UNBIASED ESTIMATION OF THE PHASE OF A SINUSOID

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ABSTRACT

Estimation of the phase of a sinusoid is an important problem in signal processing. The usual maximum likelihood estimator is biased and so can produce poor results, especially at low signal-to-noise ratios and/or short data records. It is proven that no unbiased estimator exists; based on the proof, a means of obtaining estimators with less bias than the maximum likelihood estimator are proposed.

1. INTRODUCTION

The phase of a sinusoid embedded in white Gaussian noise is a well studied problem. It is usually solved by employing the maximum likelihood estimator (MLE) [1]. Unfortunately, the MLE is biased and so can lead to poor results at low signal-to-noise ratios and/or short data records. Problems such as cycle skipping in phase-locked loops [2] are directly attributable to this bias. Other practical problems of interest, in addition to communications, are in frequency estimation via fast methods [3] and bearing estimation in line arrays [4]. All of these encounter difficulties due to a "phase wraparound", which is equivalently characterized as an estimation bias error. In this paper we investigate whether an unbiased phase estimator exists and if not, the extent to which a "nearly unbiased" estimator can be implemented.

A desirable property of a statistical point estimator of a parameter θ is that on average it yields the true value of the parameter. Specifically, given a random sample $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$, where $p(\mathbf{x};\theta), \theta \in \Theta$ is the probability density function (pdf) of $\mathbf{x}$, then the statistical estimator $\delta(\mathbf{x})$ of the parameter θ is said to be unbiased if the expected value of $\delta(\mathbf{x})$ produces θ; namely

$$E\,\delta(\mathbf{x}) = \int \delta(\mathbf{x}) p(\mathbf{x};\theta) dx = \theta, \quad \theta \in \Theta. \tag{1}$$

A number of well known techniques are available to solve integral equations in the form of (1), such as differentiating or using Fourier, Laplace or Mellin integral transforms [5]. An alternate approach used here is to determine the eigenvalues and eigenfunctions of the integral operator and solve (1) via a series solution [6].

2. PROBLEM STATEMENT

We consider the estimation of the phase θ of a sinusoid embedded in noise or

$$x[n] = A\cos[\omega_o n + \theta] + w[n],\ n = 0, 1, \ldots, N-1$$

where $w[n]$ is white Gaussian noise with unknown variance σ^2; furthermore, the amplitude A is considered unknown while the frequency ω_o is assumed to be known. All the information regarding the parameter θ is summarized by the jointly sufficient statistics given by [1]:

$$\begin{aligned} T_1 &= \sum_{n=0}^{N-1} x[n]\cos[\omega_o n], \\ T_2 &= \sum_{n=0}^{N-1} x[n]\sin[\omega_o n]. \end{aligned}$$

Consequently, all inference can be based on the joint *pdf* of $\mathbf{T} = (T_1, T_2)$ and is given by:

$$\begin{aligned} p(\mathbf{T};\theta) &= \frac{h(\theta)}{2\pi\sigma^2|\det\Sigma|^{\frac{1}{2}}} \exp\left(\frac{A}{\sigma^2}\left(T_1\cos[\theta] - T_2\sin[\theta]\right)\right) \\ &\times \exp\left(-\frac{1}{2\sigma^2}\mathbf{T}^T\Sigma^{-1}\mathbf{T}\right) \end{aligned}$$

where

$$h(\theta) = \exp\left(-\frac{A^2}{2\sigma^2}\sum_{n=0}^{N-1}\cos^2[\omega_o n + \theta]\right),$$

and

$$\Sigma = \begin{bmatrix} \sum_{n=0}^{N-1} \cos^2[n\omega_o] & \sum_{n=0}^{N-1} \cos[n\omega_o]\sin[n\omega_o] \\ \sum_{n=0}^{N-1} \cos[n\omega_o]\sin[n\omega_o] & \sum_{n=0}^{N-1} \sin^2[n\omega_o] \end{bmatrix}$$

is the covariance matrix.

