Rapid Estimation of the Range-Doppler Scattering Function

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***Abstract*—Under wide sense stationary uncorrelated scattering (WSSUS) conditions, the signal spreading due to a random channel may be described by the scattering function (SF). In an active acoustic system, the received signal is modeled as the superposition of delayed and Doppler spread replicas of the transmitted waveform. The SF completely describes the second-order statistics of a WSSUS channel and can be considered a density function that characterizes the average spread in delay and Doppler experienced by an input signal as it passes through the channel.**

The SF and its measurement will be reviewed. An estimator is proposed based on a two-dimensional (2-D) autoregressive (AR) model for the scattering function. In order to implement this estimator, we derive the conditional minimum variance unbiased estimator of the time-varying frequency response of a linear channel. Unlike conventional Fourier methods, the AR approach does not suffer from the usual convolutional smoothing due to the signal ambiguity function. Simulation results are given.

***Index Terms*—Acoustic propagation, autoregressive processes, estimation, range-Doppler, scatter channels, scattering function, scattering parameters measurement.**

I. Introduction

TRANSMISSION channels that spread the transmitted signal in both time and frequency are commonly modeled as random, time-varying, space-varying linear filters. Temporal spread is usually associated with multipath effects and frequency spread with scatterer motion. Under wide sense stationary and uncorrelated scattering (WSSUS) conditions, the scattering function completely describes the second-order statistics of the channel. It can be viewed as a density function giving the average power modulation as a function of delay and Doppler [1]. The scattering function (SF) is typically defined as being independent of the transmitted signal. However, for the underwater channel especially, this should be considered true only for signals of similar bandwidth and center frequency.

One of the common methods of identifying the channel scattering function is by deconvolving the transmitted and received signals [1]. Iterative methods for deconvolution of two-dimensional (2-D) signals or images are discussed in [1]–[3]. A direct measurement approach using specially designed waveform pairs and twin receivers is given in [4] and [5]. More recently, Hahm *et al.* reviewed the effects of simultaneous range and Doppler spreading on the results of various iterative deconvolution methods and discussed some resulting signal design issues [6].

Another method of scattering function identification is given by Jourdain in [7]. Jourdain transmits a large time-bandwidth signal and calculates the scattering function from the interambiguity between the transmitted and received signals. He then determines optimum receivers for a binary communications scheme based on the form of the scattering function. The use of pseudo-inversion of the linear system for scattering function estimation is discussed in [8], and in [9], the equivalence of the pseudo-inversion deconvolution via singular value decomposition (SVD) and the spectral division methods of deconvolution is shown.

Signal design for scattering function estimation is also a much discussed topic [10]. Persons not only considers optimization of probe signals but also calculates a lower bound for the mean square error of an unbiased estimate of the sampled scattering function [11]. The problem of designing signals for use in communications or detection systems in randomly time varying channels is addressed in [12].

The identification of the one–dimensional (1-D) range scattering function using AR models is addressed in [13] and [14] while a method for detecting multiple targets in AR reverberation is given in [15]. The use of a 2-D autoregressive noise model is used in [16] to model the temporally and spatially varying noise field seen by a hydrophone array. Low-order 1-D AR models are shown to provide good descriptions of forward-scattered signals reflected from the sea surface [17].

The need for accurate characterization of the channel scattering function is clear. Given a known scattering function, receiver and transmitter parameters may be optimized for detection [18], [19]. Recently, channel scattering functions have been used in radio communications for modeling the wideband HF (WBHF) channel impulse response over a high-latitude auroral path [20] and for the HF skywave channel [21]. Methods have been developed to predict performance of a WBHF communications system derived from the channel scattering function [22].

In this paper, we present a new method for the estimation of the channel scattering function based on AR modeling techniques. Preliminary results using this method were presented in [23]–[25]. In Section II, we model the channel as a stochastic linear time-varying system which is wide sense stationary (WSS) with uncorrelated scattering. We then define the spreading and scattering functions based on this model and finally present pictographs of the various Fourier relationships

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between various channel and channel correlation function representations. The difficulty of estimating the SF is discussed in terms of estimation of the channel transfer function or its autocorrelation function (ACF). We then review the AR model including a simple adaptation to fit the SF estimation problem into an AR spectral estimation framework. The problem is then defined in terms of the linear model [26] allowing the use of simplified matrix notation.

Estimation of the channel ACF is addressed in Section III. The conditional minimum variance unbiased (MVU) estimator and its properties are reviewed in the context of estimation of the channel's time-varying frequency response in this underdetermined problem. Using this result, the channel (ACF) is estimated and reduced to a simple ratio of the time–frequency autocorrelation functions of the transmitted and received signals.

The capabilities and limitations of the method are demonstrated through simulation in Section IV. The results of the paper are summarized in Section V.

II. Problem Formulation

A. Definitions

We first summarize the mathematical models that give rise to the scattering function estimation problem. The channel is modeled as a stochastic linear time-varying system with impulse response $h(t,\tau)$, where $h(t,\tau)$ describes the response of the system at time t to an impulse applied τ seconds prior [27], [28]. Therefore, if the input to the channel is a signal $s(t)$, then the output $x(t)$ can be written as

$$x(t) = \int_{-\infty}^{\infty} h(t,\tau)\, s(t-\tau)\, d\tau. \tag{1}$$

It is assumed that the output is the complex envelope and therefore both $s(t)$ and $h(t,\tau)$ are complex. We also assume that $h(t,\tau)$ is zero mean for all t and τ, WSS in t and, uncorrelated in τ. This embodies the usual WSSUS condition [29]. Taking the Fourier transform of $h(t,\tau)$ with respect to t yields the spreading function

$$S(\phi,\tau) = \int_{-\infty}^{\infty} h(t,\tau) \exp(-j2\pi\phi t)\, dt \tag{2}$$

which determines the amount of spread in delay τ and Doppler ϕ that an input signal undergoes in passing through a time-varying linear channel [27]. Solving for $h(t,\tau)$ and substituting into (1) yields

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\phi,\tau)\, s(t-\tau) \exp(j2\pi\phi t)\, d\tau d\phi. \tag{3}$$

We see that $x(t)$ is now represented as the weighted sum of delayed and Doppler shifted replicas of the transmitted signal. Hence, the name spreading function for $S(\phi,\tau)$ is appropriate.

Because $h(t,\tau)$ is WSS in t and uncorrelated in τ it can be shown that $S(\phi,\tau)$ is uncorrelated in both ϕ and τ so that, denoting the expectation operator by $E()$

$$E(S^*(\phi,\tau)\, S(\phi',\tau')) = E\left(|S(\phi,\tau)|^2\right) \delta(\phi'-\phi)\, \delta(\tau'-\tau) \tag{4}$$

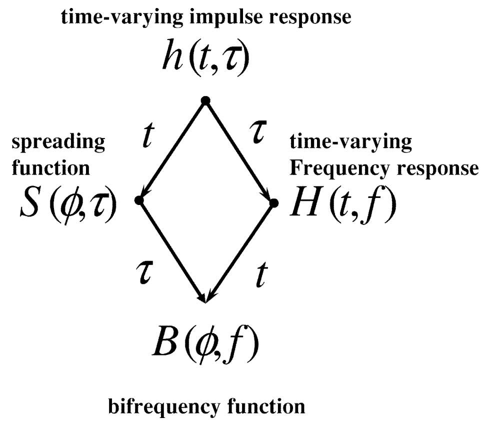

Fig. 1. Fourier transform relationships between channel function representations.

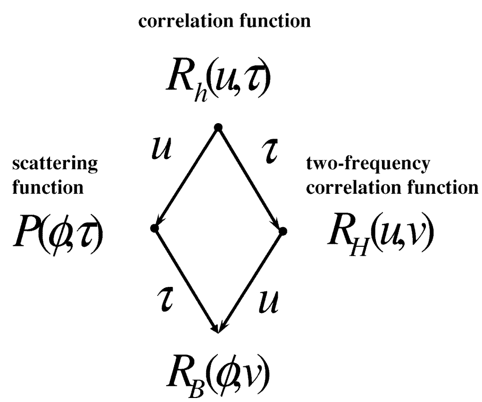

Fig. 2. Fourier transform relationships between channel correlation function representations.

where the power $E\left(|S(\phi,\tau)|^2\right)$ for a given Doppler ϕ and delay τ is defined as the scattering function

$$P(\phi,\tau) = E\left(|S(\phi,\tau)|^2\right). \tag{5}$$

By noting that the Fourier transform of $h(t,\tau)$ with respect to τ yields the time-varying frequency response (TVFR) $H(t,f)$, which is WSS in both t and f, we can define the channel ACF as

$$R_H(u,v) = E\left[H^*(t,f)\, H(t+u,f+v)\right]. \tag{6}$$

Finally, it can be shown that the scattering function is related to the channel ACF by the 2-D Fourier transform

$$P(\phi,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(u,v) \exp(-j2\pi(\phi u - \tau v))\, du dv \tag{7}$$

or equivalently, $P(\phi,\tau)$ is the power spectral density of $H(t,f)$. The channel ACF $R_H(u,v)$ is also referred to as the two-frequency correlation function [30].

Fig. 1 summarizes the Fourier relationships between the various channel model representations. Fig. 2 does the same for the corresponding correlation functions [27], [28]. The bifrequency function and its correlation function are shown for completeness but will not be discussed further.

