Tracking analysis for general linearly coupled dynamical systems

Jianlong Qiu\textsuperscript{a,b,c}, Jianquan Lu\textsuperscript{b,*}, Jinde Cao\textsuperscript{b}, Haibo He\textsuperscript{c}

\textsuperscript{a}College of Science, Linyi Normal University, Linyi 276005, China
\textsuperscript{b}Department of Mathematics, Southeast University, Nanjing 210096, China
\textsuperscript{c}Department of Electrical and Computer Engineering, University of Rhode Island, Kingston, RI 02881, USA

ABSTRACT

Tracking analysis problem is studied for general linearly coupled dynamical systems in this paper. One challenging and essential question for this issue is that: At least how many nodes should be informed about the objective tracking signal? This paper is devoted to answer this question. Two dynamical network models are considered. The first one, each individual has its own dynamics and simultaneously influenced by its neighbors' information. The dynamics of itself could be stable, periodic, semi-periodic, and chaotic. The second one, each individual update its state just according to the error states different from its communicated neighbors. The main contribution of this paper is that the minimum number of controllers is designed to force the state of each agent to the desired objective by fully utilizing the structure of the network. The convergence rate can also be estimated. The topology of the underlying network can be directed and hierarchical. Some simple criteria are given to judge whether the tracking control can be successful. In addition, numerical examples are given to show the validity of the analytical results.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

Both in nature and society, each individual is influenced by the environment, and simultaneously influence its surrounding circumstance. The investigation of complex dynamical system, which is interconnected by many subsystems, is very important. There are many examples for such dynamical system, for example arrays of chaotic systems in engineering [1], the biochemical reaction network in biology [2], a swarm of organizer in social science [3], and a group of autonomous agents [4,5]. Linearly coupled dynamical system would be a feasible model for such complex systems.

There are many interesting phenomena in such linearly coupled dynamical system, such as synchronization [6–9], swarming [10,11], consensus [12], self-organization, rendezvous problem [13], traveling waves, defect propagation, spatio-temporal chaos, etc. Among them, synchronization and consensus have been a very active area and meanwhile have many potential applications. The word “synchronization” comes from a Greek word meaning “share time”. Nowadays, it has come to be considered as “time coherence” of different process. In theoretical field, there are various kinds of synchronization concepts, e.g., complete synchronization [14–16], phase synchronization, anticipated synchronization, and partial synchronization. The concept of complete synchronization will be considered in this paper. The synchronization technique for linearly coupled dynamical systems can be very useful in many areas, such as the auto-principles for parallel image processing [17], recognizing images with strong robustness [18], secure communication [19], and pattern storing and retrieving [20,21]. Also consensus problems have a long history in computer science and the foundation of the field of distributed computing [22–24]. Partly due to recent rapidly developed technological advances in communication, computation and
engineering, consensus protocol has wide potential applications including cooperative control of unmanned aerial vehicles, scheduling of automated highway systems, coordination/formation of underwater vehicles, attitude alignment of satellite clusters and congestion control in communication networks. Therefore, investigation of the tracking control for synchronization and consensus in the general linearly coupled dynamical systems is an important and necessary step for the practical design and implement of such coupled systems.

For the linearly coupled dynamical systems, graph is one of the core concepts which is used to describe the interaction structure of the coupled system. Each agent is denoted as a node, and the connection between two agents is represented as an edge [25]. The graph is directed if the information exchange between the agents is not symmetric. A directed or undirected graph can be well exhibited by its corresponding adjacent matrix [26] and Laplacian matrix [27], which play some essential roles in the analysis of coupled dynamical systems [5]. This paper aims to study some general directed graph, the corresponding Laplacian matrix of which can be asymmetric, irreducible and m-reducible [26]. To the best of our knowledge, the tracking control issue for dynamical network with m-reducible coupling matrix has not yet been considered by other researchers.

Both for synchronization and consensus problems, the researchers have introduced a set of control laws that enable the group of agents to generate collective behavior and stable flocking motion [5,11,12,28–30,40]. However, it is easy to see that, the group’s final state solely depend on the initial value of the agents and their intrinsic dynamics. This means that these control laws cannot regulate the final collective behavior of the agents. However, the dynamical behavior of the group is sometimes inevitably influenced by some external factors in reality. Hence, only considering the interactions among the agents is not enough. Moreover, in some cases, the regulation of agents has certain target such as achieving the desired common state, or arriving at a desired destination. Therefore, the synchronization and consensus problem in linearly coupled dynamical systems with virtual leaders is an interesting and important topic. There have been some paper discussing about this issue [31–35]. Chen et al. laid a fundamental work for pinning control of complex dynamical networks in [31], in which a single controller is designed for the pinning control of connected networks. Li et al. [33] derived some conditions for stabilizing a complex dynamical network onto its homogenous state by pinning control a fraction of nodes. Shi et al. [35] studied the collective behavior dynamics of a group of agents with a virtual leader, and a set of coordination control laws are introduced to enable the group to generate the desired stable flocking motion. However, the number of controllers which are used to force all agents to the desired state is too large in most of the literatures. Hence, following question naturally arise: At least how many controllers should be designed to make tracking successful? In this paper, the minimum number of controllers will be used to achieve this objective by fully utilizing the structure of the network.

Motivated by the above discussion, tracking control for synchronization and consensus in general linearly coupled dynamical systems will be studied in this paper. By viewing the external signal as a virtual leader, we shall show that all agents eventually evolve towards the desired common state. During their evolution, only a small fraction of agents (only one for irreducible coupling case) are aware of the objective state, and the others update their states according to the information exchange between the neighbors. Under certain condition to be revealed, the objective state will be spread out to all the agents due to the information exchange between the connected agents. In [33,39], coupling relations between the agents are described by a undirected graph, which implies that the information flow between each pair of agents is mutual. However, this is not true in many realistic cases. Hence, it is reasonable to consider the coupling network as a directed graph, which means that the corresponding Laplacian matrix can be asymmetric. Furthermore, in this paper, the case that the Laplacian matrix is m-reducible is also studied. We will also consider the tracking problem in linearly coupled dynamical systems with m-reducible asymmetric fixed coupling matrix. To the best of our knowledge, the tracking control issue for dynamical network with m-reducible coupling matrix has not yet been considered by other researchers. One feature of this paper is that the desired emergent dynamical behavior, which is available for only a few fraction of agents, can be produced consistently by all agents through local communication between the dynamic agents.