We seek an unbiased estimator $\delta(\mathbf{T})$ of θ; namely

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\mathbf{T})p(\mathbf{T};\theta)d\mathbf{T} = \theta, \quad -\pi \le \theta \le \pi. \quad (2)$$

For mathematical simplicity, it is assumed that $\omega_o = \frac{2\pi k}{N}$; consequently, $\Sigma = \frac{N}{2}\mathbf{I}$, $|\det\Sigma|^{\frac{1}{2}} = \frac{N}{2}$ and
and therefore

$$\begin{aligned} p(\mathbf{T};\theta) &= \frac{h(\theta)}{\pi\sigma^2 N}\exp\left(\frac{A}{\sigma^2}\left(T_1\cos[\theta] - T_2\sin[\theta]\right)\right) \\ &\times \quad \exp\left(-\frac{1}{N\sigma^2}(T_1^2+T_2^2)\right), \end{aligned} \quad (3)$$

with

$$h(\theta) = \exp\left(-\frac{A^2N}{4\sigma^2}\right).$$

Transforming (3) to polar coordinates via $T_1 = \rho\cos[\phi]$ and $T_2 = -\rho\sin[\phi]$, where the latter minus sign has been introduced in order to subsequently obtain a symmetric kernel, yields

$$p(\rho,\phi;\theta) = \frac{h(\theta)\rho}{\pi\sigma^2 N}\exp\left(\frac{A\rho}{\sigma^2}\cos(\theta-\phi) - \frac{\rho^2}{N\sigma^2}\right). \quad (4)$$

Furthermore, it is assumed that the estimator is scale invariant; namely, the same estimate is obtained if $\mathbf{T}$ is multiplied by any constant $c > 0$. Consequently, $\delta(\rho,\phi)$ will depend only on ϕ. Under this invariance assumption, the polar form of (2) is given by the following integral equation:

$$\int_{-\pi}^{\pi}\delta(\phi)K(\theta-\phi)d\phi = \theta, \quad -\pi \le \theta \le \pi \quad (5)$$

where

$$K(\theta-\phi) = \frac{h(\theta)}{N\pi\sigma^2}\int_0^{\infty}\rho\exp\left(\frac{A\rho}{\sigma^2}\cos(\theta-\phi) - \frac{\rho^2}{N\sigma^2}\right)d\rho$$

is a 2π periodic, symmetric kernel. Furthermore using [8, 3.462.5] the kernel can be expressed in terms of $\Phi(x)$, the standard error function [8, 8.250.1]:

$$K(\theta-\phi) = \frac{h(\theta)}{\pi}\left(\frac{1}{2} + \frac{\sqrt{\pi}\,b}{4}\exp\left(\frac{b^2}{4}\right)\left(1+\Phi\left(\frac{b}{2}\right)\right)\right)$$

where

$$b = \frac{A\sqrt{N}}{\sigma}\cos(\theta-\phi).$$

3. INTEGRAL EQUATIONS

Integral equations are equations in which an unknown function appears under the integral sign. The relevant form considered here is the *Fredholm equation of the first kind* [7] with a 2π periodic, symmetric kernel $K(x,t)$; namely,

$$f(x) = \int_{-\pi}^{\pi} K(x-t)g(t)dt, \quad K(x,t) = K(t,x) \quad (6)$$

where $f(x)$ is given and $g(x)$ is an unknown function. A basic result [6] is that given a continuous real and symmetric kernel and a continuous $f(x)$, then a solution to (6) exists only if the given function $f(x)$ can be expanded in series whose basis functions are orthonormal eigenfunctions, $\Psi_k(x)$ of the kernel $K(x-t)$; namely,

$$f(x) = \sum_{k=-\infty}^{\infty} f_k\Psi_k(x) \quad (7)$$

with coefficients

$$f_k = \int_{-\pi}^{\pi} f(t)\Psi_k(t)dt\,.$$

Next, expanding the unknown function $g(x)$ in a Fourier series in terms of $\Psi_k(x)$; namely,