B. Estimation Problem

To estimate the scattering function, it is necessary to either explicitly or implicitly estimate the channel ACF due to the relationship in (7). This is difficult since the random process $H(t,f)$ is only observed via the frequency domain equivalent of (1)

$$x(t) = \int_{-\infty}^{\infty} H(t,f)\, S(f) \exp(j2\pi f t) df \tag{8}$$

where

$$H(t,f) = \int_{-\infty}^{\infty} h(t,\tau) \exp(-j2\pi f \tau)\, d\tau$$

and $S(f)$ is the Fourier transform of the signal $s(t)$ (and not the spreading function; the number of arguments easily distinguish the two). Since the signal is assumed to be the complex envelope and hence is bandlimited and since f_{M-1} is the Nyquist frequency, then (8) can be expressed in discrete form as

$$x(t_n) \approx \Delta_f \sum_{m=0}^{M-1} H(t_n, f_m)\, S(f_m) \exp(j2\pi f_m t_n) \tag{9}$$

where $H(t_n, f_m) = H(t,f)\,|_{t=t_n, f=f_m}$ for $n = 0, 1, \ldots, N-1$, and $m = 0, 1, \ldots, M-1$ and Δ_f is the frequency sampling interval. The complex envelope $S(f)$ is assumed to be defined over the $(f_0 f_{M-1})$ frequency interval. There are MN unknown samples of $H(t,f)$ but only N observed data samples so that a consistent solution cannot be found. This is known as an overspread channel [29].

To get around the consistency problem, one might use multiple returns. For each return we could transmit a sinusoid at a given frequency, say f_i. Then, each return could be used to obtain $H(t, f_i)$. For enough densely spaced frequencies, we could use this information to estimate $H(t,f)$. However, if the channel changes from ping to ping this may lead to erroneous results. Alternatively, we could transmit a broadband signal, say a sum of sinusoids and then separate the various responses using a bank of narrowband filters whose bandwidths are matched to the bandwidth of $H(t, f_i)$ or the Doppler spread of the channel. This approach and others based on it are frequently used [31]–[33]. This is equivalent to a Fourier based spectral estimator and suffers from leakage between the narrowband filters leading to a loss in resolution. Once an estimate of $H(t,f)$ is found, the ACF and/or the scattering function may be directly estimated.

Alternatively, the channel ACF may be estimated directly from the data. The theoretical relationship between the correlation function and the transmitted and received signals can be expressed in terms of the time-frequency (T-F) autocorrelation functions of the signal and the received time series, $A_s(u,v)$ and $A_x(u,v)$ [34]

$$R_H(u,v) = \frac{E[A_x(u,v)]}{A_s(u,v)} \tag{10}$$

where the T-F ACF is defined as

$$\begin{aligned} A_s(u,v) &= \int_{-\infty}^{\infty} s^*\left(t - \frac{u}{2}\right) s\left(t + \frac{u}{2}\right) \exp(-j2\pi v t)\, dt \\ &= \int_{-\infty}^{\infty} S^*\left(f - \frac{v}{2}\right) S\left(f + \frac{v}{2}\right) \exp(j2\pi u f)\, df. \end{aligned} \tag{11}$$

We note that the T-F ACF is the Fourier transform of the Wigner distribution [35]–[37]. The signal ambiguity function $\theta_s(u,v)$ is the magnitude squared of the T-F ACF

$$\theta_s(u,v) = |A_s(u,v)|^2 .$$

Using (10), one might be inclined to use the unbiased estimate

$$\widehat{R}_H(u,v) = \frac{A_x(u,v)}{A_s(u,v)} \tag{12}$$

as was done in [34]. However, the correlation estimate becomes infinite if $A_s(u,v) = 0$. This places severe restrictions on signal design for realizable signals. In Section III-B, we will show that the conditional MVU solution for $H(t_n, f_m)$ yields the correlation function estimate

$$\widehat{R}_H(u,v) = \frac{A_x(u,v)}{A_s(0,0)} \frac{A_s^*(u,v)}{A_s^*(0,0)} \tag{13}$$

which is finite for all signals.

Kailath showed [38] that the instantaneous TVFR was unambiguously measureable only if $BL \le 1$, where B is the extent of the frequency spreading of the channel, and L is the extent of the time spreading of the channel. This is the so-called underspread channel. The $BL \le 1$ criterion was later shown to be overly restrictive in [39]. It is important to note that Kailath's restrictions are based on the measurement of each instantaneous value of $H(t,f)$ using a train of impulses or a similar sampling scheme. He further showed that when we are not interested in the instantaneous values of the filter but only the statistical averages much less information is required and the $BL \le 1$ can be relaxed. For instance, he showed that the channel autocorrelation function can be unambiguously determined if $BL \le 2$ and that the average TVFR can be determined without regard to BL if ensemble averages are used. In the following section, we will invoke the autoregressive model for the scattering function to allow us to estimate the SF for the overspread channel or for $BL > 1$. Just as in the case of spectral estimation, the parameterization of the problem will allow estimation of the scattering function without the need for ensemble averaging. Because the problem is parameterized, and we assume a small number of coefficients are required to describe the scattering function, the solution should be tractable, as long as BL is not too large. We will assume without proving it that the BL requirement can be further relaxed to BL finite.

C. AR Approach

We propose a parametric approach to scattering function estimation based on autoregressive spectral modeling [40]. Since only a few parameters must be estimated for the AR approach, it often can function well when Fourier-based methods do not. Since simulations and any practical implementations must be done on a digital computer, a discussion of sampling requirements and assumptions is appropriate. As a result of sampling, the scattering function can only be estimated over the Nyquist band. Thus, we make the very practical assumptions that the multipath (delay) spread is less than L s, and the Doppler spread is less than B Hz. With these assumptions the scattering function will be estimated over the band

$$\begin{aligned} 0 \leq &\tau \leq L \\ \frac{-B}{2} \leq &\phi \leq \frac{B}{2}. \end{aligned}$$

To prevent aliasing, the ACF $R_H(u,v)$ must be sampled on a grid, where $\Delta u \leq 1/B$, and $\Delta v \leq 1/L$. We assume that $\Delta u = 1/B$ and that $\Delta v = 1/L$.

The scattering function is now written using the sampled form of (7) as

$$P(\phi,\tau) = \frac{1}{BL} \times \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_H\left(\frac{k}{B},\frac{l}{L}\right)\exp\left(-j2\pi\left(\frac{\phi k}{B}-\frac{\tau l}{L}\right)\right).$$

If we ignore the scale factor $1/BL$ and normalize the Doppler and delay by letting $f_1 = \phi/B$, $f_2 = \tau/L$, and $r'[k,l] = R_H(k/B, l/L)$, this becomes

$$P(f_1,f_2) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} r'[k,l]\exp(-j2\pi(f_1 k - f_2 l)) \tag{14}$$

$$-\frac{1}{2} < f_1 \leq \frac{1}{2}, 0 < f_2 \leq 1 \tag{15}$$

which is the usual definition of the power spectral density (PSD), except for the sign change ($-f_2$). To use standard AR estimation techniques [40], we must account for this sign change. Letting $r[k,l] = r'[k,-l]$, (14) becomes

$$P(f_1,f_2) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} r[k,l]\exp(-j2\pi(f_1 k + f_2 l))$$

which is the usual definition of the discrete-time PSD. Therefore, the usual methods of 2-D AR spectral estimation may be applied to find the AR parameters σ_u^2 and $a[k,l]$.

The spectral estimator for an $AR(p_1,p_2)$ quarter plane (QP) model is given by [40]

$$P(f_1,f_2) = \frac{\sigma_u^2}{\left|\sum_{k=0}^{p_1}\sum_{l=0}^{p_2} a[k,l]\exp(-j2\pi(f_1 k + f_2 l))\right|^2} \tag{16}$$

where p_1 and p_2 are the AR model orders. We note that the 2-D AR PSD is completely determined by $(p_1+1)(p_2+1)$ parameters. The examples that will be shown in this paper use either the 2-D autocorrelation method (ACM) or the 2-D covariance method (CM), as defined in [40]. These methods are appropriate when the AR model order is known. Recursive methods are available when one [41] or both [42] model orders are unknown.

1) Autocorrelation Method (ACM): The ACM requires an estimate of the ACF, samples of which are used in the Yule–Walker equations to estimate the AR parameters [40]. The 2-D Yule–Walker equations are

$$\begin{bmatrix} \mathbf{R}[0] & \mathbf{R}[-1] & \cdots & \mathbf{R}[-p_1] \\ \mathbf{R}[1] & \mathbf{R}[0] & \cdots & \mathbf{R}[-(p_1-1)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}[p_1] & \mathbf{R}[p_1-1] & \cdots & \mathbf{R}[0] \end{bmatrix} \begin{bmatrix} \mathbf{a}[0] \\ \mathbf{a}[1] \\ \vdots \\ \mathbf{a}[p_1] \end{bmatrix} = \begin{bmatrix} \sigma_u^2 \mathbf{e}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} \mathbf{a}[i] &= [\,a[i,0] \quad a[i,1] \quad \cdots \quad a[i,p_2]\,]^T \\ \mathbf{e}_1 &= [1 \quad 0 \quad \cdots \quad 0]^T\,(p_2+1)\times 1 \end{aligned}$$

and

$$\mathbf{R}[i] = \begin{bmatrix} r[i,0] & r[i,-1] & \cdots & r[i,-p_2] \\ r[i,1] & r[i,0] & \cdots & r[i,-(p_2-1)] \\ \vdots & \vdots & \ddots & \vdots \\ r[i,p_2] & r[i,p_2-1] & \cdots & r[i,0] \end{bmatrix}.$$

To estimate the AR parameters, we therefore need to calculate the autocorrelation function only at the lags shown in these equations using $r[k,l] = r'[k,-l] = R_H(k/B, -l/L)$.

