}_{m2m1}$

(d) $\mathbf{C}_{m1m1}$

(e) The optimal estimator of m_2 given m_1 is $\hat{m}_2 = Gm_1$. The error between m_2 and this estimate of m_2 is

$$e_2 = m_2 - Gm_1.$$

Compute C_{xe_2} and $C_{e_2e_2}$.

(f) Obtain a formula for the MMSE estimate of $\mathbf{x}$ given the measured vector $\mathbf{m}$.

(g) Use the "two-measurement" formula to obtain the MMSE estimate of $\mathbf{x}$ given the measurements m_1 and m_2.

3. Use the formulas obtained in Problem 2 with the following numerical matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \ \mathbf{R} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0.9 & \sqrt{1-0.9^2} \end{bmatrix}$$

If we want to estimate the value of $\mathbf{x}$ before any measurements are taken, the best we can do is use the mean value of $\mathbf{0}$. The error of this "mean-value estimator" is $\mathbf{x} - \mathbf{0} = \mathbf{x}$. The following Matlab code sets up the problem and plots the error for the mean-value estimator as well as for the estimators derived in parts (f) and (g) of Problem 1 (you will have to write the code for these estimators).

```
r=0.9;
A=[1 2;2 1];
R=[1 r;r 1];
L=[1 0;r sqrt(1-r^2)];
m=L*randn(2,100); %calculate 100 measurement vectors
x=A*m+randn(2,100); %calculate 100 x vectors
figure(1)
plot(x(1,:),x(2,:),'o');grid
title('Error for the mean-value estimator of x')
%
% Compute xhat1, the MMSE estimator of x given m
%
ee1=x-xhat1;
figure(2)
plot(ee1(1,:),ee1(2,:),'o');grid
title('Error for the MMSE estimator of x given m')
axis([-8 8 -8 8])
%
% Compute xhat2, the MMSE two-measurement formula for x given m1 and m2
%
ee2=x-xhat2;
figure(3)
plot(ee2(1,:),ee2(2,:),'o');grid
title('Error for MMSE two-measurement estimator of x given m1 and m2')
axis([-8 8 -8 8])
norm(xhat1-xhat2)  % shows that xhat1 and xhat2 are the same
```

4. In the derivation of the Kalman filter, guess `G1` related $\hat{\dot{\mathbf{x}}}_\epsilon(t)$ (the best estimate of $\dot{\mathbf{x}}(t)$ given data set `D1`) and $\hat{\mathbf{x}}_\epsilon(t)$ (the best estimate of $\mathbf{x}(t)$ given `D1`) by the following formula:

$$\texttt{G1:}\quad \hat{\dot{\mathbf{x}}}_\epsilon(t) = \mathbf{A}\hat{\mathbf{x}}_\epsilon(t) + \mathbf{B}_u\mathbf{u}(t),$$

where the data set `D1` is: $\mathbf{m}(\tau),\ 0 \le \tau \le t-\epsilon$. The two "best estimates" mentioned above are characterized by the following orthogonality principles:

$$\texttt{O1:}\quad E\{(\dot{\mathbf{x}}(t) - \hat{\dot{\mathbf{x}}}_\epsilon(t))\mathbf{m}^T(\tau)\} = 0,\ 0 \le \tau \le t-\epsilon$$

$$\texttt{O3:}\quad E\{(\mathbf{x}(t) - \hat{\mathbf{x}}_\epsilon(t))\mathbf{m}^T(\tau)\} = 0,\ 0 \le \tau \le t-\epsilon$$

Show that `O1` is satisfied by the estimator in `G1`, making use of the fact that `O3` characterizes the estimator $\hat{\mathbf{x}}_\epsilon(t)$.

5. In the derivation of the Kalman filter, guess `G2` related $\hat{\mathbf{m}}_\epsilon(t)$ (the best estimate of $\mathbf{m}(t)$ given data set `D1`) and $\hat{\mathbf{x}}_\epsilon(t)$ (the best estimate of $\mathbf{x}(t)$ given `D1`) by the following formula:

$$\texttt{G2 :}\quad \hat{\mathbf{m}}_\epsilon(t) = \mathbf{C}\hat{\mathbf{x}}_\epsilon(t).$$

Show that guess `G2` satisfies the orthogonality principle:

$$\texttt{O2 :}\quad E\{(\mathbf{m}(t) - \hat{\mathbf{m}}_\epsilon(t))\mathbf{m}^T(\tau)\} = 0, 0 \le \tau \le t-\epsilon$$

making use of `O3` and assumption `A5` given in class.

6. In the derivation of the Kalman filter we start with a plant model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{B}_w\mathbf{w}(t)\\ \mathbf{m}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t)\end{aligned}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are uncorrelated white noise signals with spectral density matrices $\mathbf{S}_w$ and $\mathbf{S}_v$, respectively. The Kalman filter is described by the following differential equation:

$$\dot{\hat{\mathbf{x}}}(t) \;=\; (\mathbf{A} - \mathbf{G}(t)\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}_u\mathbf{u}(t) + \mathbf{G}(t)\mathbf{m}(t)$$

Let the error vector be defined as $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$.

(a) Derive the state-space differential equation model for $\mathbf{e}(t)$ (i.e. $\dot{\mathbf{e}}(t) = ...$).

(b) Derive the differential equation for the error covariance matrix $\mathbf{\Sigma}_e(t)$ using the results from Chapter 3.