An Invariance Property of the Generalized Likelihood Ratio Test

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***Abstract*—The generalized likelihood ratio test (GLRT) is invariant with respect to transformations for which the hypothesis testing problem itself is invariant. This result from the statistics literature is presented in the context of some simple signal models. This is an important property of the GLRT in light of its widespread use and the recent interest in invariant tests applied to signal processing applications. The GLRT is derived for some examples in which the uniformly most powerful invariant (UMPI) test does and does not exist, including one in which the UMPI test exists and is not given by the GLRT.**

***Index Terms*—Signal detection, signal processing.**

I. Introduction

THE LIKELIHOOD ratio test of the statistics literature is invariant to sets of transformations for which the hypothesis test itself is invariant (e.g., [1]–[3]). The likelihood ratio test statistic is obtained by replacing the unknown parameters under each hypothesis with their maximum-likelihood estimators (MLEs). In the engineering literature (e.g., [4] and [5]) the likelihood ratio test is known as the generalized likelihood ratio test (GLRT).

That the GLRT is invariant is an important property in light of its widespread use and the increasing use of invariance principles to obtain tests for signal processing applications [6]. For example, for the class of matched subspace detectors, Scharf and Friedlander [4] note the invariance of the GLRT and show that it is uniformly most powerful invariant (UMPI). For cases in which the UMPI test does not exist, an UMPI-inspired performance bound can be used to evaluate the suboptimal performance of the GLRT (and other invariant tests).

The following references provide additional examples of invariant tests in the engineering literature. The adaptive subspace detectors of Kraut and Scharf [7], [8] extend the matched subspace detection problem to the case of unknown covariance using GLRTs. Kelly [9] provides a GLRT, for an invariant adaptive detection problem, that is invariant and UMPI. Kay and Gabriel [10] provide an invariant GLRT for a problem in which the UMPI test does not exist and compare it with the UMPI bound. Nicolls and de Jager [11] provide an example for which the GLRT is invariant, but is not the UMPI test that exists. Bose and Steinhardt [12] address an invariant problem, for which the GLRT may be intractable, and use maximal invariants to facilitate the search for reasonable tests among the class of invariant tests. Kraut and Krolik [13] generalize the invariance group of Bose and Steinhardt to obtain a scalar GLRT that is a maximal invariant for the problem, and Kraut *et al.* [14] further show that it has momotone likelihood ratio and hence is UMPI.

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II. GLRT Invariance

The problem is described in terms of a hypothesis test, with the null hypothesis denoted by $\mathcal{H}_0$ with parameter belonging to the parameter space Ω_0, and the alternate hypothesis denoted by $\mathcal{H}_1$ with parameter belonging to Ω_1. As stated in Lehmann [1] (and elsewhere), the problem of testing $\mathcal{H}_0 : \theta \in \Omega_0$ against $\mathcal{H}_1 : \theta \in \Omega_1$ remains invariant with respect to a group of transformations if the distribution remains in the same family and the parameter spaces are preserved.

We first obtain a density relationship that can be used as a necessary and sufficient condition to show that the distributions are preserved over a set of transformations $g \in G$. Initially using the notation of Lehmann [1], let x be distributed according to a probability distribution denoted by P_θ, $\theta \in \Omega$, and let $g \in G$ be a transformation acting over the sample space X. The distribution of gx belongs to the same family of distributions with perhaps a different value of the parameter, denoted $\overline{g}\theta$, which is an element of the original parameter space Ω (i.e., $\overline{g}\theta$ and θ belong to the same hypothesis). Lehmann denotes the distribution relationship as $P_\theta\{gx \in A\} = P_{\overline{g}\theta}\{x \in A\}$ for all Borel sets A. To gain some insight into the meaning of this relationship, we define an interval $A = (a, b)$, and for illustration assume that g acts monotonically (although the result holds more generally). Expanding the left-hand side gives

$$\begin{aligned} P_\theta\{gx \in A\} &= P_\theta\{gx < b\} - P_\theta\{gx < a\} \\ &= P_\theta\{x < g^{-1}b\} - P_\theta\{x < g^{-1}a\} \\ &= \int_{g^{-1}a}^{g^{-1}b} p_x(\zeta;\theta)d\zeta. \end{aligned}$$