2) Covariance Method (CM): The covariance method, on the other hand, requires an estimate of $H(t,f)$ and not its ACF. The standard formulation of the CM finds the AR parameters from (18), shown at the bottom of the page, for

$$\begin{aligned} k &= 0,1,\ldots,p_1 \\ l &= 0,1,\ldots,p_2 \\ [k,l] &\neq [0,0] \end{aligned}$$

and where $\hat{a}[0,0] = 1$ and the estimator for the white noise variance is the case $[k,l] = [0,0]$ or

$$\hat{\sigma}^2 = \frac{1}{(M-p_1)(N-p_2)} \times \sum_{i=0}^{p_1}\sum_{j=0}^{p_2}\hat{a}[i,j]\sum_{m=p_1}^{M-1}\sum_{n=p_2}^{N-1} H(m-i,n-j)\,H^*(m,n). \tag{19}$$

[40]. Letting

$$C_{HH}[i,j,k,l] = \frac{1}{(M-p_1)(N-p_2)} \times \sum_{m=p_1}^{M-1}\sum_{n=p_2}^{N-1} H(m-i,n-j)\,H^*(m-k,n-l)$$

(18) and (19) can be written in matrix form, as shown in the equation at the bottom of next the page, where

$$\frac{1}{(M-p_1)(N-p_2)}\sum_{i=0}^{p_1}\sum_{j=0}^{p_2}\hat{a}[i,j]\sum_{m=p_1}^{M-1}\sum_{n=p_2}^{N-1} H(m-i,n-j)\,H^*(m-k,n-l) = 0 \tag{18}$$

$\hat{\mathbf{b}} = \begin{bmatrix} \hat{\sigma^2} & 0 & \cdots & 0 \end{bmatrix}^T$. For convenience, we have used the colon to signify all elements of range as is done in Matlab®, i.e.,

$$\mathbf{C}_{HH}^T[0,:,0,0] = \\ [\,C_{HH}[0,0,0,0] \quad C_{HH}[0,1,0,0] \quad \cdots \quad C_{HH}[0,p_2,0,0]\,].$$

We see that each column of each submatrix, for example

$$\begin{bmatrix} \mathbf{C}_{HH}^T[0,:,0,0] \\ \mathbf{C}_{HH}^T[0,:,0,1] \\ \vdots \\ \mathbf{C}_{HH}^T[0,:,0,p_2] \end{bmatrix}$$

is a $(p_2+1)\times(p_2+1)$ block, which is Hermitian and positive semidefinite. Now, letting

$$\mathbf{C}_{HH}[k,i] = \begin{bmatrix} \mathbf{C}_{HH}^T[i,:,k,0] \\ \mathbf{C}_{HH}^T[i,:,k,1] \\ \vdots \\ \mathbf{C}_{HH}^T[i,:,k,p_2] \end{bmatrix}$$

we can write

$$\begin{bmatrix} \mathbf{C}_{HH}[0,0] & \mathbf{C}_{HH}[0,1] & \cdots & \mathbf{C}_{HH}[0,p_1] \\ \mathbf{C}_{HH}[1,0] & \mathbf{C}_{HH}[1,1] & \cdots & \mathbf{C}_{HH}[1,p_1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{HH}[p_1,0] & \mathbf{C}_{HH}[p_1,1] & \cdots & \mathbf{C}_{HH}[p_1,p_1] \end{bmatrix} \begin{bmatrix} \hat{\mathbf{a}}[0] \\ \hat{\mathbf{a}}[1] \\ \vdots \\ \hat{\mathbf{a}}[p_1] \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \quad (20)$$

This covariance matrix is Hermitian and positive semidefinite. We note that each column of the matrix has constant unconjugated lags, which is consistent with the notation used in [40] for the 1-D case.

Again, we make an adjustment in sign convention due to the fact that we are estimating a scattering function and not a true 2-D spectrum. It is simple to show that the resulting equation is exactly (20), except that

$$C_{HH}[i,j,k,l] = \frac{1}{(M-p_1)(N-p_2)} \times \sum_{m=p_1}^{M-1} \sum_{n=0}^{N-1-p_2} H(m-i,n+j)\,H^*(m-k,n+l).$$

Because both a direct estimate of $H(t,f)$ and an estimate of the ACF are required for the different methods of AR model estimation, both will be derived. We will first formulate the problem in terms of the linear model and then address the task of estimating $H(t,f)$. Once this is done, an estimate of the ACF will be found and given in terms of the received and transmitted signals as well as in terms of their time–frequency ACFs. Finally, simulation results using both methods will be presented.

D. Linear Channel Model

Starting with the generic frequency domain representation of a received signal $x(t)$, which is the result of a signal $s(t)$ propagating through a channel characterized by the time-varying frequency response $H(t,f)$, as given in (9), we formulate the problem in matrix notation with the hope of being able to solve for the time-varying frequency response given the received signal and the transmitted signal. We first let $\mathbf{A}$ be the $M\times N$ matrix

$$[\mathbf{A}]_{mn} = \Delta_f S(f_m)\exp(j2\pi f_m t_n) \quad (21)$$

so that (9) becomes

$$x(t_n) \approx \sum_{m=0}^{M-1} [\mathbf{A}]_{mn} H(t_n,f_m).$$

Now, let $\mathbf{x} = [\,x(t_0) \quad x(t_1) \quad \cdots \quad x(t_{N-1})\,]^T$, where

$$x(t_0) = \sum_{m=0}^{M-1} [\mathbf{A}]_{m0} H(t_0,f_m)$$
$$x(t_1) = \sum_{m=0}^{M-1} [\mathbf{A}]_{m1} H(t_1,f_m)$$
$$\vdots$$
$$x(t_{N-1}) = \sum_{m=0}^{M-1} [\mathbf{A}]_{m,N-1} H(t_{N-1},f_m).$$

As previously mentioned, there are MN values of $H(t,f)$ mapping into only N values of $x(t)$. Therefore, the problem of determining $H(t_n,f_m)$ from $\mathbf{x}$ is underdetermined. We now define the $M\times N$ matrix $\mathbf{H}$ as $[\mathbf{H}]_{ij} = H(t_j,f_i)$ and write the formula for $\mathbf{x}$ in matrix form, noting that the summation at time t_i is simply the inner product of the ith column of $\mathbf{A}$ with the

$$\left[\begin{array}{c|c|c|c} \mathbf{C}_{HH}^T[0,:,0,0] & \mathbf{C}_{HH}^T[1,:,0,0] & \cdots & \mathbf{C}_{HH}^T[p_1,:,0,0] \\ \mathbf{C}_{HH}^T[0,:,0,1] & \mathbf{C}_{HH}^T[1,:,0,1] & \cdots & \mathbf{C}_{HH}^T[p_1,:,0,1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{HH}^T[0,:,0,p_2] & \mathbf{C}_{HH}^T[1,:,0,p_2] & \cdots & \mathbf{C}_{HH}^T[p_1,:,0,p_2] \\ \hdashline \multicolumn{4}{c}{\vdots} \\ \hdashline \mathbf{C}_{HH}^T[0,:,p_1,0] & \mathbf{C}_{HH}^T[1,:,p_1,0] & \cdots & \mathbf{C}_{HH}^T[p_1,:,p_1,0] \\ \mathbf{C}_{HH}^T[0,:,p_1,1] & \mathbf{C}_{HH}^T[1,:,p_1,1] & \cdots & \mathbf{C}_{HH}^T[p_1,:,p_1,1] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{HH}^T[0,:,p_1,p_2] & \mathbf{C}_{HH}^T[1,:,p_1,p_2] & \cdots & \mathbf{C}_{HH}^T[p_1,:,p_1,p_2] \end{array}\right] \begin{bmatrix} \hat{\mathbf{a}}[0] \\ \hat{\mathbf{a}}[1] \\ \vdots \\ \hat{\mathbf{a}}[p_1] \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

ith column of $\mathbf{H}$ so that

$$\mathbf{x} = \underbrace{\begin{bmatrix} \mathbf{a}_0^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}_1^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{a}_{N-1}^T \end{bmatrix}}_{\mathbf{B}(N\times MN)} \underbrace{\begin{bmatrix} H(t_0,f_0) \\ \vdots \\ H(t_0,f_{M-1}) \\ \hline H(t_1,f_0) \\ \vdots \\ H(t_1,f_{M-1}) \\ \hline \vdots \\ \hline H(t_{N-1},f_0) \\ \vdots \\ H(t_{N-1},f_{M-1}) \end{bmatrix}}_{\boldsymbol{\theta}(MN\times 1)} \tag{22}$$

or $\mathbf{x} = \mathbf{B}\boldsymbol{\theta}$, where $\mathbf{A}$ has been partitioned as $\mathbf{A} = [\mathbf{a}_0 \quad \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{N-1}]$, which is $M \times N$, and $\boldsymbol{\theta}$ is a vector of the rolled out columns of $\mathbf{H}$.

In Appendix A, we use the linear channel model of (22) to show that the AR solution is identifiable as long as $p_1 \le N - 2M + 1$, $p_2 \le M/2$, and $\mathbf{S}(f_n) \neq 0$ for $n = 0, 1, \ldots, N-1$.

