Convergence of the Multidimensional Minimum Variance Spectral Estimator for Continuous and Mixed Spectra

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***Abstract*—A proof of the pointwise convergence of the multidimensional minimum variance spectral estimator as the region of data support becomes infinite is given. It is shown that an octant is sufficient to ensure that the minimum variance spectral estimator will converge to the true power spectral density. The proof is valid for 1-D, multidimensional, continuous, and mixed spectra. Another useful result is that a normalized minimum variance spectral estimator can be defined to indicate sinusoidal power for processes with a mixed spectrum. Finally, upper and lower bounds on the continuous portion of the spectral estimate are given.**

***Index Terms*—Signal resolution, signal detection.**

I. Introduction

THE minimum variance spectral estimator (MVSE), originally proposed by Capon [5], has found widespread use in time series analysis and array processing [3], [7]. In the latter case it is sometimes referred to as the minimum variance distortionless response beamformer (MVDR) [7]. One of its important properties is that it can be used to estimate the power spectral density (PSD) in multiple dimensions and for arbitrarily spaced data samples [6]. Thus, it lends itself nicely to spatial processing and temporal processing or a combination of both. In practice, only a finite region of data support is available. Thus, the question of which region of support should be used naturally arises. In order to be able to estimate the PSD without error as the size of the selected region of support increases, it is necessary to establish convergence results. Many convergence theorems are available that address various types of spectra. For 1-D sinusoids in colored noise an elegant proof is given in [8], which establishes convergence at the sinusoidal frequencies for a normalized MVSE designed to locate these sinusoidal frequency locations. For the usual MVSE, however, no convergence results are given for the continuous part of the spectrum. Further results along these lines are presented in [9]. Finally, in [10] a convergence proof for the sinusoidal frequency locations for the multichannel mixed spectrum case is given for the normalized MVSE. Again no convergence results are given for the continuous part of the spectrum for the usual MVSE.

In this paper we give a general proof that is quite straightforward, relying only on the definition of the MVSE, the use of the Cauchy–Schwarz inequality, and widely known results in Fourier series theory. The proof covers the following cases.

1) A continuous PSD, either 1-D or multidimensional.
2) A mixed PSD consisting of a continuous PSD component as well as point masses (sinusoidal contributions), either 1-D or multidimensional.

From a practical viewpoint this covers all the cases of interest except for the multichannel one. We believe that this last case can also be proven in a similar way and hence would extend the theorem in [10].

A useful result of this paper is the answer to the question of an adequate region of data support to ensure convergenece. The required region of support is the causal one in one-dimension and a quarter plane (QP) region in two dimensions, i.e., this is the availability of the data which for one dimension is $x[n]$ for $0 \leq n \leq N-1$, for two dimensions it is $x[k,l]$ for $0 \leq k \leq K-1, 0 \leq l \leq L-1$, and so forth. Then as $N \to \infty$ in one dimension and as $K \to \infty, L \to \infty$ in two dimensions, and so forth, the MVSE will converge pointwise to the true PSD at the frequencies for which the PSD is continuous. More generally one requires an octant region of support in m dimensions.

II. Definition of the MVSE

For the sake of clarity consider the m-dimensional MVSE for $m = 2$ and for $K = L$, a region of support that we will denote as $N \times N$ (although this latter choice entails no loss of generality). The extension to arbitrary m is identical except for a more complicated symbolism.

To define the MVSE we first note that it can be viewed as a scaled version of the minimum variance of a linear unbiased estimator for the complex amplitude of a sinusoid [3]. In so doing, it is assumed that the complex sinusoid is added to wide sense stationary colored noise with PSD $P(f_1,f_2)$. The variance can be shown to be given in filtering terms as

$$J^{(N)}(f_{1_0},f_{2_0}) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f_1,f_2)|^2 P(f_1,f_2) df_1 df_2 \quad (1)$$

where $H(f_1,f_2)$ is the frequency response of a linear shift invariant filter with constrained value $H(f_{1_0},f_{2_0}) = 1$ and whose coefficient support is over a QP as [3]

$$H(f_1,f_2) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h[k,l]\exp[-j2\pi(f_1 k + f_2 l)]. \quad (2)$$

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The value of $J^{(N)}(f_{1_0}, f_{2_0})$ when it is minimized over all $H(f_1, f_2)$ subject to the constraint $H(f_{1_0}, f_{2_0}) = 1$ produces the minimum value of the variance of the unbiased estimator. Using this value the MVSE is defined as

$$\hat{P}^{(N)}_{\mathrm{MVSE}}(f_{1_0}, f_{2_0}) = N^2 J^{(N)}_{\min}(f_{1_0}, f_{2_0}). \tag{3}$$

Note that a scale factor of N^2 is used to convert the variance, i.e., power, into a power spectral *density* estimator.