The organization of this paper is as follows. Some preliminaries such as definitions and lemmas are given in Section 2. Sections 3 and 4 present the main results on synchronization control and consensus control, respectively. Section 5 gives several examples, and we conclude this paper in Section 6.

Definition 1. The equilibrium point $x^*$ of a dynamical system is said to be globally exponentially stable if an arbitrary solution $x(t)$ satisfies $\|x(t) - x^*\| \leq M \cdot \|x(0) - x^*\| \cdot e^{-\alpha(t-t_0)}$ for $t \geq t_0 \geq 0$ with positive constants $M$ and $\alpha$. Positive constant $\alpha$ is called the rate of exponential convergence.

2. Preliminaries

In this section, we give some definitions and lemmas, which will be useful in the following sections.
Definition 2. Given an $n$ by $n$ matrix $P$, a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is $P$-uniformly decreasing if $(y - z)^T P (f(y) - f(z)) \leq -\eta \|y - z\|^2$ for some $\eta > 0$ and all $y, z \in \mathbb{R}^n$.

Definition 3. A square $n$ by $n$ matrix $A = (a_{ij})$ is called reducible if the indices $1, 2, \ldots, n$ can be divided into two disjoint nonempty sets $i_1, i_2, \ldots, i_l$ and $j_1, j_2, \ldots, j_r$ (with $\mu + \nu = n$) such that $a_{ij} = 0$ for $x = 1, 2, \ldots, \mu$ and $\beta = 1, 2, \ldots, \nu$. A matrix is reducible if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations.

A matrix is reducible if and only if its associated directed graph is not strongly connected.

Definition 4. A square matrix that is not reducible is said to be irreducible.

Definition 5. A $n \times n$ matrix $A = (a_{ij})$ is said to be with diffusive coupling condition if $\sum_{j=1}^{n} a_{ij} = 0$ hold for $i = 1, 2, \ldots, n$. In order to derive our main results, the following lemmas are needed.

Lemma 1 [36]. Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix with complex elements such that $|a_{ii}| \geq \sum_{j=1, j \neq i}^{n} |a_{ij}| (i = 1, 2, \ldots, n)$ with equality in at most $n - 1$ cases. Assume further that the matrix $A$ is irreducible. It follows that the determinant $-A$ is not equal to zero.

Lemma 2 ([26]; Gershgorin circle theorem). For an $n$ by $n$ matrix $A$, define $R_i = \sum_{j=1, j \neq i}^{n} |a_{ij}|$. Then each eigenvalue of $A$ is in at least one of the disks $z: -z - a_{ii} \leq R_i$.

Lemma 3 [26, 37]. For an $N$ by $N$ irreducible matrix $G$ with nonnegative off-diagonal elements satisfying the diffusive coupling condition, we have the following propositions:

(i) If $\lambda$ is an eigenvalue of $G$ and $\lambda \neq 0$, then $\text{Re}(\lambda) < 0$;

(ii) $G$ has an eigenvalue 0 with multiplicity 1 and the right eigenvector $[1, \ldots, 1]^T$;

(iii) Suppose that $\phi = [\phi_1, \phi_2, \ldots, \phi_N] \in \mathbb{R}^N$ (without loss of generality, assume $\max_{i=1, 2, \ldots, N} \{\phi_i\} = 1$) is the left eigenvector of $G$ corresponding to eigenvalue 0. Then, $\phi_i > 0$ hold for all $i = 1, 2, \ldots, N$; Furthermore, if $G$ is symmetric, then we have $\phi_i = 1$ for $i = 1, 2, \ldots, N$.

3. Synchronization control

3.1. Problem formulation

In this section, we consider a network of $q$ interacting nonlinear $n$-dimensional dynamical oscillators. We assume that the intrinsic dynamics of the oscillators are identical, which can be stable, periodic, semi-periodic and chaotic. The reason why this section is named as “synchronization control” is that the collective behavior emerged in this model (1) is usually called synchronization. The composed dynamical system can be described by the following $q \times n$ linearly coupled ordinary differential equations:

\[
\begin{align*}
\dot{x}_1(t) &= f(x_1(t)) + c \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \\
\dot{x}_2(t) &= f(x_2(t)) + c \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \\
& \vdots \\
\dot{x}_q(t) &= f(x_q(t)) + c \sum_{j=1}^{q} d_{ij} \Gamma x_j(t).
\end{align*}
\]

(1)

Or it can be written as

\[
\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \quad (i = 1, 2, \ldots, q),
\]

(2)

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ are the state variables of node $i$, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function, $c$ is the coupling strength, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the inner coupling matrix between two connected nodes with $\gamma_i > 0 (i = 1, 2, \ldots, n)$, $D = (d_{ij})_{q \times q}$ represents the outer coupling configuration of the network, where $d_{ij}$ is defined as follows: if there is a connection from node $j$ and node $i (i \neq j)$, $d_{ij} > 0$; otherwise $d_{ij} = 0 (i \neq j)$, and $\sum_{j=1}^{q} d_{ij} = 0$ for $i = 1, 2, \ldots, q$. Hence, the coupling matrix $D$ satisfies the diffusive coupling condition with nonnegative off-diagonal elements. The matrix $D$ is the Laplacian matrix with respect to the network topology [27].
Tracking signal, which will be used to lead the dynamical network, is generated by the following dynamical system with the initial value \( y_0 \)

\[
y(t) = f(y(t)), \quad y(t) \in \mathbb{R}^n.
\]  

(3)

Without loss of generality, we assume that the controller \( u(t) \) acts on the first node with state variable \( x_1(t) \) as

\[
u(t) = c \cdot d \cdot \Gamma(y(t) - x_1(t)),
\]  

(4)

where \( d > 0 \) is a constant representing feedback strength, and \( c \) is defined as in (2). It is a simple state feedback controller, which is easy and cheap to implement. Controller (4) means that only one controller is operated on the first node of the dynamical network. Alternatively, it can be regarded that only the first node, which is considered as the virtual leader, knows the target. Then the controlled dynamical network can be described as follows:

\[
\begin{align*}
    \dot{x}_1(t) &= f(x_1(t)) + c \sum_{j=1}^q d_{1j} \Gamma x_j(t) + c \cdot d \cdot \Gamma(y(t) - x_1(t)), \\
    \dot{x}_2(t) &= f(x_2(t)) + c \sum_{j=1}^q d_{2j} \Gamma x_j(t), \\
    &\quad \vdots \\
    \dot{x}_q(t) &= f(x_q(t)) + c \sum_{j=1}^q d_{qj} \Gamma x_j(t).
\end{align*}
\]

(5)

Let \( e_i(t) = x_i(t) - y(t) \) be the error state of node \( i \) between its current state and the tracking target, and \( D = (d_{ij})_{q \times q} \) be the coupling matrix of the controlled dynamical network, where

\[
\hat{d}_{ij} = \begin{cases} 
    d_{ij} - d, & i = j = 1; \\
    d_{ij}, & \text{otherwise}.
\end{cases}
\]

(6)

By subtracting (3) from (5), we have the following error dynamical system:

\[
\dot{e}_i(t) = f(x_i(t)) - f(y(t)) + c \sum_{j=1}^q \hat{d}_{ij} \Gamma e_j(t), \quad i = 1, 2, \ldots, q.
\]

(7)

Next, we shall investigate the synchronization problem of (7). Three cases will be considered in this section, in which the coupling configuration matrices \( D \) are respectively (i) irreducible symmetric, (ii) irreducible asymmetric, and (iii) \( m \)-reducible. The last case is important, and its derivation is based on the first two cases.

