Rapid Estimation of the Range-Doppler Scattering Function

Steven M. Kay, University of Rhode Island
Department of Electrical and Computer Engineering,
Kingston, RI 02881
(e-mail:kay@ele.uri.edu)

S. Bradford Doyle, Naval Undersea Warfare Center
Division Newport, RI 02841
(e-mail sbd@workmail.com)

Abstract-Under wide sense stationary uncorrelated scattering (WSSUS) conditions, the signal spreading due to a random channel may be described by the scattering function (SF). In an active acoustic system, the received signal is modeled as the superposition of delayed and Doppler spread replicas of the transmitted waveform. The SF completely describes the second-order statistics of a WSSUS channel and can be considered a density function that characterizes the average spread in delay and Doppler experienced by an input signal as it passes through the channel.

The SF and its measurement will be reviewed. An estimator is proposed based on a two-dimensional autoregressive (AR) model for the scattering function. In order to implement this estimator we derive the minimum mean square error estimator of the time-varying frequency response of a linear channel. Unlike conventional Fourier methods the AR approach does not suffer from the usual convolutional smoothing due to the signal ambiguity function. Simulation results are given.

I. INTRODUCTION

Transmission channels which spread the transmitted signal in both time and frequency are commonly modeled as random, time-varying, space-varying linear filters. Temporal spread is usually associated with multipath effects and frequency spread with scatterer motion. Under wide sense stationary and uncorrelated scattering (WSSUS) conditions the scattering function completely describes the second-order statistics of the channel. It can be viewed as a density function giving the average power modulation as a function of delay and Doppler. The SF is typically defined as being independent of the transmitted signal. However, for the underwater channel especially, this should be considered true only for signals of similar bandwidth and center frequency.

II. PROBLEM FORMULATION

A. Definitions

We first summarize the mathematical models that give rise to the scattering function estimation problem. The channel is modeled as a stochastic linear time-varying system with impulse response $h(t,\tau)$ where $h(t,\tau)$ describes the response of the system at time *t* to an impulse applied τ seconds prior [1][2]. Therefore, if the input to the channel is a signal *s(t)*, then the output, *x(t)* can be written as

$$x(t)=\int_{-\infty}^{\infty} h(t,\tau)s(t-\tau)d\tau . \tag{1}$$

This work was supported by Office of Naval Research (ONR) contract N000140010871, and by a Naval Undersea Warfare Center In-House Laboratory Independent Research Grant under ONR Program Element #0601152N.

It is assumed that the output is the complex envelope and therefore both *s(t)* and $h(t,\tau)$ are complex. We also assume that $h(t,\tau)$ is zero mean for all *t* and τ, WSS in *t* and, uncorrelated in τ. This embodies the usual WSSUS condition [3]. Taking the Fourier transform of $h(t,\tau)$ with respect to *t* yields the spreading function

$$S(\phi,\tau)=\int_{-\infty}^{\infty} h(t,\tau)\exp(-j2\pi\phi t)dt \tag{2}$$

which determines the amount of spread in delay τ and frequency ϕ that an input signal undergoes in passing through a time-varying linear channel [1]. Solving for $h(t,\tau)$ and substituting into (1) yields

$$x(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\phi,\tau)s(t-\tau)\exp(j2\pi\phi t)d\tau d\phi. \tag{3}$$

We see that *x(t)* is now represented as the sum of delayed and Doppler shifted replicas of the transmitted signal. Hence the name spreading function for $S(\phi,\tau)$ is appropriate.