To show that this is indeed the MVSE, although possibly expressed in unfamiliar terms, we reformulate the MVSE for one dimension. (For two dimensions the form of the MVSE in terms of matrices can be found in [3]). Thus, we have from (1)–(3) that

$$\begin{aligned}\hat{P}^{(N)}_{\mathrm{MVSE}}(f_0) &= N J^{(N)}_{\min}(f_0)\\ J^{(N)}(f_0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P(f) df\\ H(f) &= \sum_{k=0}^{N-1} h[k]\exp[-j2\pi f k]\end{aligned}$$

where $H(f_0) = 1$. Now let $\mathbf{h} = [h[0]\, h[1] \ldots h[N-1]]^T$, where T denotes transpose, and $\mathbf{e}(f_0) = [1\ \exp(j2\pi f_0) \ldots \exp(j2\pi f_0(N-1)]^T$. Then it can be shown that

$$J^{(N)}(f_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P(f) df = \mathbf{h}^H \mathbf{R} \mathbf{h}$$

where $\mathbf{R}$ is the $N \times N$ autocorrelation matrix and H denotes conjugate transpose. Hence, the MVSE is found by minimizing $J^{(N)}(f_0) = \mathbf{h}^H \mathbf{R} \mathbf{h}$ subject to the constraint that $\mathbf{h}^H \mathbf{e}(f_0) = 1$. The solution is well known to be

$$J^{(N)}_{\min}(f_0) = \frac{1}{\mathbf{e}^H(f_0)\mathbf{R}^{-1}\mathbf{e}(f_0)}$$

and when scaled by N produces the MVSE as

$$\hat{P}^{(N)}_{\mathrm{MVSE}}(f_0) = \frac{N}{\mathbf{e}^H(f_0)\mathbf{R}^{-1}\mathbf{e}(f_0)} \tag{4}$$

which is the usual expression [3].

III. Proof of Convergence

The main theorem asserts that as $N \to \infty$ in (4) the MVSE converges to the true PSD at those points at which the PSD is continuous. For the other points of a mixed spectrum, i.e., one that contains point masses, the use of (4) will produce infinity at the frequencies of the sinusoids. If one wishes to obtain the powers of the sinusoids at the sinusoidal frequencies, then the *normalized* version [3]

$$\hat{P}^{(N)'}_{\mathrm{MVSE}}(f_0) = \frac{1}{\mathbf{e}^H(f_0)\mathbf{R}^{-1}\mathbf{e}(f_0)} \tag{5}$$

will produce the power of the sinusoids at the sinusoidal frequency locations and zero for the frequencies at which the PSD is continuous as $N \to \infty$.

Theorem 3.1 (Pointwise Convergence of the Multidimensional MVSE): Let $P(f_1, f_2, \ldots, f_m)$ denote a multidimensional power spectral density (PSD) given by

$$\begin{aligned}P(f_1, f_2, \ldots, f_m) &= P_c(f_1, f_2, \ldots, f_m)\\ &\quad + \sum_{i=1}^{p} P_i \delta(f_1 - f_{1_i}, f_2 - f_{2_i}, \ldots, f_m - f_{m_i})\end{aligned} \tag{6}$$

where $P_c(f_1, f_2, \ldots, f_m)$ is the continuous part of the PSD and the remaining Dirac impulses represent the point masses at $(f_{1_i}, f_{2_i}, \ldots, f_{m_i})$. $P_c(f_1, f_2, \ldots, f_m)$ is assumed continuous and periodic with period one in each variable, i.e., continuous as a function on the m-torus (with the basic frequency m-cube defined as $-1/2 \le f_1 \le 1/2, -1/2 \le f_2 \le 1/2, \ldots, -1/2 \le f_m \le 1/2$). It is further assumed that $P_c(f_1, f_2, \ldots, f_m) \ge \alpha > 0$.