### 3.2. Irreducible symmetric case

The case that the configuration coupling matrix \( D \) is irreducible symmetric is studied in this subsection, which means that the network topology is strongly connected and undirected.

**Theorem 1.** Suppose that the configuration coupling matrix \( D \) is irreducible and symmetric, and \( f(x) - Q x \) is \( P \)-uniformly decreasing with \( P = \text{diag}\{p_1, \ldots, p_n\} \) and \( Q = \text{diag}\{q_1, q_2, \ldots, q_n\} \), where \( p_j > 0 \). Then the whole dynamical network will globally exponentially synchronize the tracking signal if \( c \lambda_{\text{max}}(D) P + P Q - \eta I_n < 0 \).

**Proof.** By using Lemma 2, one can get that \( \dot{\lambda}(\hat{D}) < 0 \). Since \( \det(\hat{D}) \neq 0 \) according to Lemma 1, we can conclude that \( \lambda(\hat{D}) < 0 \). Hence, it is reasonable to require that \( c \lambda_{\text{max}}(D) P + P Q - \eta I_n < 0 \). Consider the candidate Lyapunov function as follows:

\[
V(t) = \frac{1}{2} \sum_{i=1}^q e_i^T(t) P e_i(t).
\]

(8)

By calculating the time derivative of \( V(t) \) along the solution of (7), one can obtain that:

\[
\begin{align*}
    V(t)|_{(7)} &= \sum_{i=1}^q e_i^T(t) P e_i(t) = \sum_{i=1}^q e_i^T(t) \left[ f(x_i(t)) - f(y(t)) \right] + c \sum_{j=1}^q \hat{d}_{ij} \Gamma e_j(t) \\
    &= \sum_{i=1}^q e_i^T(t) \left[ f(x_i(t)) - Q x_i(t) \right] - \left( f(y(t)) - Q y(t) \right)] + \sum_{i=1}^q e_i^T(t) P Q e_i(t) + c \sum_{i=1}^q e_i^T(t) P \left( \sum_{j=1}^q \hat{d}_{ij} \Gamma e_j(t) \right) \\
    &\leq -\eta \sum_{i=1}^q e_i^T(t) e_i(t) + \sum_{i=1}^q e_i^T(t) P Q e_i(t) + c \sum_{i=1}^q e_i^T(t) P \left( \sum_{j=1}^q \hat{d}_{ij} \Gamma e_j(t) \right),
\end{align*}
\]

(9)

where Definition 2 is used.
Further deriving the last term of (9), we have the following equality:

\[
\begin{align*}
\sum_{j=1}^{n} e_i^T(t) P \left( \sum_{l=1}^{n} \frac{q}{j} \gamma_l e_j(t) \right) & = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{q}{j} \gamma_l e_j^T(t) P e_j(t) \\
& = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{q}{j} \gamma_l \left( \sum_{k=1}^{n} p_{kj} e_k(t) e_j(t) \right) \\
& = \sum_{j=1}^{n} \sum_{k=1}^{n} p_{kj} \left( \frac{q}{j} \gamma_l e_k(t) e_j(t) \right) \\
& = \sum_{j=1}^{n} p_{kj} \left( \frac{q}{j} \gamma_l \left( e_k(t) \right)^T \right) \\
& = \sum_{j=1}^{n} p_{kj} \left( \frac{q}{j} \gamma_l \left( e_k(t) \right)^T \right) \\
& = \sum_{j=1}^{n} \sum_{k=1}^{n} p_{kj} \left( \frac{q}{j} \gamma_l \left( e_k(t) \right)^T \right) \\
& = \sum_{j=1}^{n} \left( \frac{q}{j} \gamma_l \left( e_k(t) \right)^T \right) \\
\end{align*}
\]

where \( e_k(t) = (e_{k1}(t), e_{k2}(t), \ldots, e_{kn}(t))^T \) is a column vector formed by the \( k \)th component of each individual.

By substituting equality (10) into (9), one can obtain that:

\[
\begin{align*}
\dot{V}(t)|_{\gamma_l} & \leq \sum_{j=1}^{n} e_i^T(t) (C \lambda_{\text{max}}(D) P + \gamma_l - \gamma_l n) e_i(t).
\end{align*}
\]

Since \( C \lambda_{\text{max}}(D) P + \gamma_l - \gamma_l n < 0 \), there exists a positive constant \( \epsilon > 0 \) such that \( \epsilon C \lambda_{\text{max}}(D) P + \gamma_l - \gamma_l n \leq -\epsilon \gamma_l n \). Hence, we obtain that \( \dot{V}(t)|_{\gamma_l} \leq -\epsilon \sum_{j=1}^{n} e_i^T(t) e_i(t) \leq -2V(t)/\lambda_{\text{max}}(P) \). Hence, it can be concluded that \( \frac{1}{2} \lambda_{\text{min}}(P) \|e_i(t)\|^2 \leq V(t) \leq V(0) \cdot e^{-2t/\lambda_{\text{max}}(P)} \), which implies that \( \|e_i(t)\| \leq \sqrt{2V(0)/\lambda_{\text{min}}(P)} \cdot e^{-t/\lambda_{\text{max}}(P)} \) for \( i = 1, 2, \ldots, p \). This completes the proof of this theorem. □

**Remark 1.** The function of matrix \( Q \) in Theorem 1 is quite similar with the Lipschitz constant in Lipschitz continuous systems, which is used to bound the nonlinear vector-valued function \( f(x) \). If the dynamical system \( \dot{x}(t) = f(x(t)) \) is asymptotically stable, then matrix \( Q \) could be negative definite. Otherwise, \( Q \) could be positive definite.