Because $h(t,\tau)$ is WSS in *t* and uncorrelated in τ it can be shown that $S(\phi,\tau)$ is uncorrelated in both ϕ and τ so that, denoting the expectation operator by $E(\bullet)$,

$$E\left(S^*(\phi,\tau)S(\phi',\tau')\right)=E\left(|S(\phi,\tau)|^2\right)\delta(\phi'-\phi)\delta(\tau'-\tau) \tag{4}$$

where the power, $E\left(|S(\phi,\tau)|^2\right)$, for a given Doppler ϕ and delay τ is defined as the scattering function

$$P(\phi,\tau)=E\left(|S(\phi,\tau)|^2\right). \tag{5}$$

By noting that the Fourier transform of $h(t,\tau)$ with respect to τ yields the time-varying frequency response (TVFR) $H(t, f)$, which is WSS in both *t* and *f*, we can define the channel ACF as

$$R_H(u,v)=E\left[H^*(t,f)H(t+u,f+v)\right]. \tag{6}$$

Finally, it can be shown that the scattering function is related to the channel ACF by the two-dimensional Fourier transform

$$P(\phi,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(u,v) e^{-j2\pi(\phi u - \tau v)} du dv \qquad (7)$$

or equivalently, $P(\phi,\tau)$ is the power spectral density of $H(t, f)$. The channel ACF, $R_H(u,v)$, is also referred to as the two-frequency correlation function [4].

B. The Estimation Problem

To estimate the scattering function it is necessary to either explicitly or implicitly estimate the channel ACF due to the relationship in (7). This is difficult since the random process $H(t, f)$ is only observed via the frequency domain equivalent of (1)

$$x(t) = \int_{-\infty}^{\infty} H(t, f) S(f) \exp(j2\pi f t) df \qquad (8)$$

where

$$H(t, f) = \int_{-\infty}^{\infty} h(t,\tau) \exp(-j2\pi f\tau) d\tau$$

and $S(f)$ is the Fourier transform of the signal $s(t)$ (and not the spreading function - the number of arguments easily distinguish the two). If the signal is bandlimited and f_{M-1} is the Nyquist frequency or higher, then (8) can be expressed in discrete form as

$$x(t_n) \approx \Delta_f \sum_{m=0}^{M-1} H(t_n, f_m) S(f_m) \exp(j2\pi f_m t_n) \qquad (9)$$

for $n=0,1,\ldots,N-1$. There are MN unknown samples of $H(t,f)$ but only N observed data samples so that a consistent solution cannot be found. This is known as an overspread channel [5].

The theoretical relationship between the correlation function and the transmitted and received signals can be expressed in terms of the time-frequency (T-F) autocorrelation functions of the signal and the received time series, $A_S(u,v)$ and $A_X(u,v)$ [6]

$$R_H(u,v) = \frac{E[A_x(u,v)]}{A_s(u,v)} \qquad (10)$$

where the T-F ACF is defined as

$$\begin{aligned} A_s(u,v) &= \int_{-\infty}^{\infty} s^*(t - \tfrac{u}{2}) s(t + \tfrac{u}{2}) e^{-j2\pi v t} dt \\ &= \int_{-\infty}^{\infty} S^*(f - \tfrac{v}{2}) S(t + \tfrac{v}{2}) e^{j2\pi u f} df. \end{aligned} \qquad (11)$$

We note that the signal ambiguity function, $\theta_S(u,v)$, is the magnitude squared of the T-F ACF

$$\theta_s(u,v) = |A_s(u,v)|^2.$$

Using (10), one might be inclined to use the unbiased estimate

$$\hat{R}_H(u,v) = A_x(u,v) / A_s(u,v) \qquad (12)$$

as was done in [6]. However, the correlation estimate becomes infinite if $A_s(u,v) = 0$. This places severe restrictions on signal design for realizable signals. In [7] we show that the minimum mean square error (MMSE) solution for $H(t,f)$, is

$$\hat{H}(t_n, f_m) = \frac{S^*(f_m) \exp(-j2\pi f_m t_n) x(t_n)}{\Delta_f \sum_{k=0}^{M-1} |S(f_k)|^2} \qquad (13)$$

for n=0, 1, …, N-1, m=0, 1, …, M-1. Using (13) it can be shown that the correlation function estimate becomes [7]

$$\hat{R}_H(u,v) = \frac{A_x(u,v) A_s^*(u,v)}{(A_s(0,0))^2} \qquad (14)$$

which is finite for all signals. Substituting the continuous time-frequency version of (13) into (8) yields an identity after integration proving that this is a valid solution to the estimation problem. In fact, it can also be shown that (13) is the solution of minimum norm. We note that the MMSE estimate of the time-varying frequency response (13) is deterministic in the frequency direction (dependent only on the transmitted signal) and random in the time direction. The solution of (13) will be used throughout the rest of the paper.