Letting $\eta = g\zeta$ and changing variables, this integral can be written as $\int_a^b p_x(g^{-1}\eta;\theta)|(dg^{-1}\eta)/d\eta|d\eta$. A similar expansion of $P_{\overline{g}\theta}\{x \in A\}$ gives $\int_a^b p_x(\eta;\overline{g}\theta)d\eta$. Therefore, we have that $p_x(g^{-1}\eta;\theta)|dg^{-1}\eta/d\eta| = p_x(\eta;\overline{g}\theta)$. Changing the dummy variable and moving the Jacobian to the other side of the equality for notation purposes, we write

$$p_x(x;\theta) = p_x(gx;\overline{g}\theta)\left|\frac{dgx}{dx}\right|. \quad (1)$$

It is now shown that given a problem that is invariant with respect to a set of transformations, say all $g \in G$, then the GLRT for that problem is also invariant. The GLRT is obtained by replacing the unknown parameters under each hypothesis by their MLEs under that hypothesis. This can be written as

$$L(x)=\frac{\max_{\theta\in\Omega_1} p(x;\theta)}{\max_{\theta\in\Omega_0} p(x;\theta)}.$$

To see that the GLRT is invariant with respect to transformations gx, for $g\in G$, consider

$$L(gx)=\frac{\max_{\theta\in\Omega_1} p(gx;\theta)}{\max_{\theta\in\Omega_0} p(gx;\theta)}.$$

Replacing the parameter notation θ by $\overline{g}\theta$ (with $\overline{g}$ an invertible transformation) and inserting a factor to the numerator and denominator that will be used later gives

$$L(gx)=\frac{\max_{\overline{g}\theta\in\Omega_1} p(gx;\overline{g}\theta)\left|\frac{dgx}{dx}\right|}{\max_{\overline{g}\theta\in\Omega_0} p(gx;\overline{g}\theta)\left|\frac{dgx}{dx}\right|}.$$

Note that $|dgx/dx|$ can be inserted here, since it is assumed not to depend on θ. Using the invariance relationship of (2) gives the ratio

$$L(gx)=\frac{\max_{\overline{g}\theta\in\Omega_1} p(x;\theta)}{\max_{\overline{g}\theta\in\Omega_0} p(x;\theta)}=\frac{\max_{\theta\in\overline{g}^{-1}\Omega_1} p(x;\theta)}{\max_{\theta\in\overline{g}^{-1}\Omega_0} p(x;\theta)}.$$

And since the parameter spaces are preserved, $g\theta$ and θ belong to the same hypotheses

$$L(gx)=\frac{\max_{\theta\in\Omega_1} p(x;\theta)}{\max_{\theta\in\Omega_0} p(x;\theta)}=L(x).$$

And hence, $L(gx)=L(x)$ for all $g\in G$. Given that the hypothesis testing problem is invariant, then the GLRT is invariant.

III. Examples

Three examples are provided. All three are invariant hypothesis testing problems. The GLRTs are derived and are found to be invariant as expected. The notation used for these examples is first summarized. The vector $\mathbf{x}$ and its elements are the original N data samples. The vector $\mathbf{y}$ and its elements are the result of some group transformation acting on the original samples. The vector $\mathbf{m}$ is used for maximal invariants, and L denotes a test statistic.

Example 1: Detection of a DC Level of Unknown Amplitude in Gaussian Noise of Unknown Variance (Special Case of a Gaussian Linear Model [5]): GLRT is UMPI.

This detection problem formulated as a hypothesis test can be written as

$$\mathcal{H}_0: x[n]=w[n]$$
$$\mathcal{H}_1: x[n]=A+w[n]$$

for $n=0,1,\ldots,N-1$, where A is a deterministic unknown parameter that may be less than or greater than zero, and the $w[n]$ are white Gaussian noise (WGN) samples of unknown variance σ^2. A UMP test does not exist for this problem, since the hypothesis test is two-sided. This example is a special case of a problem studied in [6] and [15].

The probability density functions (pdfs) under the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by

$$\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]$$

and

$$\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

and are denoted by $p_{\mathbf{x}}(\mathbf{x};\mathcal{H}_0)$ and $p_{\mathbf{x}}(\mathbf{x};\mathcal{H}_1)$, respectively.