**Remark 2.** Any node in dynamical network with irreducible symmetric coupling matrix can be chosen as the leader to make \( \lambda_{\text{max}}(D) < 0 \) according to Lemma 1. By referring to [33], the node \( j \) with maximum out-degree \( \sum_{i=1}^{n} d_{ij} \) should be a good choice. The best node chosen as leader should be the one which can minimize the largest eigenvalue of the controlled coupling matrix \( \hat{D} \).

**Remark 3.** The criterion is given in terms of the coupling strength \( c \) and the eigenvalue \( \lambda_{\text{max}}(D) \), which means that there is a requirement on the large interconnected system itself. The entire coupled systems should be tightly connected (mathematically the criteria in Theorem 1 should be satisfied) to ensure that all the individuals can be controlled by only controlling one of them, and it is quite similar with the classic concept “controllability” in control theory.

**Remark 4.** The control issue considered in this paper can be regarded as a Single-Input Multiple-Output problem, since the controller is only added on one out of \( q \) subsystems.

### 3.3. Irreducible asymmetric case

In this subsection, we assume that the configuration coupling matrix \( D = (d_{ij})_{q \times q} \) is asymmetric and irreducible, which implies that the structure of the network is strongly connected and directed.

Let \( \phi = (\phi_1, \phi_2, \ldots, \phi_q)^T \) be the left eigenvector with respect to the eigenvalue zero of configuration matrix \( D \). It should be noted that \( \phi_i > 0 \) for \( i = 1, 2, \ldots, q \) according to Lemma 3. Without loss of generality, we can assume that \( \max_i \{ \phi_i \} = 1 \). Denote \( \Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_q) \), and matrix \( \hat{D} \) is defined as (6).

Construct a symmetric matrix as

\[
H = (h_{ij})_{q \times q} = \frac{1}{2} \left( \Phi \hat{D} + \hat{D}^T \Phi \right).
\]

**Theorem 2.** Suppose that the configuration coupling matrix \( D \) is irreducible and asymmetric, and \( f(x) - Qx \) is \( P \)-uniformly decreasing with \( P = \text{diag}(p_1, \ldots, p_n) \) and \( Q = \text{diag}(q_1, q_2, \ldots, q_n) \), where \( p_i > 0 \). By using the controller (4) operated on the first node, the whole dynamical network will globally exponentially approach the tracking signal, if \( q_j + c_j \lambda_{\text{max}}(H) < 0 \) for \( j = 1, 2, \ldots, n \). Furthermore, the convergence rate of the synchronization process is larger than \( \frac{1}{\lambda_{\text{max}}(P)} \).

**Proof.** According to the construction of matrix \( H \), one can get that the matrix \( H \) has the following properties:

- \( h_{ii} < 0, h_{ij} \geq 0 \) for \( i \neq j, i, j = 1, 2, \ldots, q \);
- \( |h_{11}| > \sum_{j=2}^{q} |h_{1j}| \);
- \( |h_{ii}| = \sum_{j=1, j \neq i}^{q} |h_{ij}|, i = 2, 3, \ldots, q. \)
Due to the irreducibility of configuration coupling matrix $D$ and $\phi_i > 0$, it can be concluded that matrix $H$ is also irreducible. Hence, according to the Lemmas 1 and 2, we can obtain that $\lambda_{\text{max}}(H) < 0$.

Since $c > 0$, $\gamma > 0$ and $\lambda_{\text{max}}(H) < 0$, one can get that $c_1^{ij}\gamma_{\text{max}}(H) < 0$. Hence, it is reasonable to require the condition that $q_j + c_1^{ij}\gamma_{\text{max}}(H) \leq 0$. Choose the following Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{n} q_i e_i^T(t)Pe_i(t). \quad (13)$$

The time derivative of $V(t)$ along the solution of (7) can be expressed as follows:

$$\dot{V}(t)_{(7)} = \sum_{i=1}^{n} q_i e_i^T(t)P \dot{e}_i(t) = \sum_{i=1}^{n} q_i \dot{e}_i^T(t)P\left\{ f(x_i(t)) - f(y(t)) + c \sum_{j=1}^{n} d_{ij} \dot{e}_j(t) \right\}$$

$$= \sum_{i=1}^{n} q_i e_i^T(t)P\left[ f(x_i(t)) - Qx_i(t) - f(y(t)) - Qy(t) \right] + \sum_{i=1}^{n} q_i e_i^T(t)PQe_i(t) + c \sum_{i=1}^{n} q_i e_i^T(t)P \left( \sum_{j=1}^{n} d_{ij} \dot{e}_j(t) \right). \quad (14)$$

By some algebra calculations, we have the following equalities:

$$\sum_{i=1}^{n} q_i e_i^T(t)PQe_i(t) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} p_j q_j e_{ij}^2(t) \right) = \sum_{i=1}^{n} p_j q_j \left( \sum_{i=1}^{n} q_i e_{ij}^2(t) \right) = \sum_{i=1}^{n} p_j q_j (e_i^T(t))^T \Phi e_i(t), \quad (15)$$

and

$$c \sum_{i=1}^{n} q_i e_i^T(t)P \left( \sum_{j=1}^{n} d_{ij} \dot{e}_j(t) \right) = c \sum_{i=1}^{n} \sum_{j=1}^{n} e_i^T(t)P \dot{e}_j \sum_{j=1}^{n} q_j d_{ij} \dot{e}_j(t) = c \sum_{i=1}^{n} \sum_{j=1}^{n} q_j d_{ij} \dot{e}_j(t) (\sum_{k=1}^{n} p_k \gamma_k e_k(t) e_k^T(t))$$

$$= c \sum_{i=1}^{n} \sum_{k=1}^{n} q_j d_{ij} \dot{e}_j(t) (\sum_{k=1}^{n} p_k \gamma_k e_k(t) e_k^T(t)) = c \sum_{i=1}^{n} \sum_{k=1}^{n} p_k \gamma_k (\sum_{j=1}^{n} q_j d_{ij} \dot{e}_j(t)) e_k^T(t)$$

$$= \frac{1}{2} c \sum_{i=1}^{n} \sum_{k=1}^{n} p_k \gamma_k (\sum_{j=1}^{n} q_j d_{ij} \dot{e}_j(t)) e_k^T(t) \Phi \tilde{D} \Phi = c \sum_{i=1}^{n} p_j q_j (e_i^T(t))^T H e_i(t), \quad (16)$$

where $e_i^T(t) = (e_{ij}(t), e_{2j}(t), \ldots, e_{qj}(t))^T$ is a column vector formed by the $j$th component of each individual.