The AR Approach

We propose a parametric approach to scattering function estimation based on autoregressive spectral modeling. Since only a few parameters must be estimated for the AR approach, it often can function well when Fourier based methods cannot. Since simulations and any practical implementation must be done on a digital computer, a discussion of sampling requirements and assumptions is appropriate. As a result of sampling, the scattering function can only be estimated over the Nyquist band. Thus, we make the very practical assumptions that the multipath (delay) spread is less than L seconds and the Doppler spread is less than B Hz. With these assumptions the scattering function will be estimated over the band

$$0 \le \tau \le L$$
$$-B/2 \le \phi \le B/2$$

To prevent aliasing in the SF, $R_H(u,v)$, must be sampled on a grid where $\Delta_u \le 1/B$ and $\Delta_v \le 1/L$. We assume that $\Delta_u = 1/B$ and $\Delta_v = 1/L$.

The scattering function is now written using the sampled form of (7)

$$P(\phi,\tau) = \frac{1}{BL}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_H\left(\frac{k}{B},\frac{l}{L}\right) e^{\left(-j2\pi\left(\frac{\phi k}{B}-\frac{\tau l}{L}\right)\right)}. \quad (15)$$

If we ignore the scale factor *1/BL* and normalize the Doppler and delay by letting $f_1 = \phi/B$ and $f_2 = \tau/L$ and $r'[k,l] = R_H\left(\frac{k}{B},\frac{l}{L}\right)$ this becomes

$$P(f_1,f_2) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} r'[k,l]\exp(-j2\pi(f_1 k - f_2 l))$$
$$-1/2 < f_1 < 1/2,\ 0 < f_2 < 1$$

which is the usual definition of the power spectral density (PSD) except for the sign change ($-f_2$). To use standard AR estimation techniques [8] we must account for this sign change. Letting $r[k,l] = r'[k,-l]$, (15) becomes

$$P(f_1,f_2) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} r[k,l]\exp(-j2\pi(f_1 k + f_2 l))$$

which is the usual definition of the discrete-time PSD. Therefore, the usual methods of 2-dimensional AR spectral estimation may be applied to find the AR parameters σ_u^2 and $a[k,l]$. The spectral estimator for an AR[p_1,p_2] quarter plane (QP) model is given by [8]

$$P(f_1,f_2) = \frac{\sigma_u^2}{\left|\sum_{k=0}^{p_1}\sum_{l=0}^{p_2} a[k,l]\exp(-j2\pi(f_1 k + f_2 l))\right|^2}. \quad (16)$$

The examples that will be shown in this paper use either the 2-D autocorrelation method (ACM) or the 2-D covariance method (CM) as defined in [8].

The Autocorrelation Method (ACM)

The ACM requires an estimate of the ACF, samples of which are plugged into the Yule-Walker equations used to estimate the AR parameters [8]. The 2-D Yule-Walker equations are

$$\begin{bmatrix} \mathbf{R}[0] & \mathbf{R}[-1] & \cdots & \mathbf{R}[-p_1] \\ \mathbf{R}[1] & \mathbf{R}[0] & \cdots & \mathbf{R}[-(p_1-1)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}[p_1] & \mathbf{R}[p_1-1] & \cdots & \mathbf{R}[0] \end{bmatrix} \begin{bmatrix} \mathbf{a}[0] \\ \mathbf{a}[1] \\ \vdots \\ \mathbf{a}[p_1] \end{bmatrix} = \begin{bmatrix} \sigma_u^2 \mathbf{e}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