III. Estimation Procedures

A. Conditional MVU Estimator

In sonar, the return from distributed interference is known as reverberation. The scattering field can be modeled as a spatial Poisson random process and, where there are a large number of scatterers, the received envelope will be a zero mean complex Gaussian random process [30]. We therefore model the received data using the complex Bayesian linear model based on the result of (22). Let $\mathbf{x} = \mathbf{B}\boldsymbol{\theta} + \mathbf{w}$ denote the complex Bayesian linear model, where $\boldsymbol{\theta} \sim CN(\boldsymbol{\mu}_\theta, \mathbf{C}_\theta)$, $\mathbf{w} \sim CN(\mathbf{0}, \sigma^2\mathbf{I})$, $\boldsymbol{\theta}$ and $\mathbf{w}$ are independent, $\mathbf{B}$ is $N \times p$ with $N < p$, and the rank of $\mathbf{B}$ is N [26]. Note that the matrix $\mathbf{B}$ in our problem is usually denoted as $\mathbf{H}$ in the formulation of the linear model. Except in Appendix B, we will use $\mathbf{B}$ so that the linear model formulation will not be confused with the TVFR. In Appendix B, we show that the conditional MVU estimator for a linear function of the parameters of a complex Bayesian linear model implies the following estimator for $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}} = \mathbf{B}^H \left(\mathbf{B}\mathbf{B}^H\right)^{-1} \mathbf{x}. \tag{23}$$

We note that this estimator is identical to the Moore–Penrose inverse of (22)[43] and could also be derived using singular value decomposition or regression approaches. However, we prefer the conditional MVU solution due to its optimality properties. The expression for $\hat{\boldsymbol{\theta}}$ is expanded in terms of the transmitted and received signals in Appendix C. The result of this expansion is the explicit solution for the estimate of the TVFR given as

$$\widehat{H}(t_n, f_m) = \frac{1}{\Delta_f \sum_{j=0}^{M-1} |S(f_j)|^2} S^*(f_m) \exp(-j2\pi f_m t_n)\, x(t_n) \tag{24}$$

for

$$n = 0, 1, \ldots, N-1$$
$$m = 0, 1, \ldots, M-1.$$

This can be written in continuous time frequency as

$$\widehat{H}(t,f) = \frac{1}{\varepsilon} S^*(f) \exp(-j2\pi f t)\, x(t) \tag{25}$$

where the energy in the signal is

$$\varepsilon = \int_{-B/2}^{B/2} |S(f)|^2 df.$$

Substituting (25) into (8) yields an identity after integration proving that this is a valid solution to the estimation problem. In fact, it can also be shown that (25) is the solution of minimum norm. We note that the conditional MVU estimator of the time-varying frequency response (25) is deterministic in the frequency direction (dependent only on the transmitted signal) and random in the time direction. The estimate of (24) can be directly used in the covariance method. The autocorrelation function estimator of the TVFR, which is used in the autocorrelation method, is derived in the next section.

B. Estimating the Autocorrelation Function

We now have a direct estimate of the TVFR of the channel. This in general will be a noisy estimate as it is from a single measurement of an overspread channel. Although the direct estimate may be poor, it is possible that it may contain enough information to estimate the scattering function. Both a direct Fourier estimate and the ACM estimate require that the autocorrelation function be estimated.

The 2-D autocorrelation function was defined in (6) as

$$R_H(u,v) = E[H^*(t,f)\, H(t+u, f+v)]$$

and exists if $H(t,f)$ is WSS in both time (space) and frequency. Assuming ergodicity, we will estimate the ACF by replacing the expected value with integration over time and frequency to yield

$$\widehat{R}_H(u,v) = \frac{1}{BL} \int_{-B/2}^{B/2} \int_0^L \widehat{H}^*(t,f)\, \widehat{H}(t+u, f+v)\, dt df. \tag{26}$$

Using (25) in (26) and neglecting the constant factor $1/BL$, we have (See Appendix D)

$$\widehat{R}_H(u,v) = \frac{A_x(u,v)\, A_s^*(u,v)}{(A_s(0,0))^2}.$$

We see that this estimate of the ACF does not require that the signal ambiguity function be nonzero as did the estimate of (12).

Next, to implement $\widehat{R}_H(u,v)$ in the discrete domain, we assume that the samples of $x(t)$ are available for $t = n\Delta_t$. Recalling that to prevent aliasing the ACF $R_H(u,v)$ must be sampled on a grid where $\Delta u \le 1/B$ and $\Delta v \le 1/L$, we can estimate the ACF in the discrete domain using (11) as

$$\begin{aligned}
&\widehat{R}_H\left(\frac{k}{B},\frac{l}{L}\right)\\
=&\frac{A_x\left(\frac{k}{B},\frac{l}{L}\right)A_s^*\left(\frac{k}{B},\frac{l}{L}\right)}{\left(A_s\left(0,0\right)\right)^2}\approx\Delta_t\\
&\cdot\sum_{n=0}^{N-1}x^*\left(n\Delta_t-\frac{k}{2B}\right)x\left(n\Delta_t+\frac{k}{2B}\right)\exp\left(-j2\pi n\frac{l}{L}\Delta_t\right)\\
&\cdot\Delta_t\sum_{n=0}^{N-1}s\left(n\Delta_t-\frac{k}{2B}\right)s^*\left(n\Delta_t+\frac{k}{2B}\right)\exp\left(j2\pi n\frac{l}{L}\Delta_t\right)\\
&\cdot\frac{1}{\left[\Delta_t\sum_{n=0}^{N-1}\left|s\left(n\Delta_t\right)\right|^2\right]^2}.
\end{aligned}$$

Clearly, it is also required that $1/\left(2B\right)$ be a multiple of the sampling rate Δ_t, or we must have $1/\left(2B\right)=m\Delta_t$ for m, which is an integer. Finally, for use in the ACM, we use the discrete ACF estimator

$$\begin{aligned}
&\widehat{r}\left[k,l\right]\\
=&\widehat{R}_H\left(\frac{k}{B},\frac{-l}{L}\right)\\
=&\sum_{n=0}^{N-1}x^*\left(\left(n-mk\right)\Delta_t\right)x\left(\left(n+mk\right)\Delta_t\right)\exp\left(j2\pi n\frac{l}{L}\Delta_t\right)\\
&\cdot\sum_{n=0}^{N-1}s\left(\left(n-mk\right)\Delta_t\right)s^*\left(\left(n+mk\right)\Delta_t\right)\exp\left(-j2\pi n\frac{l}{L}\Delta_t\right)\\
&\cdot\frac{1}{\left(\sum_{n=0}^{N-1}\left|s\left(n\Delta_t\right)\right|^2\right)^2}
\end{aligned}\tag{27}$$

which can be used to estimate the ACF on an appropriate grid for use in the ACM (17).

IV. Simulation Results

In 2-D AR spectral estimation, all causal AR models are based on a region of support which is either the nonsymmetric half plane (NSHP) or the quarter plane (QP). In general, only the NSHP will yield the correct PSD if the region of support is infinite. However, it has been observed from simulations that, for sinusoidal signals in noise, spectral estimators based on the NSHP perform poorly, possibly because the required model order is too high [40]. All of the results presented herein utilize a 2-D quarter plane (QP) autoregressive (AR) model. Estimates using the ACM and CM are compared. A comparison of results using the NSHP and QP is beyond the scope of this paper and is an area of future work.

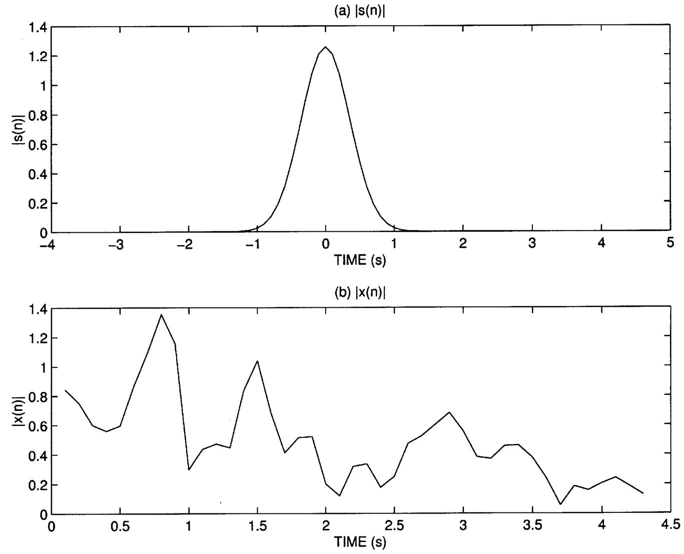

Fig. 3. (a) Magnitude of transmitted Gaussian envelope. (b) Magnitude of received signal envelope.

To demonstrate the validity of this approach the results of a number of simulations are presented. We will assume all data is sampled in delay and Doppler at intervals of Δ_τ and Δ_ϕ, respectively. In the simulation we define a known scattering function, P, with maximum time spread L and maximum Doppler spread B. We also define a known transmit waveform with time support T. The samples of a realization of the spreading function are zero mean complex Gaussian variables with variance $P\left(k\Delta_\phi,l\Delta_\tau\right)$ so that $S\left(k\Delta_\phi,l\Delta_\tau\right)=z_{kl}\sqrt{P\left(k\Delta_\phi,l\Delta_\tau\right)}$, where $z_{kl}\sim\mathcal{CN}\left(0,1\right)$ and all the z_{kl}s are uncorrelated. The received signal is calculated using a discrete version of (3), shown at the bottom of the page, for $0\le n\le\left(T+L\right)/\Delta_t$. Note that in this expression, samples of the transmit waveform $s(m)$ are needed over the range $[-L\Delta_\tau/\Delta_t,T+L]$. If a transmitted signal is given over an interval from 0 to T, we zero-pad outside the interval. For a known analytical expression such as a Gaussian pulse, the signal is calculated over the entire range.