Since $f(x) - Qx$ is $P$-uniformly decreasing, substitution of equalities (15) and (16) into (14) implies that:

$$\dot{V}(t)_{(7)} \leq -\eta \sum_{i=1}^{n} q_i e_i^T(t) e_i(t) + \sum_{j=1}^{n} p_j q_j (e_i^T(t))^T \left[ q_j \Phi + c_1^{ij} H \right] e_i(t) \quad (17)$$

Since $p_j > 0$, $\max_{1 \leq i \leq q} \{ \phi_i \} = 1$, and $q_j + c_1^{ij}\gamma_{\text{max}}(H) \leq 0$ for $j = 1, 2, \ldots, n$, we have:

$$\dot{V}(t)_{(7)} \leq -\eta \sum_{i=1}^{n} q_i e_i^T(t) e_i(t) \leq -2\eta V(t) / \lambda_{\text{max}}(P).$$

Therefore, we can conclude that $\frac{1}{\eta} \min_{1 \leq i \leq q} \{ \phi_i \} / \lambda_{\text{max}}(P) ||e||^2 \leq V(t) \leq V(0) e^{2 \eta \min_{1 \leq i \leq q} \{ \phi_i \} / \lambda_{\text{max}}(P)}$, and further that $||e_i(t)|| \leq \sqrt{\frac{2V(0)}{\min_{1 \leq i \leq q} \{ \phi_i \} \lambda_{\text{max}}(P)}} \cdot e^{\frac{2\eta}{\lambda_{\text{max}}(P)} t}$. Hence, the states of all individuals $x_i(t) (i = 1, 2, \ldots, q)$ exponentially approach the tracking signal $y(t)$ with convergence rate larger than $\frac{\eta}{\lambda_{\text{max}}(P)}$. \qed

**Remark 5.** It should be noted that the balanced matrix (i.e. $\sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} d_{ji}$ for $i = 1, 2, \ldots, q$), which has been studied in [5], is a special asymmetric matrix with its left eigenvector $\phi = (1,1,\ldots,1)^T$. Hence, the result about balanced network can be easily obtained by substituting $H = (h_0)_{q \times q} = \frac{1}{2}(\tilde{D} + \tilde{D}^T)$.\]

3.4. m-Reducible

In this subsection, we consider the case of m-reducible matrix.

**Definition 6.** For an irreducible square matrix $B$ with nonnegative off-diagonal elements, the quantity $\alpha(B)$ is defined as follows: decompose $B$ uniquely as $B = L + D$, where $L$ is a zero row sum matrix and $D$ is a diagonal matrix. Let $\phi$ be the unique left eigenvector of $L$ with respect to eigenvalue zero with $\max_L \{ \phi_k \} = 1$. Let $\Phi = \text{diag}(\phi)$. Then $\alpha(B) = \frac{1}{2} \lambda_{\text{max}}(\Phi B + B^T \Phi)$.\]
Definition 7. Consider a reducible matrix $D$ of order $N$. The matrix is $N$-reducible if it is diagonal. For $1 \leq m < N$, the matrix $D$ is $m$-reducible if it is not $(m + 1)$-reducible and it can be rewritten in the following Frobenius normal form after certain permutations:

$$
D = \begin{bmatrix}
D_1 & D_{12} & \cdots & D_{1k} & D_{1,k+1} & D_{1,k+2} & \cdots & D_{1,k+m} \\
0 & D_2 & \cdots & D_{2k} & D_{2,k+1} & D_{2,k+2} & \cdots & D_{2,k+m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & D_{k,1} & D_{k,2} & \cdots & D_{k,k+m} \\
0 & 0 & \cdots & 0 & 0 & D_{k+1,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_{k+m}
\end{bmatrix},
$$

(18)

where $D_r (r = 1, 2, \ldots, k + m)$ are square irreducible matrices, and for each $r \leq k$, there exists $s > r$ such that $D_s \neq 0$.

Remark 6. The permutation process of the general Laplacian matrix into Frobenius normal form corresponds to decomposing the graph into maximally strongly connected subgraphs. It is a standard problem in graph algorithms, which can be solved in linear time [38].

Remark 7. It should be noted that the concept “$m$-reducible” is quite general in Laplacian matrix theory. If one defines the irreducible matrix as 0-reducible matrix, then any Laplacian matrix of its corresponding network (including connected and disconnected) can be represented by $m$-reducible matrix with $m \geq 0$.

Let irreducible matrix $D_r = (d_{ij}^r) \in \mathbb{R}^{q_r \times q_r}$, $N_r = \sum_{s=r}^{k} q_s$ for $r = 1, 2, \ldots, k + m$, and $N_0 = 0$. The dimension of matrix $D$ is $N_{k+m}$. Then the composed dynamical system with $m$-reducible coupling matrix (18) can be decomposed into $k + m$ subsystems denoted by $\mathcal{S}_r = \{N_{r-1} + 1, N_{r-1} + 2, \ldots, N_r\}$ for $r = 1, 2, \ldots, k + m$:

$$
\mathcal{S}_r: \dot{x}_t(t) = f(x_t(t)) + c \sum_{j \in \mathcal{S}_r} d_{ij} \Gamma x_j(t) + c \sum_{j \in \mathcal{S}_r, \nu > r} d_{ij} \Gamma x_j(t), \quad i \in \mathcal{S}_r, \quad r = 1, 2, \ldots, k.
$$

(19)

and

$$
\mathcal{S}_{\omega}: \dot{x}_t(t) = f(x_t(t)) + c \sum_{j \in \mathcal{S}_{\omega}} d_{ij} \Gamma x_j(t), \quad i \in \mathcal{S}_{\omega}, \quad \omega = k + 1, \ldots, k + m.
$$

(20)

By the definition of $m$-reducible matrix, for each $r \leq k$, there exists $s > r$ such that $D_s \neq 0$, hence one has that subsystem $\mathcal{S}_\omega (\omega = 1, \ldots, k)$ must be influenced by some other subsystems. From (20), it can be concluded that the subsystem $\mathcal{S}_{\omega} (\omega = k + 1, \ldots, k + m)$ is not influenced by the others. In other words, the subsystem $\mathcal{S}_{\omega}$ for $k+1 \leq \omega \leq k + m$ is independent from the others and self-contained. Moreover, the self-synchronized states of these $m$ subsystems, which depends on the initial conditions, network topology and dynamical behavior of each node, are very difficult to be predicted, particulary for chaotic individuals. Therefore, at least $m$ controllers are needed to make the whole system approach the desired state. Then a question arises naturally, can we just use the minimum number (m here) of controllers to control this dynamical network? The answer is “Yes”, and it will be shown in the following.