where

$$\mathbf{a}[i] = [a[i,0] \quad a[i,1] \quad \cdots \quad a[i,p_2]]^T$$
$$\mathbf{e}_1 = [1 \quad 0 \quad \cdots \quad 0]^T \quad (p_2+1)\times 1$$

and

$$\mathbf{R}[i] = \begin{bmatrix} r[i,0] & r[i,-1] & \cdots & r[i,-p_2] \\ r[i,1] & r[i,0] & \cdots & r[i,-(p_2-1)] \\ \vdots & \vdots & \ddots & \vdots \\ r[i,p_2] & r[i,p_2-1] & \cdots & r[i,0] \end{bmatrix}. \quad (17)$$

To estimate the AR parameters, we therefore need to calculate the autocorrelation function only at the lags shown in these equations using $r[i,j] = r'[k,-l] = R_H\left(\frac{k}{B},\frac{-l}{L}\right)$. Assuming ergodicity, we estimate the elements of the autocorrelation function defined in (6) using

$$\hat{r}[i,j] = \frac{1}{MN}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\hat{H}^*(t_n,f_m)\hat{H}\left(t_n+\frac{k}{B}, f_m+\frac{-l}{L}\right) \quad (18)$$

The Covariance Method (CM)

The covariance method, on the other hand, requires an estimate of $H(t,f)$ and not its ACF. In [7] we show that the CM equations are

$$\begin{bmatrix} \mathbf{C}[0,0] & \mathbf{C}[0,1] & \cdots & \mathbf{C}[0,p_1] \\ \mathbf{C}[1,0] & \mathbf{C}[1,1] & \cdots & \mathbf{C}[1,p_1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}[p_1,0] & \mathbf{C}[p_1,1] & \cdots & \mathbf{C}[p_1,p_1] \end{bmatrix} \begin{bmatrix} \mathbf{a}[0] \\ \mathbf{a}[1] \\ \vdots \\ \mathbf{a}[p_1] \end{bmatrix} = \begin{bmatrix} \sigma_u^2 \mathbf{e}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad (19)$$

where

$$\mathbf{C}[k,i] = \begin{bmatrix} \mathbf{C}_{HH}^T[i,:,k,0] \\ \mathbf{C}_{HH}^T[i,:,k,1] \\ \vdots \\ \mathbf{C}_{HH}^T[i,:,k,p_2] \end{bmatrix}$$

and the colon denotes the entire range of the index, i.e.

$$\mathbf{C}_{HH}^{T}[0,:,0,0]=$$
$$\left[C_{HH}[0,0,0,0] \quad C_{HH}[0,1,0,0] \quad \cdots \quad C_{HH}[0,p_2,0,0]\right]$$

and each element is calculated using

$$\mathbf{C}_{HH}[i,j,k,l]=\sum_{m=p_1}^{M-1}\sum_{n=0}^{N-1-p_2}H(m-i,n-j)H^*[m-k,n+l].$$

SIMULATION RESULTS

In 2-dimensional AR spectral estimation, all causal AR models are based on a region of support that is either the nonsymmetric half plane (NSHP) or the quarter plane (QP). In general only the NSHP will yield the correct PSD if the region of support is infinite. However, it has been observed from simulations that, for sinusoidal signals in noise, spectral estimators based on the NSHP perform poorly, possibly because the required model order is too high [8]. All of the results presented herein utilize a 2D quarter plane (QP) autoregressive (AR) model. Estimates using the ACM and CM are compared. A comparison of results using the NSHP and QP is beyond the scope of this paper and is an area of future work.