The first example is for a known AR scattering function with $p_1=p_2=1$, which we denote AR(1,1) [defined by (16)] with time spread $L=4$ s and Doppler spread support $B=5$ Hz, which is interrogated by a Gaussian probe pulse of rms duration $T=0.357$ s. The rms bandwidth of a Gaussian pulse is $W=1/\sqrt{2}T$, which is 1.98 Hz in this case. The scattering function is characterized by the AR coefficients $a\left[0,0\right]=1.0000$, $a\left[0,1\right]=0.1854-0.5706j$, $a\left[1,0\right]=-0.7000$, $a\left[1,1\right]=-0.1298+0.3994j$. The received signal, whose bandwidth is $1.98+5=6.98$ Hz, is sampled at 10 Hz, and the SF is sampled in Doppler in 0.25-Hz increments. Fig. 3 shows examples of the envelopes of the transmitted and received signals for this case. Note that the analytical expression for a Gaussian

$$x\left(n\Delta_t\right)=\sum_{k=-B/(2\Delta_\phi)}^{B/(2\Delta_\phi)}\sum_{l=0}^{L/\Delta_t}S\left(k\Delta_\phi,l\Delta_\tau\right)s\left(n\Delta_t-l\Delta_\tau\right)\exp\left(j2\pi k\Delta_\phi l\Delta_\tau\right)$$

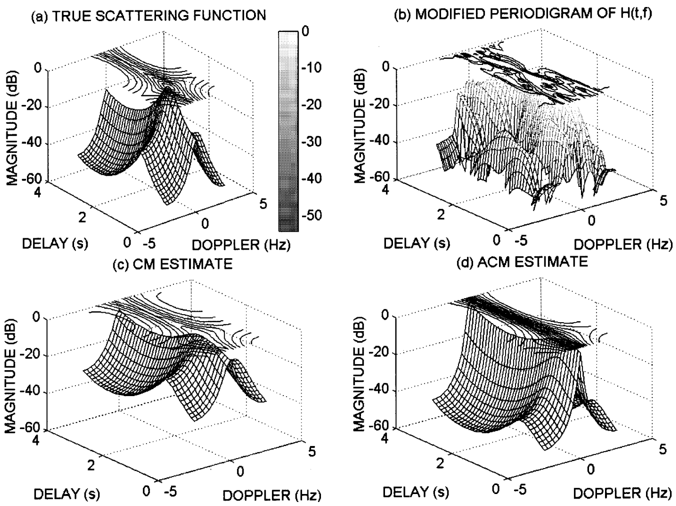

Fig. 4. (a) True scattering function used in the simulation. (b) One-ping Fourier estimate. (c) One-ping AR(1,1) CM estimate. (d) One-ping AR(1,1) ACM estimate.

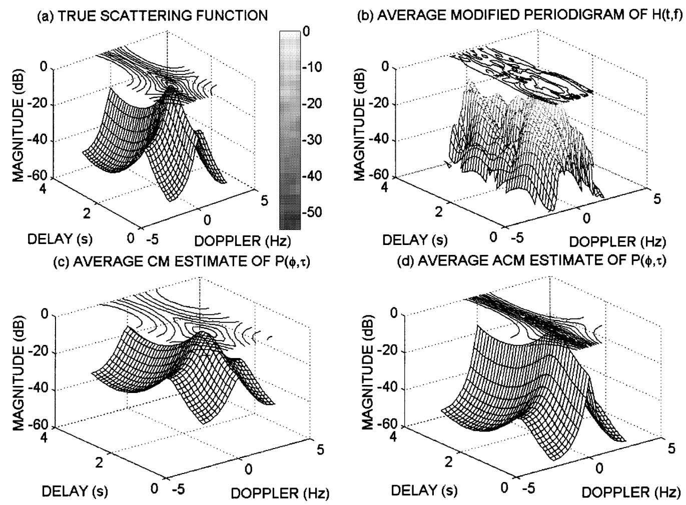

Fig. 6. (a) True scattering function. (b) Average of 50 Fourier estimates. (c) Average of 50 AR(1,1) CM estimates. (d) Average of 50 AR(1,1) ACM estimates.

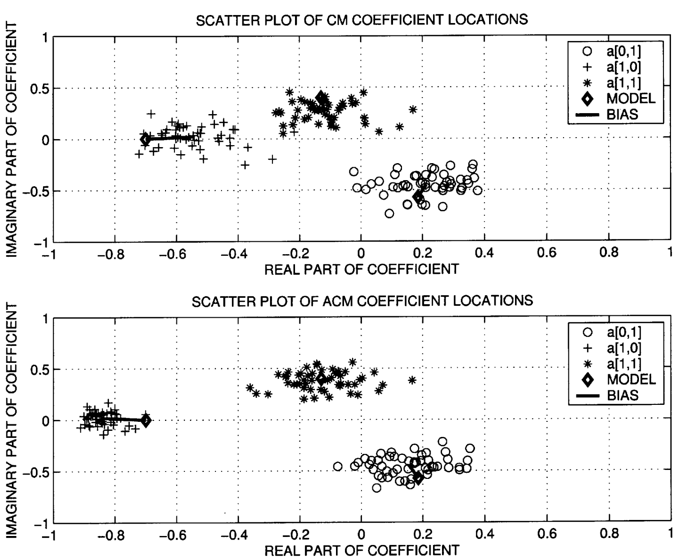

Fig. 5. Scatter plot for 50 realizations of 1-ping AR coefficient locations using both the ACM and CM estimators.

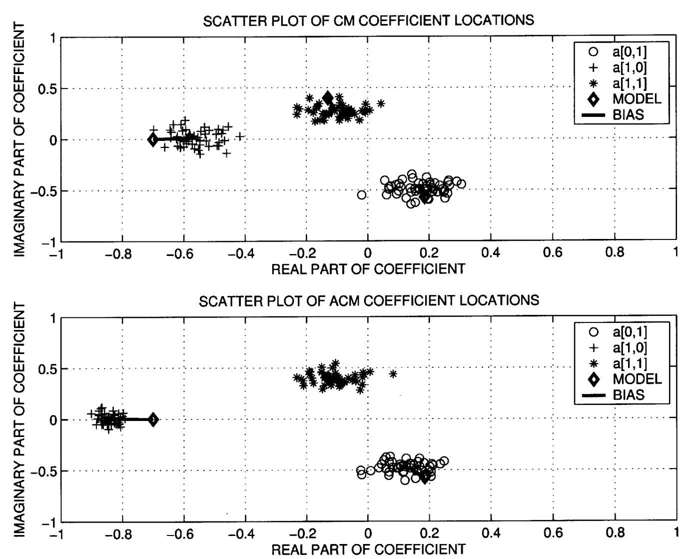

Fig. 7. Scatter plots of 50 AR[1,1] coefficient locations for both CM and ACM estimators. Three pings are averaged to form each estimate.

pulse was used to generate the transmitted signal over the range $[-L\Delta_\tau/\Delta_t, T+L]$, and the received signal has time support only over the range [0, L]. Fig. 4 shows the known SF and the single ping estimates for various estimators. All contour plots are shown on identical axes and contours are given in decibels. The Fourier estimate is formed by calculating the 2-D periodogram of the conditional MVU estimate of the TVFR (24). This is followed by AR(1,1) estimates using both the CM and the ACM estimators. Clearly, the AR estimators give higher resolution and more accurate estimates of the scattering function for this simple case.

Scatter plots of AR parameter estimate locations for 50 realizations of the two AR(1,1) estimators are shown in Fig. 5. Solid lines on the graph are drawn from the actual model locations to the average of the 50 realizations. In almost all cases, the average location of each parameter estimate is biased toward the origin. The one notable exception is for the $a[1,1]$ coefficient using the ACM. The exact cause of this bias is a matter of future investigation. It is also notable that in this case the ACM estimates of the $a[1,0]$ coefficient have significantly less scatter than the CM estimates. The average scattering function estimates for these 50 single ping realizations are shown in Fig. 6. Qualitatively, the average of CM results appears to match the true scattering function better than the average of the ACM results.

Although we wish to estimate the scattering function with a single ping, the use of multiple pings will improve the accuracy of the estimates if the channel can be considered stationary over the time spanned by the multiple pings. Fig. 7 shows a similar scatter plot for a case where three pings are used to form the estimate. Here, the conditional MVU estimate of the TVFR is calculated and the corresponding correlation functions [(27) into (17) for ACM or (24) into (20) for CM] for AR estimation is formed for each ping. The correlation functions are then averaged before finally calculating the AR parameters. We see that the variance of the estimates is significantly reduced although the bias remains. Although it is beyond the scope of this paper, this indicates that multiping and/or recursive estimation schemes may provide robust estimates in environments where some stability may be assumed from ping to ping.

V. Conclusions

A novel method of scattering function estimation based on autoregressive spectral modeling has been proposed. The current implementation of this method uses the conditional MVU estimate of the TVFR, given a known input waveform and the received envelope. Preliminary simulation results exhibit promise of obtaining high-resolution estimates of the scattering function from a single ping. The results also indicate that the covariance method may be slightly more accurate on average than the autocorrelation method. However, no claims of optimality can be made regarding the current estimator. Attempts by these authors to calculate the maximum likelihood estimate using the EM algorithm have failed due to the extreme computational and storage requirements of the algorithm. Continuing research is focused on improving this technique using optimal estimators and waveforms and the use of quarter plane versus nonsymmetric half plane estimators.

Appendix A
Identifiability of the AR Parameters

Assume that we have access to $\mathbf{C}_{xx} = E\left(\mathbf{x}\mathbf{x}^H\right)$, where $\mathbf{x} = \mathbf{B}\boldsymbol{\theta}$ is given by (22). Can we determine the AR parameters describing $\mathbf{C}_{\theta\theta}$, where

$$\mathbf{C}_{xx} = \mathbf{B}\mathbf{C}_{\theta\theta}\mathbf{B}^H \tag{28}$$

and $\mathbf{B}$ is defined in (22), $\mathbf{C}_{xx}$ is $N \times N$, $\mathbf{B}$ is $N \times MN$, and $\mathbf{C}_{\theta\theta}$ is $MN \times MN$? $\mathbf{C}_{\theta\theta}$ is Hermitian and block Toeplitz consisting of N unique blocks of size $M \times M$, which are themselves Toeplitz. Therefore, each block has $2M-1$ unique complex elements. Thus, to describe $\mathbf{C}_{\theta\theta}$, we need $N(2M-1)$ coefficients. As we now show, an AR parameterization of $\mathbf{C}_{\theta\theta}$ allows us to identify $\mathbf{C}_{\theta\theta}$ from $\mathbf{C}_{xx}$. This says that an estimate of $\mathbf{C}_{\theta\theta}$ is possible based on $\hat{\mathbf{C}}_{xx} = \mathbf{x}\mathbf{x}^H$.