By adding one controller (totally $m$ controllers) to any node (i.e. the first controller of each subsystem $\mathcal{S}_{\omega} (\omega = k + 1, \ldots, k + m)$ with state variable $x_{N_{\omega-1}+1}$ in the form of:

$$
u_0(t) = c \cdot d \cdot \Gamma (y(t) - x_{N_{\omega-1}+1}), \quad \omega = k + 1, \ldots, k + m.
$$

(21)

we can get the controlled dynamical system described by (19) and (22):

$$
\mathcal{S}_{\omega}: \begin{cases}
\dot{x}_{N_{\omega-1}+1} = f(x_{N_{\omega-1}+1}) + c \sum_{j \in \mathcal{S}_{\omega}} d_{N_{\omega-1}+1,j} \Gamma x_j(t) + c \cdot d \cdot \Gamma (y(t) - x_{N_{\omega-1}+1}), \\
\dot{x}_t(t) = f(x_t(t)) + c \sum_{j \in \mathcal{S}_{\omega}} d_{ij} \Gamma x_j(t), \quad i \in \mathcal{S}_{\omega}, \quad i \neq N_{\omega-1} + 1,
\end{cases}
$$

for $\omega = k + 1, \ldots, k + m$.

(22)

Defining $e_t(t) = x_t(t) - y(t)$, we can get the dynamical behavior of error state $e_t(t)$ expressed as follows:

$$
\mathcal{S}: \dot{e}_t(t) = f(x_t(t)) - f(y(t)) + c \sum_{j \in \mathcal{S}_{\omega}} d_{ij} \Gamma e_j(t) + c \sum_{j \in \mathcal{S}_{\omega}, \nu > r} d_{ij} \Gamma e_j(t), \quad i \in \mathcal{S}_r, \quad r = 1, 2, \ldots, k.
$$

(23)

and

$$
\mathcal{S}_{\omega}: \dot{e}_t(t) = f(x_t(t)) - f(y(t)) + c \sum_{j \in \mathcal{S}_{\omega}} d_{ij} \Gamma e_j(t), \quad i \in \mathcal{S}_{\omega}, \quad \omega = k + 1, \ldots, k + m.
$$

(24)
where
\[
\dot{d}_j = \begin{cases} 
  d_{ij} - d, & \text{for } i = j = N_{\alpha - 1} + 1, \quad \omega = k + 1, \ldots, k + m; \\
  d_{ij}, & \text{otherwise}.
\end{cases}
\]

Hence, the coupling matrix of error dynamical system is
\[
\tilde{D} = \begin{bmatrix}
D_1 & D_{12} & \cdots & D_{1k} & D_{1,k+1} & D_{1,k+2} & \cdots & D_{1,k+m} \\
0 & D_2 & \cdots & D_{2k} & D_{2,k+1} & D_{2,k+2} & \cdots & D_{2,k+m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_k & D_k,_{k+1} & D_k,_{k+2} & \cdots & D_k,_{k+m} \\
0 & 0 & \cdots & 0 & \tilde{D}_{k-1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \tilde{D}_{k-2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \tilde{D}_{k+m}
\end{bmatrix}
\]
\[
\tilde{D}_{\omega} = \begin{bmatrix}
\tilde{d}_{11} - d & \tilde{d}_{12} & \cdots & \tilde{d}_{1\omega} \\
\tilde{d}_{21} & \tilde{d}_{22} & \cdots & \tilde{d}_{2\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{d}_{\omega,1} & \tilde{d}_{\omega,2} & \cdots & \tilde{d}_{\omega,\omega}
\end{bmatrix} \in \mathbb{R}^{q_1,q_\omega} \quad \text{for } \omega = k + 1, \ldots, k + m.
\]

**Theorem 3.** Suppose that the coupling matrix $D$ of dynamical network (1) is $m$-reducible in the form of (18). $f(x) - Qx$ is $P$-uniformly decreasing with $P = \text{diag}(p_1,p_2,\ldots,p_n)$ and $Q = \text{diag}(q_1,q_2,\ldots,q_n)$, where $p_j > 0$. Then the state of each node in the dynamical system will converge to the tracking signal $y(t)$ under the controllers (21), if $q_j + c_j \lambda_{\text{max}} \leq 0$, where $\lambda_{\text{max}} = \max\{\lambda(D_1),\ldots,\lambda(D_k),\lambda(D_{k+1}),\ldots,\lambda(D_{k+m})\}$, and $\lambda(\cdot)$ is defined in Definition 6.

**Proof.** By using the result in Theorem 2, one can conclude that the controlled subsystem $S_{\alpha 0}(\omega = k + 1, \ldots, k + m)$ can be globally exponentially synchronized with the tracking signal $y(t)$, that is the final synchronized state of the subsystem $S_{\alpha 0}(\omega = k + 1, \ldots, k + m)$ satisfies the following equation:
\[
\dot{y}(t) = f(y(t)) + O(e^{-\epsilon t}), \quad \text{for some } \epsilon > 0.
\]
Then, according to the zero-sum rows, i.e. $\sum_{j \in S_k} d_{ij} + \sum_{j \in S_k, j > k} d_{ij} = 0 (i \in S_k)$, the $k$th error subsystem can be described as follows:
\[
E_k : \dot{e}_k(t) = f(x_k(t)) - f(y(t)) + c \sum_{j \in S_k} d_{ij} f_j(t) + c \sum_{j \in S_k, j > k} d_{ij} f_j(t) = f(x_k(t)) - f(y(t)) + c \sum_{j \in S_k} d_{ij} f_j(t) + O(e^{-\epsilon t}), \quad i \in S_k.
\]

Also by Theorem 2, the $k$th subsystem can also exponentially synchronize with the tracking signal $y(t)$.

By induction, we can conclude that the following coupled subsystems $S_{\alpha r}(r = k - 1, \ldots, 1)$ can also exponentially synchronize with the tracking signal. Therefore, the composed system with $m$-reducible coupling matrix (18) will exponentially synchronize with the tracking signal under the controllers (21). The proof of this theorem is ended. \(\square\)

**Remark 8.** The network with $m$-reducible Laplacian matrix can be disconnected if $m \geq 2$. For example, if one takes $m = 2$, $D_{k+1} = 0$ ($i = 1, 2, \ldots, k$), and $D_{2,i} \neq 0$ ($i = 1, 2, \ldots, k$), then the corresponding network with such Laplacian matrix is disconnected with two components.