To demonstrate the validity of this approach the results of a number of simulations are presented. We will assume all data is sampled in delay and Doppler at intervals of Δ_τ and Δ_ϕ, respectively. In the simulation we define a known scattering function, P, with maximum time spread L and maximum Doppler spread B. We also define a known transmit waveform with time support T. The samples of a realization of the spreading function are zero mean complex Gaussian variables with variance $P(k\Delta_\phi,l\Delta_\tau)$ or $S(k\Delta_\phi,l\Delta_\tau)=z_{kl}\sqrt{P(k\Delta_\phi,l\Delta_\tau)}$ where $z_{kl}\sim CN(0,1)$. The received signal is calculated using a discrete version of (3),

$$x(\Delta_t)=\sum_{k=-/(2\Delta_\phi)}^{B/(2\Delta_\phi)}\sum_{l=0}^{L/\Delta_\tau}S(k\Delta_\phi,l\Delta_\tau)s(n\Delta_t-l\Delta_\tau)e^{j2\pi k\Delta_\phi l\Delta_\tau}$$

for $0\le n\le (T+L)/\Delta_\tau$. Note that in this calculation samples of the transmit waveform are needed over the range [-L, T+L]. If a transmitted signal is given over an interval from 0 to T we zero-pad outside the interval. For a known analytical expression such as a Gaussian pulse, the signal is calculated over the entire range.

The first example is for a known AR(1,1) SF (defined by (16)) with time spread L=4 s and Doppler spread support B=4 Hz. which is interrogated by a Gaussian probe pulse of duration T=0.357 s. The bandwidth of a Gaussian pulse is $W=1/(T\sqrt{2})$ which is 1.98 Hz in this case. The scattering function is characterized by the AR coefficients a[0,0]=1.0000, a[0,1]=0.1854-0.5706j, a[1,0]=0.7000, a[1,1]=-0.1298+0.3994j. Fig. 1 shows examples of the envelopes of the transmitted and received signals for this case. Note that the transmitted signal used for the analysis has time support over the range [-L,T+L] and the received signal has time support only over the range [0,L]. All contour plots are shown on identical axes and contours are given in dB. Fig. 2 shows the known SF and the single ping estimates for various estimators. The Fourier estimate is formed by calculating the 2-D periodogram of the MMSE estimate of the TVFR or (13). This is followed by AR(1,1) estimates using both the CM and the ACM estimators. Clearly the AR estimators give higher resolution, more accurate estimates of the scattering function for this simple case.

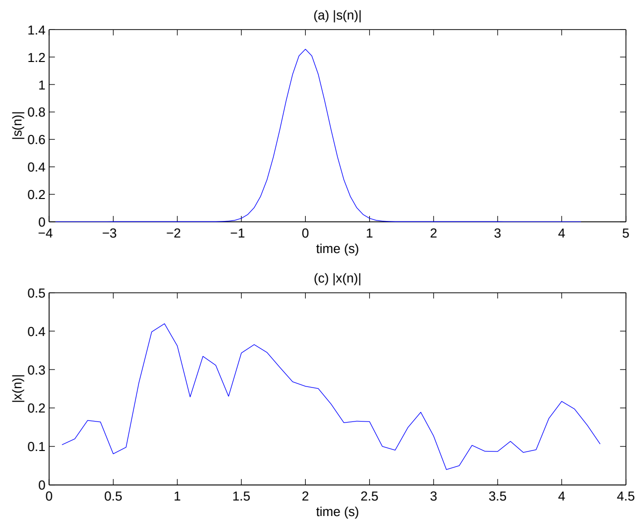

Fig. 1. (a) Magnitude of transmitted Gaussian envelope. (b) Magnitude of received signal envelope.