We now expand (28) as

$$\begin{aligned}
\mathbf{C}_{xx} &= \begin{bmatrix} \mathbf{a}_0^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}_1^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{a}_{N-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{R}_0 & \mathbf{R}_1 & \cdots & \mathbf{R}_{N-1} \\ \mathbf{R}_1^H & \mathbf{R}_0 & \cdots & \mathbf{R}_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{N-1}^H & \mathbf{R}_{N-2}^H & \cdots & \mathbf{R}_0 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \mathbf{a}_0^* & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_1^* & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{a}_{N-1}^* \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a}_0^T\mathbf{R}_0\mathbf{a}_0^* & \mathbf{a}_0^T\mathbf{R}_1\mathbf{a}_1^* & \cdots & \mathbf{a}_0^T\mathbf{R}_{N-1}\mathbf{a}_{N-1}^* \\ \mathbf{a}_1^T\mathbf{R}_1^H\mathbf{a}_0^* & \mathbf{a}_1^T\mathbf{R}_0\mathbf{a}_1^* & \cdots & \mathbf{a}_1^T\mathbf{R}_{N-2}\mathbf{a}_{N-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{N-1}^T\mathbf{R}_{N-1}^H\mathbf{a}_0^* & \mathbf{a}_{N-1}^T\mathbf{R}_{N-2}^H\mathbf{a}_1^* & \cdots & \mathbf{a}_{N-1}^T\mathbf{R}_0\mathbf{a}_{N-1}^* \end{bmatrix}
\end{aligned}$$

where $\mathbf{R}_i$ has $2M-1$ unique complex elements. Let $\mathbf{z}_m$ denote the mth diagonal of $\mathbf{C}_{xx}$ so that

$$\mathbf{z}_m = \begin{bmatrix} \mathbf{a}_0^T\mathbf{R}_m\mathbf{a}_m^* \\ \mathbf{a}_1^T\mathbf{R}_m\mathbf{a}_{m+1}^* \\ \vdots \\ \mathbf{a}_{N-1-m}^T\mathbf{R}_m\mathbf{a}_{N-1}^* \end{bmatrix}$$

for $m = 0, 1, \ldots, N-1$. Each $\mathbf{z}_m$ is $(N-m) \times 1$. Therefore, the estimates of different $\mathbf{R}_m$'s can be decoupled. Since $\mathbf{z}_m$ is $(N-m) \times 1$, each vector $\mathbf{z}_m$ can be expressed as a system of linear equations in the $2M-1$ unknown parameters of $\mathbf{R}_m$. These equations are assumed consistent. In addition, assume that m is chosen so that $N - m \geq 2M - 1$, and therefore, there is a unique solution, as we now show.

We now assume we are looking for an $AR(p_1, p_2)$ solution and derive an explicit solution for $\mathbf{R}_m$. Let

$$\mathbf{R}_m = \begin{bmatrix} r[m,0] & r[m,-1] & \cdots & r[m,-p_2] \\ r[m,1] & r[m,0] & \cdots & r[m,-(p_2-1)] \\ \vdots & \vdots & \ddots & \vdots \\ r[m,p_2] & r[m,p_2-1] & \cdots & r[m,0] \end{bmatrix}$$

for $m = -p_1, -p_1-1, \ldots, p_1-1, p_1$. Now, expand the kth element of $\mathbf{z}_m$, which is $\mathbf{a}_k^T\mathbf{R}_m\mathbf{a}_{m+k}^*$, as

$$\begin{aligned}
z_m[k] = {} & a_{m+k}^*[0] \sum_{l'=0}^{p_2} a_k[l'] \, r[m, l'] \\
& + a_{m+k}^*[1] \sum_{l'=0}^{p_2} a_k[l'] \, r[m, l'-1] \\
& \vdots \\
& + a_{m+k}^*[p_2] \sum_{l'=0}^{p_2} a_k[l'] \, r[m, l'-p_2]
\end{aligned}$$

for $k = 0, 1, \ldots, N-m-1$. Collecting the r's yields

$$\begin{aligned}
z_m[k] = {} & \sum_{l'=0}^{p_2} r[m, l'] \sum_{q=0}^{p_2-l'} a_{m+k}^*[q] \, a_k[q+l'] \\
& + \sum_{l'=-p_2}^{-1} r[m, l'] \sum_{q=-l'}^{p_2} a_{m+k}^*[q] \, a_k[q+l'].
\end{aligned}$$

Let $l = l' + p_2$, and define

$$\eta_{m,k}[l] = \begin{cases} \displaystyle\sum_{q=0}^{2p_2-l} a_{m+k}^*[q-p_2] \, a_k[q+l-p_2], & l \geq p_2 \\ \displaystyle\sum_{q=p_2-l}^{p_2} a_{m+k}^*[q] \, a_k[q+l-p_2], & l < p_2 \end{cases} \tag{29}$$

for $0 \leq l \leq 2p_2$, and $\mathbf{r}_m = [r[m,-p_2]\, r[m,-p_2+1] \cdots r[m,p_2]]^T$. Now, $z_m[k] = \eta_{m,k}^T \mathbf{r}_m$, and

$$\mathbf{z}_m = \underbrace{\begin{bmatrix} \eta_{m,0}^T \\ \eta_{m,1}^T \\ \vdots \\ \eta_{m,N-1-m}^T \end{bmatrix}}_{\mathbf{H}_m} \mathbf{r}_m \tag{30}$$

where $\mathbf{z}_m$ is $(N-m)\times 1$, $\mathbf{H}_m$ is $(N-m)\times(2p_2+1)$, and $\mathbf{r}_m$ is $(2p_2+1)\times 1$. This is naturally partitioned as $\mathbf{H}_m = [\mathbf{H}_m^- \mathbf{H}_m^+]$, where $\mathbf{H}_m^-$ and $\mathbf{H}_m^+$ correspond with $l < p_2$ and $l \geq p_2$, respectively.

In order to show that (30) is uniquely solvable, we must show that $\mathbf{H}_m$ is of full column rank. We note that by assumption, $\mathbf{z}_m$ is in the range space of $\mathbf{H}_m$. Clearly, a bounding requirement is $N-m \geq 2p_2+1$ or $p_2 \leq (N-m-1)/2$. We recall from (21) that $a_k[l] = [\mathbf{A}]_{lk} = \Delta_f S(f_l)\exp(j2\pi f_l t_k)$.

Now, expand (29) into the first equation shown at the bottom of the page. We want to show that the columns of H_m are independent. Note that two of the terms in each sum do not depend on the column index l. For later convenience, we write these as vectors depending on m and q and indexed by k so that we have the second equation at the bottom of the page. Therefore

$$\begin{aligned} \mathbf{H}_m^+[k,l] =& \Delta_f^2 \sum_{q=0}^{p_2-l} S(f_{q+l})\,\mathrm{diag}(\mathbf{c}_{mq})\,\mathbf{E}_q^+ \\ l =& 0,1,\ldots,p_2 \\ \mathbf{H}_m^-[k,l] =& \Delta_f^2 \sum_{q=p_2-l}^{p_2} S(f_{q+l-p_2})\,\mathrm{diag}(\mathbf{d}_{mq})\,\mathbf{E}_q^- \\ l =& 0,1,\ldots,p_2-1 \end{aligned}$$

and we have the third equation at the bottom of the page. $\mathbf{E}_q^-$ is defined similarly, except that each of the frequency indicies is reduced by p_2, i.e., $\exp(j2\pi f_{0+q}t_0)$ becomes $\exp(j2\pi f_{0+q-p_2}t_0)$.

Now, assuming $f_i \neq f_j \forall i \neq j,\ i,j = 0,1,\ldots,2p_2$, any $\mathbf{E}_q^+$ is of full column rank, and the set $\mathbf{E}_q^+$ is linearly independent. In addition, since $\mathbf{E}_q^+$ is of full column rank, diag $(\mathbf{c}_{mq})\,\mathbf{E}_q^+$ is of full column rank as long as $\mathbf{c}_{mq}[k] \neq 0 \forall k$. This is equivalent to requiring that $S(f_q) \neq 0$ for $q = p_2, P_2+1, \ldots, 2p_2$. The set diag $(\mathbf{c}_{mq})\,\mathbf{E}_q^+$ is therefore also independent and forms a basis for $\mathbf{H}_m^+$. $\mathbf{H}_m^+$ is therefore of full column rank.

The same argument holds for $\mathbf{H}_m^-$, adding the requirement that $S(f_q) \neq 0$ for $q = 0,1,\ldots,p_2-1$. Clearly, $\mathbf{H}_m^-$ will be of full column rank and $\mathbf{E}_q^+$, $\mathbf{E}_q^-$ are independent sets, and therefore, $\mathbf{H}_m = [\mathbf{H}_m^- \mathbf{H}_m^+]$ will be of full column rank.

In summary, we have shown that $\mathbf{R}_m$ is identifiable if $N - m \geq 2M-1$, and an $AR(p_1,p_2)$ solution is identifiable as long as $p_1 \leq N-2M+1$, $p_2 \leq (N-p_1-1)/2$, and $\mathbf{S}(f_n) \neq 0$ for $n = 0,1,\ldots,2p_2$.