4. Consensus control

In this section, we study the control of collective behavior in multi-agent systems. The dynamical behavior of each agent in the system is only influenced by its neighbors’ information, but has no intrinsic dynamic itself. The model of such multi-agent system is described by the following linearly coupled ordinary differential equations [13,5]:
\[
\begin{align*}
    \dot{x}_1(t) &= \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \\
    \dot{x}_2(t) &= \sum_{j=1}^{q} d_{2j} \Gamma x_j(t), \\
    &\vdots \\
    \dot{x}_q(t) &= \sum_{j=1}^{q} d_{qj} \Gamma x_j(t).
\end{align*}
\] (30)

Or
\[
\dot{x}_i(t) = \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \quad i = 1, 2, \ldots, q.
\] (31)

where \(x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n\) are the state variables of agent \(i\), \(\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)\) is the inner coupling matrix between two connected nodes with \(\gamma_i > 0 (i = 1, 2, \ldots, n)\), \(D = (d_{ij})_{q \times q}\) represents the outer coupling configuration of the agents, where \(d_{ij}\) is defined as follows: if there is a connection from node \(j\) and node \(i (i \neq j)\), \(d_{ij} > 0\); otherwise \(d_{ij} = 0 (i \neq j)\), and \(\sum_{j=1}^{q} d_{ij} = 0\) for \(i = 1, 2, \ldots, q\). The matrix \(D\) is the Laplacian matrix with respect to the network topology.

**Remark 9.** In fact, the consensus requirement on the model (30) can be regarded as the gradient decent algorithm to minimize the energy function \(E(x_1, \ldots, x_q) = \frac{1}{2} \sum_{i=1}^{q} d_{ii} (x_i - \bar{x})^T \Gamma (x_i - \bar{x})\) without any constraint.

In this section, we will try to control the final consensus state of the coupled multi-agents by using the minimum number of controllers. Let the objective final state be:
\[
z(t) = z_0.
\] (32)

By defining the error state as \(e_i(t) = x_i(t) - z_0\), one can get the following error dynamical system due to the fact that \(\sum_{j=1}^{q} d_{ij} z_0 = 0\):
\[
\dot{e}_i(t) = \sum_{j=1}^{q} d_{ij} \Gamma e_j(t), \quad i = 1, 2, \ldots, q.
\] (33)

In the following, we will consider the control of multi-agents system (31) with respectively (i) symmetric irreducible coupling matrix; (ii) asymmetric irreducible coupling matrix; and (iii) \(m\)-reducible coupling matrix.

Suppose that the coupling matrix of multi-agents is symmetric (or asymmetric) and irreducible. Without loss of generality, the first agent is chosen as the controlled one with controller:
\[
    u(t) = \gamma \cdot \Gamma e_1(t),
\] (34)
then the controlled error multi-agents system can be expressed as follows:
\[
\dot{e}_i(t) = \sum_{j=1}^{q} d_{ij} \Gamma e_j(t), \quad i = 1, 2, \ldots, q.
\] (35)

where coupling matrix \(\bar{D} = (d_{ij})_{q \times q}\) is defined as in (6). Following the similar line as the proof of Theorems 1 and 2, one can get the following results.

**Theorem 4.** Suppose that the coupling matrix \(D\) of the multi-agents system (31) is symmetric and irreducible. Under the controller (34) acted on the first agent, the states of all agents will exponentially approach the desired final state \(z_0\) with convergence rate \(\lambda_{\max}(\bar{D}) \cdot \min_{k} \{\gamma_k\}\).

**Theorem 5.** Suppose that the coupling matrix \(D\) of the multi-agents system (31) is asymmetric and irreducible. By adding the controller (34) on the first agent, the states of the agents will go to the desired state \(z_0\) with convergence rate \(\lambda_{\max}(H) \cdot \min_{k} \{\gamma_k\}\), where \(H\) is defined in (12).

It has been revealed that the final consensus state of model (30) is totally determined by the initial state of the agents and the network topology [5], and average consensus is achieved for multi-agents with undirected or balanced coupling structure [5]. However, usually the final consensus state cannot be guaranteed to be the objective state we need. Hence, one control is required for each self-contained subsystems. Therefore, when the coupling matrix \(D\) of multi-agents dynamical system (31) is \(m\)-reducible in the form of (18), at least \(m\) controllers are needed to force the states of all agents to the desired final state \(z_0\). Without loss of generality, \(m\) controllers \(u_{\omega}(t) (\omega = k + 1, \ldots, k + m)\) can be respectively operated on the first agent of the \(m\) subsystems \((k + 1)^{th}, \ldots, (k + m)^{th}\) in the form of:
\[
u_{\omega}(t) = \gamma \cdot \Gamma (z_0 - x_{\omega_{k+1}}), \quad \text{for} \quad \omega = k + 1, \ldots, k + m.
\] (36)
Then we can get the error dynamical system (35) with coupling matrix (26). Similar with the proof of Theorem 3, we can get the following result.

**Theorem 6.** Suppose that the coupling matrix $D$ of the multi-agents system (31) is $m$-reducible as shown in (18). Under the action of controllers (36), all the agents’ states will go to the desired final state $z_0$ exponentially with convergence rate $-\alpha_{\text{max}} \cdot \min_k \{\gamma_k\}$, where $-\alpha_{\text{max}}$ is defined in Theorem 3.

**Remark 10.** It should be noted that the following statement is the main difference between the models (1) and (30). The dynamical behavior in (1) is governed not only by the intrinsic nonlinear dynamics at each node but also by the diffusion due to the spatial coupling between connected nodes. However, for model (30), each node has no inherent dynamic, and its dynamical behavior is only determined by the difference between its current state and the state values of its neighbors.

5. Numerical examples

In this section, three numerical examples will be given to show the validity of the theoretical results obtained in the previous sections to judge whether the coupled dynamical system can be controlled to the desired objective. A three-dimensional dynamical system is chosen as the node of the coupled dynamical network, the dynamical behavior of which is described by the following ordinary differential equation:

$$
\dot{y}(t) = -Ay(t) + Bg(y(t)),
$$

(37)

where $y(t) = (y_1(t), y_2(t), y_3(t))^T \in \mathbb{R}^3$, and $g(y(t)) = (g(y_1(t)), g(y_2(t)), g(y_3(t)))^T$ with $g(s) = \frac{1}{2}(s + 1/|s|)$. The parameters of this system are taken as $A = I_3$, and

$$
B = \begin{bmatrix}
    1.25 & -3.2 & -3.2 \\
    -3.2 & 1.1 & -4.4 \\
    -3.2 & 4.4 & 1.0 \\
\end{bmatrix}.
$$

(38)

As pointed out in [39], the dynamical system (37) with such given parameters has a double-scrolling chaotic attractor as shown in Fig. 1 with initial condition $y(0) = (0.2, 0.2, 0.2)^T$. This chaotic attractor $y(t)$ will be chosen as the tracking signal in the numerical simulations.