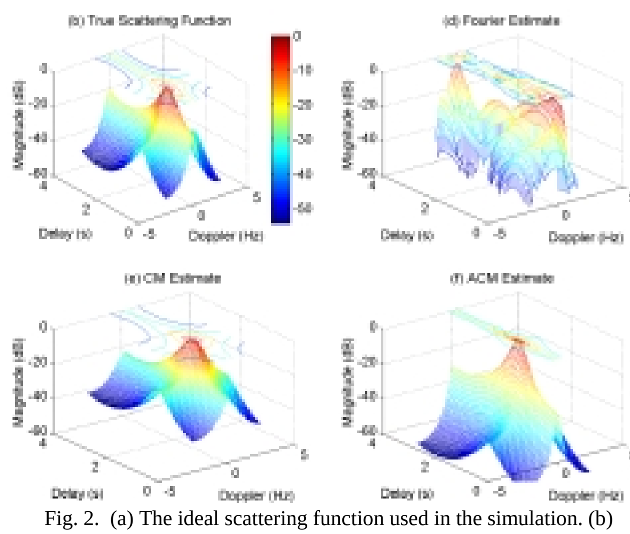

Fig. 2. (a) The ideal scattering function used in the simulation. (b) The 1-ping Fourier estimate. (c) The 1-ping AR(1,1) CM estimate. (d) The 1-ping AR(1,1) ACM estimate.

Scatter plots of AR parameter locations for 50 realizations of the two AR(1,1) estimators are shown in Fig. 3. Solid lines on the graph are drawn from the actual model locations to the average of the 50 realizations. In almost all cases the average location of each parameter estimate is biased toward the origin. The one notable exception is for the *a*[1,1] coefficient using the ACM. The exact cause of this bias is a matter of future investigation. It is also notable that in this case the ACM estimates of the *a*[1,0] coefficient have significantly less scatter than the CM estimates. The average scattering function estimates for these 50 single ping realizations are shown in Fig. 4.

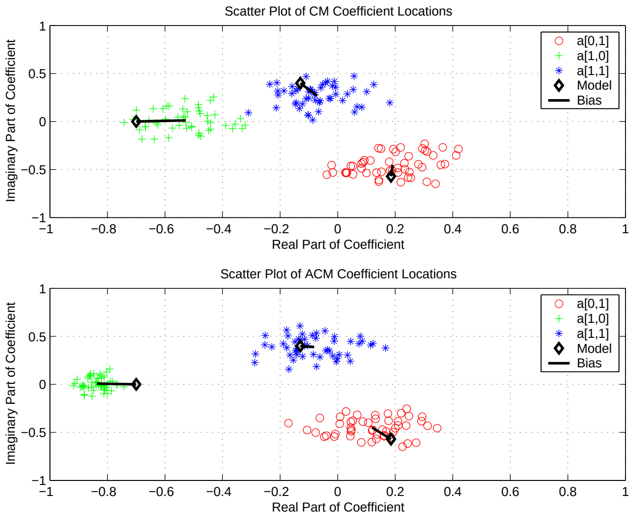

Fig. 3. Scatter plot for 50 realizations of 1-ping AR coefficient locations using both the ACM and CM estimators.

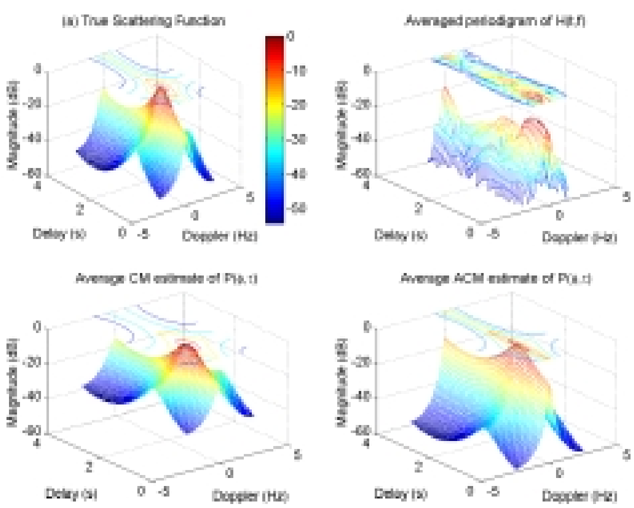

Fig. 4. (a) The true scattering function. (b) The average of 50 Fourier estimates. (c) The average of 50 AR(1,1) CM estimates. (d) The average of 50 AR(1,1) ACM estimates.