Appendix B
Conditional MVU Estimator for a Linear Function of the Parameters of a Complex Bayesian Linear Model

Let $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ denote the complex Bayesian linear model, where $\boldsymbol{\theta} \sim CN(\boldsymbol{\mu}_\theta, \mathbf{C}_\theta)$ and is $p\times 1$, $\mathbf{w} \sim CN(\mathbf{0}, \sigma^2\mathbf{I})$ and is $N\times 1$, $\boldsymbol{\theta}$ and $\mathbf{w}$ are independent, $\mathbf{H}$ is $N \times p$ with $N < p$, and the rank of $\mathbf{H}$ is N [26]. The conditional PDF of $\mathbf{x}$ is

$$p(\mathbf{x}|\boldsymbol{\theta}) = \frac{1}{\pi^N \sigma^{2N}} \exp\left(-\frac{1}{\sigma^2}(\mathbf{x}-\mathbf{H}\boldsymbol{\theta})^H(\mathbf{x}-\mathbf{H}\boldsymbol{\theta})\right).$$

By attempting to find a conditional estimator of $\boldsymbol{\theta}$, we avoid the need for prior knowledge. However, because $N < p$, there are an infinite number of $\boldsymbol{\theta}$ that yield the same $\mathbf{H}\boldsymbol{\theta}$. Hence, $\boldsymbol{\theta}$ is not identifiable. A reasonable approach is to estimate a linear function of $\boldsymbol{\theta}$, say $\alpha = \mathbf{P}^H\boldsymbol{\theta}$, where $\mathbf{P}$ is $p\times 1$. Then, from this result we infer a reasonable estimator for $\boldsymbol{\theta}$. Note that ultimately we do not use $\hat{\boldsymbol{\theta}}$ directly but only a function of $\boldsymbol{\theta}$, specifically, a

---

$$\eta_{m,k}[l] = \Delta_f^2 \begin{cases} \sum_{q=0}^{2p_2-l} S^*(f_{q-p_2})\exp(-j2\pi f_{q-p_2}t_{m+k})\,S(f_{q+l-p_2})\exp(j2\pi f_{q+l-p_2}t_k), & l \geq p_2 \\ \sum_{q=p_2-l}^{p_2} S^*(f_q)\exp(-j2\pi f_q t_{m+k})\,S(f_{q+l-p_2})\exp(j2\pi f_{q+l-p_2}t_k), & l < p_2. \end{cases}$$

$$\eta_{m,k}[l] = \Delta_f^2 \begin{cases} \sum_{q=0}^{2p_2-l} \mathbf{c}_{mq}[k]\,S(f_{q+l-p_2})\exp(j2\pi f_{q+l-p_2}t_k), & l \geq p_2 \\ \sum_{q=p_2-l}^{p_2} \mathbf{d}_{mq}[k]\,S(f_{q+l-p_2})\exp(j2\pi f_{q+l-p_2}t_k), & l < p_2. \end{cases}$$

$$\mathbf{E}_q^+ = \begin{bmatrix} \exp(j2\pi f_{0+q}t_0) & \exp(j2\pi f_{1+q}t_0) & \cdots & \exp(j2\pi f_{p_2+q}t_0) \\ \exp(j2\pi f_{0+q}t_1) & \exp(j2\pi f_{1+q}t_1) & \cdots & \exp(j2\pi f_{p_2+q}t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(j2\pi f_{0+q}t_{N-m-1}) & \exp(j2\pi f_{1+q}t_{N-m-1}) & \cdots & \exp(j2\pi f_{p_2+q}t_{N-m-1}) \end{bmatrix}$$

Hermitian function used as an ACF estimator. The restriction to a linear function is made for mathematical tractability.

Now, our goal is to find the conditional MVU estimator of α. Similar results are available for real linear models in [44]–[46], where α is referred to as an estimable function of $\boldsymbol{\theta}$.

Theorem 1: Let $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ denote the complex Bayesian linear model where $\boldsymbol{\theta} \sim CN(\boldsymbol{\mu}_\theta, \mathbf{C}_\theta)$ and is $p\times 1$, $\mathbf{w} \sim CN(\mathbf{0}, \sigma^2\mathbf{I})$ and is $N\times 1$, $\boldsymbol{\theta}$ and $\mathbf{w}$ are independent, $\mathbf{H}$ is $N\times p$ with $N < p$, and the rank of $\mathbf{H}$ is N. Let $\alpha = \mathbf{P}^H\boldsymbol{\theta}$, where $\mathbf{P}$ is $p\times 1$, and $\mathbf{P}$ lies in the range space of $\mathbf{H}^H$. Then, the conditional MVU estimator of α is given by

$$\hat{\alpha} = \mathbf{P}^H\mathbf{H}^H\left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{x}.$$

Proof: Let $\hat{\alpha} = \mathbf{L}^H\mathbf{x} + g(\mathbf{x})$, where g is some complex valued function of $\mathbf{x}$, and $\mathbf{L}$ is $N\times 1$ and complex. For $\hat{\alpha}$ to be unbiased, we must have

$$\begin{aligned} E(\hat{\alpha}) =& \mathbf{L}^H E(\mathbf{x}) + E(g(\mathbf{x})) = \alpha \;\forall\boldsymbol{\theta} \\ =& \mathbf{L}^H\mathbf{H}\boldsymbol{\theta} + E(g(\mathbf{x})) = \mathbf{P}^H\boldsymbol{\theta}\;\forall\boldsymbol{\theta}. \end{aligned}$$

For this to hold for all $\boldsymbol{\theta}$, we must have $E(g(\mathbf{x})) = 0$ (let $\boldsymbol{\theta} = \mathbf{0}$) and $\mathbf{L}^H\mathbf{H} = \mathbf{P}^H$ or $\mathbf{P} = \mathbf{H}^H\mathbf{L}$ so that $\mathbf{P}$ must lie in the range space of $\mathbf{H}^H$. To minimize the variance

$$\begin{aligned} &\mathrm{var}(\hat{\alpha}) \\ =& E\left[|\hat{\alpha} - E(\hat{\alpha})|^2\right] \\ =& E\left[\left|\mathbf{L}^H\mathbf{x} + g(\mathbf{x}) - \mathbf{L}^H\mathbf{H}\boldsymbol{\theta} - E(g(\mathbf{x}))\right|^2\right] \\ =& E\left[\left|\mathbf{L}^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta}) + g(\mathbf{x})\right|^2\right] \text{ since } E(g(\mathbf{x})) = 0 \\ =& E\left[\left(\mathbf{L}^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta}) + g(\mathbf{x})\right)\left((\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H\mathbf{L} + g^*(\mathbf{x})\right)\right] \end{aligned}$$

where * denotes conjugation. Finally

$$\begin{aligned} \mathrm{var}(\hat{\alpha}) =& \mathbf{L}^H\sigma^2\mathbf{I}\mathbf{L} + \mathbf{L}^H E\left[(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\, g^*(\mathbf{x})\right] \\ &+ E\left[g(\mathbf{x})(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H\right]\mathbf{L} + \mathbf{E}\left[|g(\mathbf{x})|^2\right]. \end{aligned} \tag{31}$$

Next, we prove that $E\left[g(\mathbf{x})(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H\right] = \mathbf{0}^T$. Since $E(g(\mathbf{x})) = 0\;\forall\boldsymbol{\theta}$

$$\begin{aligned} &E(g(\mathbf{x})) \\ =& \int g(\mathbf{x})\frac{1}{\pi^N\sigma^{2N}}\exp\left(-\frac{1}{\sigma^2}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right)d\mathbf{x} \\ =& 0. \end{aligned}$$

We set the complex gradient of $E(g(\mathbf{x})) = 0$ or [26], as shown in the equation at the bottom of the page, but $(\partial/\partial\boldsymbol{\theta})\,||(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})||^2 = -\left[\mathbf{H}^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right]^*$ [26] so that

$$\begin{aligned} \mathbf{0} =& \int g(\mathbf{x})\left(-\frac{1}{\sigma^2}\right)\left[-\mathbf{H}^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right]^* p(\mathbf{x}|\boldsymbol{\theta})\,d\mathbf{x} \\ \mathbf{0}^T =& \int g(\mathbf{x})\left[\mathbf{H}^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right]^H p(\mathbf{x}|\boldsymbol{\theta})\,d\mathbf{x} \\ =& \int g(\mathbf{x})(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H p(\mathbf{x}|\boldsymbol{\theta})\,d\mathbf{x}\,\mathbf{H}. \end{aligned}$$

Since $\mathbf{H}^H$ is full rank, we must have

$$E\left[g(\mathbf{x})(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H\right] = \int g(\mathbf{x})(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H p(\mathbf{x}|\boldsymbol{\theta})\,d\mathbf{x} = \mathbf{0}^T.$$

Substituting this result into (31) yields

$$\mathrm{var}(\hat{\alpha}) = \sigma^2\mathbf{L}^H\mathbf{L} + \mathbf{E}\left[|g(\mathbf{x})|^2\right] \geq \sigma^2\mathbf{L}^H\mathbf{L}$$

with equality iff $g = 0$.