Denote $f(y(t)) = -Ay(t) + Bg(y(t))$. It can be obtained that $f(y) - Qy$ is $P$-uniformly decreasing with $P = I_3, Q = \text{diag}(7, 7, 7)$ and $\eta = 0.9$.

In the following simulations, we consider the linearly coupled dynamical system:

$$
\dot{x}_i(t) = -Ax_i(t) + Bg(x_i(t)) + c \sum_{j=1}^{q} d_{ij} \Gamma x_j(t), \quad i = 1, 2, \ldots, q,
$$

(39)

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T, \Gamma = \text{diag}(3, 4, 4)$, the coupling strength $c = 5$, and $D = (d_{ij})_{q \times q}$ is the coupling matrix. The feedback strength $d$ in controller is taken as $d = 3$. Three numerical examples will be presented to show the efficiency of the de-
signed controllers, the coupling matrix $D$ of which are, respectively, irreducible symmetric, irreducible asymmetric, and 2-reducible.

**Example 1** (Irreducible symmetric). In this example, a symmetric irreducible coupling matrix $D = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ is chosen to represent the topology of the network. By only using one controller (4), we can get the error coupled dynamical system (7) with coupling matrix $B = D$. One has that $\lambda_{\max}(B) = -0.5505$. Then according to Theorem 1, we can conclude that the three coupled dynamical systems $x_i(t) (i = 1,2,3)$ will exponentially synchronize with the tracking signal $y(t)$ with initial condition $y(0) = (0.2,0.2,0.2)^T$. Numerical simulation is given in Fig. 2 with the initial values of the dynamical network randomly taken from $[-1,1]$.

**Example 2** (Irreducible asymmetric). The linearly coupled dynamical system with irreducible asymmetric coupling matrix is considered in this example. The coupling matrix $D$ is taken as $D = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -3 & 1 \\ 3 & 2 & -5 \end{bmatrix}$, the left eigenvector corresponding to the zero eigenvalue is $\phi = (0.7732,1.0000,0.6310)^T$. The coupling matrix $B$ of error dynamical system (7) is $B = \begin{bmatrix} -6 & 1 & 2 \\ 2 & -3 & 1 \\ 3 & 2 & -5 \end{bmatrix}$. Then one gets $\lambda_{\max}(B) = -0.7159$. According to Theorem 2, we can conclude that the whole dynamical network will globally exponentially approach the tracking signal $y(t)$. By randomly taking the initial conditions from $[-1,1]$, the computer simulation is shown in Fig. 3.

**Example 3** (2-Reducible). In this example, connected dynamical network with 8 nodes is considered. We choose a 2-reducible coupling matrix $D$ as follows:

$$D = \begin{bmatrix} -6 & 1 & 2 & 0 & 0 & 0 & 3 & 0 \\ 2 & -6 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & -3 \end{bmatrix} \tag{40}$$

![Irreducible symmetric coupling](image1)

**Fig. 2.** State error of the controlled dynamical system with irreducible symmetric coupling matrix in Example 1.
By adding the controllers (21) to the 4th and 6th nodes coupled system (the first node of subsystems 2 and 3) with 2-reducible coupling matrix, we can get the error dynamical system (39) with coupling matrix $\tilde{D}$ as shown below:

$$
\tilde{D} = \begin{bmatrix}
-6 & 1 & 2 & 0 & 0 & 0 & 3 & 0 \\
2 & -6 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -7 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -10 & 2 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & -3
\end{bmatrix}.
$$

Fig. 3. State errors of the controlled dynamical system with irreducible asymmetric coupling matrix in Example 2.

Fig. 4. State errors of the controlled dynamical system with 2-reducible coupling matrix (40).
In detail, one has \[ b_{D_1} = \frac{1}{C_0^6} \frac{1}{2} \] and \[ b_{D_2} = \frac{1}{C_0^7} \] Then according to Definition 6, we can get that: \[ a_{\text{max}} = \max \{ -0.6446, -0.9210, -0.9665 \} = -0.6446. \]

Hence, under the controllers (21), the connected dynamical system (39) with 2-reducible coupling matrix (40) can be globally exponentially synchronized with the tracking signal \( y(t) \). Fig. 4 shows the error states \( e_i(t) = x_i(t) - y(t) \) and the state variables \( x_i(t) \) for \( i = 1, 2, \ldots, 8 \), where the initial values of the coupled systems are randomly chosen from \((-1, 1)\). The numerical results verify our analytical results efficiently.

**Remark 11.** Only one controller is not valid for the tracking control of the entire dynamical network with coupling matrix (40) due to the existence of two self-governed subsystems; see Fig. 5. When only one controller is used for the third subsystem, Fig. 5 shows that tracking control is valid only for subsystem 3, but not for subsystems 1 and 2. This remark demonstrates that \( m \) is the minimum number of controllers for the tracking control of dynamical network with \( m \)-reducible coupling matrix.

### 6. Conclusion

The minimum number of controllers have been designed to force the coupled identical dynamical systems to track the desired state, which can be an equilibrium point, periodic orbit, or even chaotic attractor. Two different models are studied respectively. The desired objective behavior, which is available only for a few fraction of agents, can be produced consistently by all agents through local communication between the dynamic agents. The coupling matrix of the interconnected dynamical systems is not restricted to be symmetric and irreducible. The connected system with asymmetric or/and \( m \)-reducible coupling matrix has been investigated. We have also estimated the convergence rate of the coupled dynamical system to the desired tracking signal. Tracking control for the coupled non-identical dynamical systems is a very interesting and important issue, which will the motivation of our further research.
Acknowledgements

This work was jointly supported by the National Natural Science Foundation of China under Grant No. 6071130237, the Postdoctoral Sustentation Fund of Jiangsu Province under Grant No. 0901004B, the Natural Science Foundation of Shandong Province under Grant Nos. Y2008A32 and Y2007A17, and the International Cultivation Project for Young College Teachers of Shandong Province.

References