Although we wish to estimate the scattering function with a single ping, the use of multiple pings will improve the accuracy of the estimates if the channel can be considered stationary over the time spanned by the multiple pings. Fig. 5 shows a similar scatter plot for a case where three pings are used to form the estimate. Here, the MMSE estimate of the TVFR is calculated and the corresponding correlation functions ((18) into (17) for ACM or (13) into (19) for CM) for AR estimation is formed for each ping. The correlation functions are then averaged before finally calculating the AR parameters. We see that the variance of the estimates is significantly reduced although the bias remains. Although it is beyond the scope of this paper, this indicates that multiping and/or recursive estimation schemes may provide robust estimates in environments where some stability may be assumed from ping to ping.

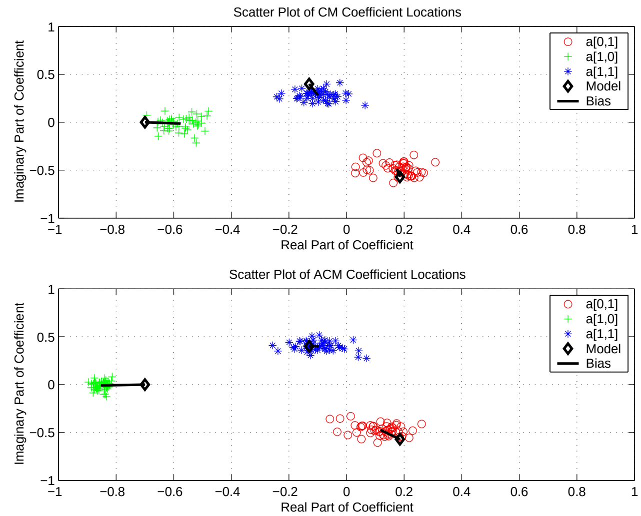

Fig. 5. Scatter plots of 50 AR[1,1] coefficient locations for both CM and ACM estimators. 3 pings are averaged to form each estimate.

Conclusions

A novel method of scattering function estimation based on autoregressive spectral modeling has been proposed. The current implementation of this method uses the MMSE estimate of the TVFR given a known input waveform and the received data. Preliminary simulation results exhibit promise of obtaining high-resolution estimates of the scattering function from a single ping. The results also indicate that the correlation method may be slightly more accurate on average than the autocorrelation method. However, no claims of optimality can be made regarding the current estimator. Attempts by these authors to calculate the maximum likelihood estimate using the EM algorithm have failed due to the extreme computational and storage requirements of the algorithm. Continuing research is focused on improving this technique using optimal estimators and waveforms and the use of quarter plane versus nonsymmetric half plane estimators.

REFERENCES

[1] L. J. Ziomek, *Approach to underwater acoustic detection and signal design*, Ph.D. thesis, Penn. State University, University Park Applied Research Lab, Oct. 1981.

[2] L. J. Ziomek, *Underwater acoustics: A linear systems theory approach.* New York: Academic, 1985.

[3] P. A. Bello, "Characterization of randomly time-variant linear channels," *IEEE Trans. On Comm. Theory*, vol. CS-11, pp. 360-393, Dec. 1963.

[4] H. L. Van Trees, *Detection, estimation, and modulation theory, Part III: Radar-sonar signal processing and Gaussians signals in noise.* New York: John Wiley & Sons, Inc, 1971

[5] P. A. Bello, “Measurement of random time-variant linear channels,” *IEEE Trans. On Info. Theory*, vol. IT-14, pp. 684-693, Sept. 1968.
[6] S. M. Kay, “Estimation of the range-Doppler scattering function based on autoregressive modeling,” tech. Rep., Signal Processing Systems, Middletown, RI 02840, June 1993
[7] S. M. Kay & S. B. Doyle, “Rapid estimation of the range-Doppler scattering function,” unpublished.
[8] S. M. Kay, *Modern spectral estimation: Theory & application.* Englewood Cliffs, NJ: Prentice Hall, 1988.