Finally, to minimize $\mathrm{var}(\hat{\alpha})$ over $\mathbf{L}$ with $\mathbf{P} = \mathbf{H}^H\mathbf{L}$, note that $\mathbf{H}$ is full rank so that $\mathbf{P} = \mathbf{H}^H\mathbf{L}$ is a consistent set of equations, and $\mathbf{P}$ is constrained to lie in the range space of $\mathbf{H}^H$. The general solution is

$$\mathbf{L} = \underbrace{\left(\mathbf{H}^H\right)^-\mathbf{P}}_{\boldsymbol{\xi}_1} + \underbrace{\left(\mathbf{I} - \left(\mathbf{H}^H\right)^-\mathbf{H}^H\right)\mathbf{z}}_{\boldsymbol{\xi}_2}$$

where $\mathbf{z}$ is $N\times 1$ and arbitrary, and $\left(\mathbf{H}^H\right)^-$ is the generalized inverse of $\mathbf{H}^H$ [44]. However

$$\boldsymbol{\xi}_1^H\boldsymbol{\xi}_2 = \mathbf{P}^H\left(\mathbf{H}^H\right)^{-H}\left(\mathbf{I} - \left(\mathbf{H}^H\right)^-\mathbf{H}^H\right)\mathbf{z}$$

and using $\left(\mathbf{H}^H\right)^- = \left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{H}$ (since $\mathbf{H}^H$ is full rank) [43], [44]

$$\begin{aligned} \boldsymbol{\xi}_1^H\boldsymbol{\xi}_2 =& \mathbf{P}^H\left(\left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{H}\right)^H\left(\mathbf{I} - \left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{H}\mathbf{H}^H\right)\mathbf{z} \\ =& \mathbf{0} \end{aligned}$$

so that

$$\mathbf{L}^H\mathbf{L} = ||\boldsymbol{\xi}_1||^2 + ||\boldsymbol{\xi}_2||^2 \geq ||\boldsymbol{\xi}_1||^2$$

with equality iff $\mathbf{z} = \mathbf{0}$. This implies that the optimal $\mathbf{L}$ is

$$\mathbf{L}_{opt} = \left(\mathbf{H}^H\right)^-\mathbf{P} = \left(\mathbf{H}\mathbf{H}^H\right)^{-1}\mathbf{H}\mathbf{P}$$

$$\frac{\partial E(g(\mathbf{x}))}{\partial\boldsymbol{\theta}} = \int g(\mathbf{x})\frac{1}{\pi^N\sigma^{2N}}\left(-\frac{1}{\sigma^2}\right)\frac{\partial}{\partial\boldsymbol{\theta}}||(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})||^2\exp\left(-\frac{1}{\sigma^2}(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})^H(\mathbf{x} - \mathbf{H}\boldsymbol{\theta})\right)d\mathbf{x} = \mathbf{0}$$

and finally

$$\hat{\alpha} = \mathbf{L}_{opt}^{H}\mathbf{x} = \mathbf{P}^{H}\mathbf{H}^{H}\left(\mathbf{H}\mathbf{H}^{H}\right)^{-1}\mathbf{x}.$$

We take $\hat{\boldsymbol{\theta}} = \mathbf{H}^{H}\left(\mathbf{H}\mathbf{H}^{H}\right)^{-1}\mathbf{x}$ as our estimator of $\boldsymbol{\theta}$ and note that $\mathbf{P}^{H}\hat{\boldsymbol{\theta}}$ is the unique conditional MVU estimator of $\mathbf{P}^{H}\boldsymbol{\theta}$. ■

To avoid confusing the $\mathbf{H}$ used to define the linear model with the TVFR, we replace $\mathbf{H}$ with $\mathbf{B}$ as in (22) so that the conditional MVU estimator of the rolled out TVFR becomes $\hat{\boldsymbol{\theta}} = \mathbf{B}^{H}\left(\mathbf{B}\mathbf{B}^{H}\right)^{-1}\mathbf{x}$.

APPENDIX C
EXPLICIT SOLUTION OF $\hat{\boldsymbol{\theta}}$

We now explicitly determine $\hat{\boldsymbol{\theta}}$ for the TVFR using (22) and (23). Thus

$$\begin{aligned}\mathbf{B}\mathbf{B}^{H} &= \begin{bmatrix} \mathbf{a}_0^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}_1^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{a}_{N-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_0^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}_1^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{a}_{N-1}^T \end{bmatrix}^{H} \\ &= \underbrace{\begin{bmatrix} \mathbf{a}_0^H\mathbf{a}_0 & 0 & \cdots & 0 \\ 0 & \mathbf{a}_1^H\mathbf{a}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{a}_{N-1}^H\mathbf{a}_{N-1} \end{bmatrix}}_{N\times N}\end{aligned}$$

and

$$\left(\mathbf{B}\mathbf{B}^{H}\right)^{-1}\mathbf{x} = \underbrace{\begin{bmatrix} \frac{x(t_0)}{\mathbf{a}_0^H\mathbf{a}_0} \\ \vdots \\ \frac{x(t_{N-1})}{\mathbf{a}_{N-1}^H\mathbf{a}_{N-1}} \end{bmatrix}}_{N\times 1}.$$

Using this in (23) produces

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \frac{[\mathbf{a}_0]_0^* x(t_0)}{\mathbf{a}_0^H\mathbf{a}_0} \\ \vdots \\ \frac{[\mathbf{a}_0]_{M-1}^* x(t_0)}{\mathbf{a}_0^H\mathbf{a}_0} \\ \hline \frac{[\mathbf{a}_1]_0^* x(t_1)}{\mathbf{a}_1^H\mathbf{a}_1} \\ \vdots \\ \frac{[\mathbf{a}_1]_{M-1}^* x(t_1)}{\mathbf{a}_1^H\mathbf{a}_1} \\ \hline \vdots \\ \hline \frac{[\mathbf{a}_{N-1}]_0^* x(t_{N-1})}{\mathbf{a}_{N-1}^H\mathbf{a}_{N-1}} \\ \vdots \\ \frac{[\mathbf{a}_{N-1}]_{M-1}^* x(t_{N-1})}{\mathbf{a}_{N-1}^H\mathbf{a}_{N-1}} \end{bmatrix}$$

where $[\mathbf{a}_i]_j = [\mathbf{A}]_{ji}$, $i = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, M-1$. From the original definition of $\mathbf{A}$(21)

$$\begin{aligned}[\mathbf{a}_i]_j &= \Delta_f S(f_j)\, e^{j2\pi f_j t_i} \\ \mathbf{a}_i^H\mathbf{a}_i &= \Delta_f^2 \sum_{j=0}^{M-1} |S(f_j)|^2.\end{aligned}$$

We note that this last expression holds for all i. The conditional MVU estimate now becomes

$$\hat{\boldsymbol{\theta}} = \frac{1}{\Delta_f \sum_j |S(f_j)|^2} \times \begin{bmatrix} S^*(f_0)\exp(-j2\pi f_0 t_0)\, x(t_0) \\ \vdots \\ S^*(f_{M-1})\exp(-j2\pi f_{M-1} t_0)\, x(t_0) \\ \hline S^*(f_0)\exp(-j2\pi f_0 t_1)\, x(t_1) \\ \vdots \\ S^*(f_{M-1})\exp(-j2\pi f_{M-1} t_1)\, x(t_1) \\ \hline \vdots \\ \hline S^*(f_0)\exp(-j2\pi f_0 t_{N-1})\, x(t_{N-1}) \\ \vdots \\ S^*(f_{M-1})\exp(-j2\pi f_{M-1} t_{N-1})\, x(t_{N-1}) \end{bmatrix}.$$

APPENDIX D
DERIVATION OF THE ACF IN TERMS OF AMBIGUITY FUNCTIONS

Since we defined the theoretical autocorrelation function in terms of time–frequency autocorrelation functions in (10), it is instructive to rewrite the estimate $\widehat{R}_H(u, v)$ in a similar fashion by substituting (25) into (26). We ignore the constant factor $1/BL$ as we will normalize our results.

$$\begin{aligned}\widehat{R}_H(u, v) &= \int_{-B/2}^{B/2}\int_0^L \widehat{H}^*(t, f)\,\widehat{H}(t+u, f+v)\,dtdf \\ &= \frac{1}{\varepsilon^2}\int_{-B/2}^{B/2}\int_0^L S(f)\, e^{j2\pi ft} x^*(t) \\ &\quad \cdot S^*(f+v)\, e^{-j2\pi(f+v)(t+u)} x(t+u)\, dtdf \\ &= \frac{1}{\varepsilon^2}\int_{-B/2}^{B/2}\int_0^L S(f)\, x^*(t)\, S^*(f+v)\, e^{j2\pi ft} \\ &\quad \cdot e^{-j2\pi(ft+fu+vt+vu)} x(t+u)\, dtdf \\ &= \frac{e^{-j2\pi vu}}{\varepsilon^2}\int_{-B/2}^{B/2}\int_0^L S(f)\, S^*(f+v)\, e^{-j2\pi(fu+vt)} \\ &\quad \cdot x^*(t)\, x(t+u)\, dtdf \\ &= \frac{e^{-j2\pi vu}}{\varepsilon^2}\int_0^L x^*(t)\, x(t+u)\, e^{-j2\pi vt} dt \\ &\quad \cdot \int_{-B/2}^{B/2} S(f)\, S^*(f+v)\, e^{-j2\pi fu} df.\end{aligned}$$

Now, recalling the definition of the time–frequency autocorrelation function (11) and letting $t' = t + u/2$ and $f' = f + v/2$ and assuming that the signal is truly time limited and bandlimited

$$\begin{aligned}
&\widehat{R}_H(u,v)\\
&=\frac{e^{-j2\pi vu}}{\varepsilon^2}\\
&\quad\cdot\int_{-\infty}^{\infty} x^*\left(t'-\frac{u}{2}\right)x\left(t'+\frac{u}{2}\right)e^{-j2\pi vt'}e^{j2\pi vu/2}dt'\\
&\quad\cdot\int_{-\infty}^{\infty} S\left(f'-\frac{v}{2}\right)S^*\left(f'+\frac{v}{2}\right)e^{-j2\pi f'u}e^{j2\pi vu/2}df'\\
&=\frac{e^{-j2\pi vu}e^{j2\pi vu/2}e^{j2\pi vu/2}A_x(u,v)\,A_s^*(u,v)}{\varepsilon^2}\\
&=\frac{A_x(u,v)\,A_s^*(u,v)}{(A_s(0,0))^2}.
\end{aligned}$$

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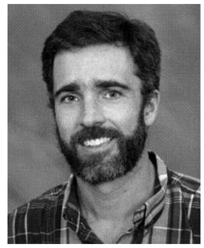

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