$$g(x) = \sum_{k=-\infty}^{\infty} g_k\Psi_k(x)$$

and substituting into (6) we have that

$$\begin{aligned} \int_{-\pi}^{\pi} K(x-t)g(t)dt &= \int_{-\pi}^{\pi} K(x-t)\sum_{k=-\infty}^{\infty} g_k\Psi_k(t)dt \\ &= \sum_{k=-\infty}^{\infty} g_k\int_{-\pi}^{\pi} K(x-t)\Psi_k(t)dt \\ &= \sum_{k=-\infty}^{\infty} g_k\lambda_k\Psi_k(x) \end{aligned} \quad (8)$$

where

$$\lambda_k = \int_{-\pi}^{\pi} K(x-t)\Psi_k(t)dt.$$

Examples of the kernel $K(t)$ for values of $\beta = \frac{A^2N}{\sigma^2}$ are shown in Figure 1. Equating (7) with (8), it can be seen that the solution to (6) can be expressed as a series in terms of the Fourier series coefficients of $f(x)$ and the eigenvalues and eigenfunctions of the kernel [6]; namely

$$g(x) = \sum_{k=-\infty}^{\infty} \frac{f_k}{\lambda_k}\Psi_k(x) \tag{9}$$

provided

$$\sum_{k=-\infty}^{\infty} \left|\frac{f_k}{\lambda_k}\right|^2 < \infty.$$

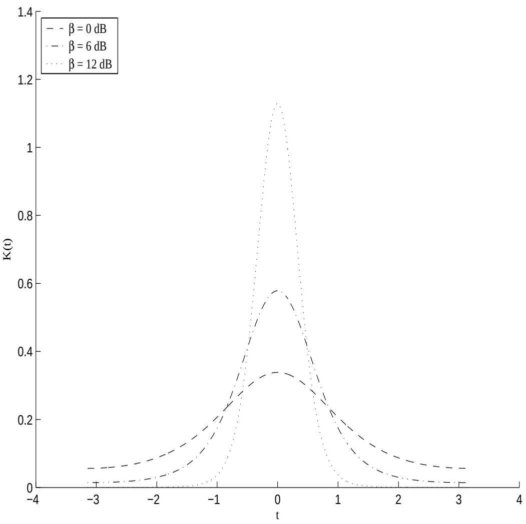

Fig. 1. Integral equation kernel.

We note that the kernel is an analytic and periodic function of θ; consequently, this kernel can generate only analytic and periodic functions of θ, regardless of $\delta(\phi)$. However, the desired right hand side (RHS) of (5); namely θ, is analytic but not periodic. Conversely, if RHS is periodically extended, that extension is not analytic. Consequently, no exact solution exists. This is rigorously proven by demonstrating that a series solution does not exist.

It is easily shown [7] that the eigenfunctions of $K(\theta-\phi)$ are complex exponentials, $\exp(\imath k\phi)$ and consequently the eigenvalues are

$$\lambda_k \;=\; \int_{-\pi}^{\pi} K(t)\exp(-\imath kt)dt\,.$$

Furthermore, the complex Fourier series representation of θ is

$$\theta = \sum_{k=-\infty}^{\infty} \theta_k \exp(\imath k\theta)$$

where the complex Fourier coefficients θ_k satisfy the relationship

$$\begin{aligned}\theta_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t\exp(-\imath kt)dt \\ &= \begin{cases}\frac{\imath(-1)^k}{k} & k \neq 0 \\ 0 & k = 0\,.\end{cases}\end{aligned}$$

Therefore from (9), the series expression for the unbiased estimate of phase is given by

$$\delta(\phi) = \sum_{k=-\infty}^{\infty} \frac{\theta_k}{\lambda_k}\exp(\imath k\phi) \tag{10}$$

provided $\sum_{k=-\infty}^{\infty}\left|\frac{\theta_k}{\lambda_k}\right|^2 < \infty$.