Defining the MVSE as

$$\hat{P}^{(N)}_{\mathrm{MVSE}}(f_1, f_2, \ldots, f_m) = N^m J^{(N)}_{\min}(f_1, f_2, \ldots, f_m)$$

and also defining the normalized MVSE as

$$\hat{P}^{(N)'}_{\mathrm{MVSE}}(f_1, f_2, \ldots, f_m) = J^{(N)}_{\min}(f_1, f_2, \ldots, f_m)$$

then

1) See the bottom of the page, and
2)

$$\lim_{N\to\infty} \hat{P}^{(N)'}_{\mathrm{MVSE}}(f_1, f_2, \ldots, f_m) = \begin{cases} P_i, & \text{if } (f_1, f_2, \ldots, f_m) = (f_{1_i}, f_{2_i}, \ldots, f_{m_i})\\ 0, & \text{otherwise}\end{cases}$$

Proof: The lower bound result used in the proof is based on the work in [1] where it is used for the 1-D case. For sake of simplicity in notation we consider only the 2-D case with the m-dimensional case requiring only a slight change in notation. We break the proof up into two parts, the first for frequencies at which the PSD is continuous and the second for the frequencies at which point masses reside.

First consider a given arbitrary frequency (f_{1_0}, f_{2_0}) at which the PSD is continuous and hence is equal to $P_c(f_{1_0}, f_{2_0})$. Let $H_{\mathrm{opt}}(f_1, f_2)$ be the frequency response that minimizes (1) subject to the constraint that $H(f_{1_0}, f_{2_0}) = 1$. Then we have

$$\begin{aligned}1^2 &= |H_{\mathrm{opt}}(f_{1_0}, f_{2_0})|^2\\ &= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_{\mathrm{opt}}[k,l]\exp[-j2\pi(f_{1_0}k + f_{2_0}l)]\right|^2\\ &= \left|\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} H_{\mathrm{opt}}(f_1, f_2)\right.\end{aligned}$$

$$\lim_{N\to\infty} \hat{P}^{(N)}_{\mathrm{MVSE}}(f_1, f_2, \ldots, f_m) = \begin{cases} \infty, & \text{if } (f_1, f_2, \ldots, f_m) = (f_{1_i}, f_{2_i}, \ldots, f_{m_i}), \text{for some } i\\ P_c(f_1, f_2, \ldots, f_m), & \text{otherwise}\end{cases}$$

$$\times \exp[j2\pi(f_1 k + f_2 l)] df_1 df_2 \times \exp[-j2\pi(f_{1_0} k + f_{2_0} l)] \Bigg|^2$$

$$= \Bigg| N \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} H_{\text{opt}}(f_1, f_2) \frac{1}{\sqrt{N}} \times \sum_{k=0}^{N-1} \exp[-j2\pi(f_{1_0} - f_1)k] \frac{1}{\sqrt{N}} \times \sum_{l=0}^{N-1} \exp[-j2\pi(f_{2_0} - f_2)l] df_1 df_2 \Bigg|^2$$

and defining

$$E^{(N)}(f) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp[-j2\pi f k]$$

this becomes

$$1^2 = \Bigg| N \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} H_{\text{opt}}(f_1, f_2) \sqrt{P_c(f_1, f_2)} E^{(N)} \times (f_{1_0} - f_1) E^{(N)}(f_{2_0} - f_2) \frac{1}{\sqrt{P_c(f_1, f_2)}} df_1 df_2 \Bigg|^2$$

$$\le N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| H_{\text{opt}}(f_1, f_2) \right|^2 P_c(f_1, f_2) df_1 df_2 \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\left| E^{(N)}(f_{1_0} - f_1) \right|^2 \left| E^{(N)}(f_{2_0} - f_2) \right|^2}{P_c(f_1, f_2)} df_1 df_2$$

$$\le N^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| H_{\text{opt}}(f_1, f_2) \right|^2 P(f_1, f_2) df_1 df_2 \times \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\left| E^{(N)}(f_{1_0} - f_1) \right|^2 \left| E^{(N)}(f_{2_0} - f_2) \right|^2}{P_c(f_1, f_2)} df_1 df_2$$

with the second to last step due to the Cauchy–Schwarz inequality and the last step due to the increased output of the filter due to the sinusoidal contributions. Now let $F_1^{(N)}(f) = |E^{(N)}(f)|^2$, which is just

$$F_1^{(N)}(f) = \frac{1}{N} \left(\frac{\sin(\pi f N)}{\sin(\pi f)} \right)^2$$

and is recognized as the 1-D Fejer kernel. As a result we have from above that

$$N^2 J_{\min}^{(N)}(f_{1_0}, f_{2_0}) \ge \frac{1}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F_1^{(N)}(f_{1_0} - f_1) F_1^{(N)}(f_{2_0} - f_2)}{P_c(f_1, f_2)} df_1 df_2}.$$

Note that the product of two 1-D Fejer kernels F_1 can be written as the 2-D Fejer kernel F_2 [2] so that we have

$$\hat{P}_{\text{MVSE}}^{(N)}(f_{1_0}, f_{2_0}) = N^2 J_{\min}^{(N)}(f_{1_0}, f_{2_0}) \ge \frac{1}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2)}{P_c(f_1, f_2)} df_1 df_2} \tag{7}$$

a lower bound on the MVSE.