The Fourier coefficients θ_k explicitly decay as $1/k$. In general, the rate of decay of Fourier coefficients is related to the smoothness of the function [9]. In particular, the derivatives of the kernel with respect to θ exist for all orders and it is easily shown that

$$\begin{aligned}|\lambda_k| &= \frac{1}{k^m}\left|\int_{-\pi}^{\pi}\frac{d^mK(t)}{dt^m}\exp(\imath kt)dt\right| \\ &\leq \frac{1}{k^m}\int_{-\pi}^{\pi}\left|\frac{d^mK(t)}{dt^m}\right|\exp(\imath kt)dt\,.\end{aligned}$$

This implies that λ_k decays faster than $1/k$; consequently

$$\sum_{k=-\infty}^{\infty}\left|\frac{f_k}{\lambda_k}\right|^2 \to \infty$$

and therefore no unbiased estimate of phase $\delta(\phi)$ exists.

4. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimate of sinusoidal phase is given by [1]

$$\hat{\theta} = -\arctan\left(\frac{T_2}{T_1}\right). \tag{11}$$

Applying the expectation operator we obtain

$$
\begin{aligned}
E\hat{\theta} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\theta} p(T_1, T_2; \theta) dT_1 dT_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\arctan\left(\frac{T_2}{T_1}\right) p(T_1, T_2; \theta) dT_1 dT_2\,.
\end{aligned}
$$

Transforming this expression to polar coordinates via $T_1 = \rho\cos[\phi]$ and $T_2 = -\rho\sin(\phi)$ yields

$$
\begin{aligned}
E\hat{\theta} &= \int_{-\pi}^{\pi}\int_{0}^{\infty} \phi p(\rho, \phi; \theta) d\rho\, d\phi \\
&= \int_{-\pi}^{\pi} \phi K(\theta - \phi) d\phi\,. \qquad (12)
\end{aligned}
$$

Expanding ϕ in terms of the eigenfunctions of $K(\theta - \phi)$; namely, a Fourier series and substituting into (12):

$$
\begin{aligned}
E\hat{\theta} &= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \phi_k \exp(\imath k\phi) K(\theta - \phi) d\phi \\
&= \sum_{k=-\infty}^{\infty} \phi_k \int_{-\pi}^{\pi} \exp(\imath k\phi) K(\theta - \phi) d\phi \\
&= \sum_{k=-\infty}^{\infty} \theta_k \lambda_k \exp(\imath k\theta) \qquad (13)
\end{aligned}
$$

where we have used the fact that $\phi_k = \theta_k$.

It is seen that the expected value of the MLE depends on the reciprocal of the eigenvalues of $\delta(\phi)$ and is the source of the MLE's bias. Examples of the average MLE, illustrating this bias, are shown in Figure 2. Future work will consider estimators of the form

$$
\Delta(\phi) = \sum_{k=-\infty}^{\infty} \frac{w_k \theta_k}{\lambda_k} \exp(\imath k\phi),
$$

where the weights w_k are the Fourier coefficients of a window function w. The weights are chosen such that the estimator obtained from the convolution of w with θ has less bias than the MLE, at the possible expense of variance.

5. CONCLUSIONS

In this paper we have shown that an unbiased estimate of the phase of a sinsusoid embedded in white Gaussian noise does not exist. The functional form used to obtain this result was compared to the average MLE and reveals that the bias of the MLE is induced by the eigenvalues of a smoothing kernel.

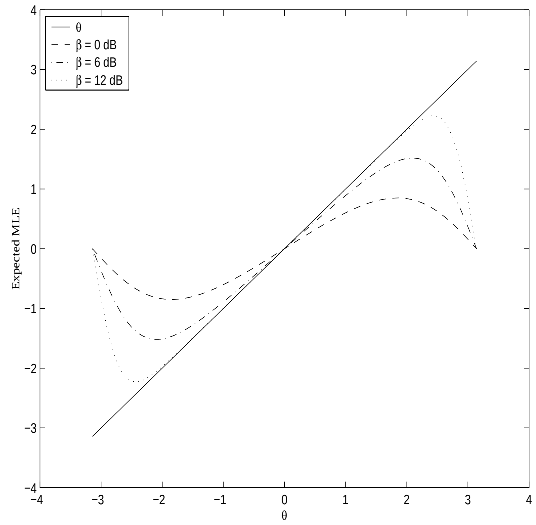

Fig. 2. Expected maximum likelihood estimate.

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