Next, let $\tilde{h}[k,l] = \frac{1}{N^2} \exp[j2\pi(f_{1_0} k + f_{2_0} l)]$ and note that it satisfies the constraint since its Fourier transform is

$$\tilde{H}(f_1, f_2) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \frac{1}{N^2} \times \exp[j2\pi((f_{1_0} - f_1)k + (f_{2_0} - f_2)l)]$$

and therefore $\tilde{H}(f_{1_0}, f_{2_0}) = 1$. As a result, $\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |\tilde{H}(f_1, f_2)|^2 P(f_1, f_2) df_1 df_2$ can be no smaller than $J_{\min}^{(N)}(f_{1_0}, f_{2_0})$ and therefore

$$J_{\min}^{(N)}(f_{1_0}, f_{2_0}) \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\tilde{H}(f_1, f_2)|^2 P(f_1, f_2) df_1 df_2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Bigg| \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \frac{1}{N^2} \times \exp[j2\pi((f_{1_0} - f_1)k + (f_{2_0} - f_2)l)] \Bigg|^2 \times P(f_1, f_2) df_1 df_2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2)}{N^2} P(f_1, f_2) df_1 df_2.$$

Thus, we have that

$$\hat{P}_{\text{MVSE}}^{(N)}(f_{1_0}, f_{2_0}) = N^2 J_{\min}^{(N)}(f_{1_0}, f_{2_0}) \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2) P(f_1, f_2) df_1 df_2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2) P_c(f_1, f_2) df_1 df_2 + \sum_{i=1}^{p} P_i F_2^{(N)}(f_{1_0} - f_{1_i}, f_{2_0} - f_{2_i})$$

and finally, we have bounded the MVSE as

$$\frac{1}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2)}{P_c(f_1, f_2)} df_1 df_2} \le \hat{P}_{\text{MVSE}}^{(N)}(f_{1_0}, f_{2_0}) \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2) P_c(f_1, f_2) df_1 df_2 + \sum_{i=1}^{p} P_i F_2^{(N)}(f_{1_0} - f_{1_i}, f_{2_0} - f_{2_i}). \tag{8}$$

The multidimensional Fejer's theorem [2] states that for a continuous function $g(f_1, f_2)$ we have the limit

$$\lim_{N \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(f_1, f_2) F_2^{(N)}(\xi_1 - f_1, \xi_2 - f_2) df_1 df_2 = g(\xi_1, \xi_2).$$

Also, for $(f_1, f_2) \neq (0, 0)$, it is known that

$$\lim_{N \to \infty} F_2^{(N)}(f_1, f_2) = 0. \tag{9}$$

Now because $P_c(f_1, f_2)$ is continuous and bounded away from zero by assumption, $1/P_c(f_1, f_2)$ is also continuous, and therefore Fejer's theorem applies. As a result the limit of both sides of the inequality in (8) is $P_c(f_{1_0}, f_{2_0})$ and thus by the sandwich theorem [4]

$$\lim_{N\to\infty} N^2 J_{\min}^{(N)}(f_{1_0}, f_{2_0}) = P_c(f_{1_0}, f_{2_0}) = P(f_{1_0}, f_{2_0})$$

for all (f_{1_0}, f_{2_0}) for which $P(f_1, f_2)$ is continuous.

Next assume that (f_{1_0}, f_{2_0}) is at a point mass, say (f_{1_k}, f_{2_k}). Consider an arbitrary $H(f_1, f_2)$ such that $H(f_{1_0}, f_{2_0}) = 1$. Then,

$$\begin{aligned} &J^{(N)}(f_{1_0}, f_{2_0}) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f_1,f_2)|^2 P(f_1,f_2)df_1 df_2 \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f_1,f_2)|^2 P_c(f_1,f_2)df_1 df_2 \\ &\quad + \sum_{i=1}^{p} P_i |H(f_{1_i}, f_{2_i})|^2 \\ &\geq P_k |H(f_{1_k}, f_{2_k})|^2 = P_k \end{aligned}$$

Hence, $N^2 J^{(N)}(f_{1_0}, f_{2_0}) \geq N^2 P_k$ for all $H(f_1, f_2)$ and therefore

$$\hat{P}_{\mathrm{MVSE}}^{(N)}(f_{1_0}, f_{2_0}) = N^2 J_{\min}^{(N)}(f_{1_0}, f_{2_0}) \geq N^2 P_k$$

and clearly approaches infinity as $N \to \infty$. This completes the proof for the usual MVSE.

Next consider the normalized version of the MVSE used to locate the sinusoidal frequencies. From (8) for a frequency at which the PSD is continuous we have

$$\begin{aligned} &\hat{P}_{\mathrm{MVSE}}^{(N)\prime}(f_{1_0}, f_{2_0}) \\ &\leq \frac{1}{N^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_{1_0} - f_1, f_{2_0} - f_2) \\ &\quad \times P_c(f_1,f_2)df_1 df_2 \\ &\quad + \frac{1}{N^2}\sum_{i=1}^{p} P_i F_2^{(N)}(f_{1_0} - f_{1_i}, f_{2_0} - f_{2_i}). \end{aligned} \tag{10}$$

Since the integral converges by the previous results and using (9) we see that the right-hand-side converges to zero. On the other hand, for a frequency corresponding to a point mass, say (f_{1_k}, f_{2_k}), we have that

$$\begin{aligned} &\hat{P}_{\mathrm{MVSE}}^{(N)\prime}(f_{1_k}, f_{2_k}) \\ &= \min J^{(N)}(f_{1_k}, f_{2_k}) \\ &= \min \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f_1,f_2)|^2 P_c(f_1,f_2)df_1\, df_2 \\ &\quad + \sum_{i=1}^{p} P_i |H(f_{1_i}, f_{2_i})|^2 \geq P_k \end{aligned}$$

since $H(f_{1_k}, f_{2_k}) = 1$. Finally, from (10)

$$\begin{aligned} &\hat{P}_{\mathrm{MVSE}}^{(N)\prime}(f_{1_k}, f_{2_k}) \\ &\leq \frac{1}{N^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_{1_k} - f_1, f_{2_k} - f_2) \\ &\quad \times P_c(f_1,f_2)df_1 df_2 \\ &\quad + \frac{1}{N^2}\sum_{i=1, i\neq k}^{p} P_i F_2^{(N)}(f_{1_k} - f_{1_i}, f_{2_k} - f_{2_i}) \\ &\quad + P_k \underbrace{\frac{F_2^{(N)}(0,0)}{N^2}}_{=1}. \end{aligned}$$

As before the right-hand-side converges to zero except for the last term, which is P_k. This completes the proof. ■

IV. Upper and Lower Bounds on the MVSE

Note that from (8) upper and lower bounds for the MVSE for the frequencies at which the PSD is continuous can be written as

$$\begin{aligned} &\frac{1}{\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{F_2^{(N)}(f_1-u_1, f_2-u_2)}{P_c(u_1,u_2)} du_1 du_2} \\ &\quad \leq \hat{P}_{\mathrm{MVSE}}^{(N)}(f_1, f_2) \\ &\quad \leq \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_2^{(N)}(f_1-u_1, f_2-u_2) P_c(u_1,u_2) du_1 du_2 \end{aligned}$$

and noting that both bounds represent 2-D convolutions we have

$$\begin{aligned} \frac{1}{F_2^{(N)}(f_1,f_2) \star\star \frac{1}{P_c(f_1,f_2)}} &\leq \hat{P}_{\mathrm{MVSE}}^{(N)}(f_1, f_2) \\ &\leq F_2^{(N)}(f_1,f_2) \star\star P_c(f_1,f_2). \end{aligned} \tag{11}$$

The 2-D Fejer kernel is given by

$$F_2^{(N)}(f_1,f_2) = \frac{1}{N^2}\left(\frac{\sin(\pi f_1 N)}{\sin(\pi f_1)}\right)^2 \left(\frac{\sin(\pi f_2 N)}{\sin(\pi f_2)}\right)^2$$

and clearly as $N \to \infty$ produces a 2-D Dirac delta function necessary for convergence. However, for finite N, bounds can easily be obtained using this expression. It is clear that from (11) the bounds will be tighter when the PSD has a small dynamic range. In the extreme but uninteresting case of $P_c(f_1, f_2) = c_1$, where c_1 is a constant, it is seen that the bounds are the same and are both equal to c_1.

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