The problem is invariant under the group of scale transformations defined by the set $G=\{g: g\mathbf{x}=c\mathbf{x},\ \text{for } c\neq 0\}$. The pdf of this transformed set of samples is Gaussian, and the parameter spaces before and after the transformation are the same. For example, the mean is zero under $\mathcal{H}_0$ and nonzero under $\mathcal{H}_1$ for any element of the set G. It is easy to see in this case that the density conditions of (1) are met.

The GLRT is obtained by substituting the MLEs of A and σ^2 under each hypothesis into these pdfs and then constructing the likelihood ratio. The MLEs are $\hat{A}=(1/N)\sum_{n=0}^{N-1}x[n]$, $\hat{\sigma}_0^2=(1/N)\sum_{n=0}^{N-1}x^2[n]$, and $\hat{\sigma}_1^2=(1/N)\sum_{n=0}^{N-1}(x[n]-\hat{A})^2$, where hats denote estimators and subscripts denote the hypothesis to which it belongs. Substituting these into the pdf equations $p_{\mathbf{x}}(\mathbf{x};\mathcal{H}_0)$ and $p_{\mathbf{x}}(\mathbf{x};\mathcal{H}_1)$ above, constructing the likelihood ratio, and simplifying, gives the statistic $\hat{\sigma}_1^2/\hat{\sigma}_0^2$. We will use a monotonically increasing function of this as the GLRT

$$L_{G1}(\mathbf{x})=\left|\frac{\sum_{n=0}^{N-1}x[n]}{\left(\sum_{n=0}^{N-1}x^2[n]\right)^{1/2}}\right|>\gamma_{G1}$$

where the threshold γ_{G1} is selected to satisfy the probability of the false-alarm requirement. The GLRT is scale invariant as expected, since $L_{G1}(c\mathbf{x})=L_{G1}(\mathbf{x})$ for $c\neq 0$

One approach to deriving the UMPI test is by constructing a likelihood ratio using the pdfs of a maximal invariant statistic. A maximal invariant for this problem is

$$\mathbf{m}=\begin{bmatrix}\frac{x[0]}{x[N-1]} & \frac{x[1]}{x[N-1]} & \cdots & \frac{x[N-2]}{x[N-1]}\end{bmatrix}^T.$$

We need the density of $\mathbf{m}$ under each hypothesis. To obtain the density of $\mathbf{m}$, we first transform the original data $\mathbf{x}$ to $\mathbf{m}_1=[(x[0]/x[N-1])\ (x[1]/x[N-1])\ \cdots\ (x[N-2]/x[N-1])\ x[N-1]]^T$, which is of the same dimension as the original $\mathbf{x}$ and then marginalize over $x[N-1]$ to obtain the pdf of the maximal invariant $\mathbf{m}$.

Using the standard formula for transformation of a random variable, we first obtain the pdf for $\mathbf{m}_1$, $p_{\mathbf{m}_1}(\mathbf{m}_1;\mathcal{H}_1)$, which is

$$\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[\frac{-1}{2\sigma^2}\left(\sum_{n=0}^{N-2}(m[n]m[N-1]-A)^2+(m[N-1]-A)^2\right)\right]|m[N-1]|^{N-1}$$

where $|m[N-1]|^{N-1}$ is the Jacobian factor. The exponential factor can be expanded as

$$\exp\left[\frac{-1}{2\sigma^2}\left(m^2[N-1]\left(\sum_{n=0}^{N-2}m^2[n]+1\right)-2Am[N-1]\left(\sum_{n=0}^{N-2}m[n]+1\right)+NA^2\right)\right].$$

Using this, marginalize the pdf of $\mathbf{m}_1$ over $m[N-1]$, and making the obvious extension for $\mathcal{H}_0$, the ratio $p_{\mathbf{m}}(\mathbf{m};\mathcal{H}_1)/p_{\mathbf{m}}(\mathbf{m};\mathcal{H}_0)$ is as in (2), shown at the bottom of the page. Applying a change of variables where we let $z = (\gamma^2(\sum_{n=0}^{N-2} m^2[n] + 1))^{1/2}$, and making the substitutions and cancelations between denominator and numerator, the ratio becomes as in (3), shown at the bottom of page. This is increasing in $|(\sum_{n=0}^{N-2} m[n] + 1)/(\sum_{n=0}^{N-2} m^2[n] + 1)^{1/2}|$ and in terms of the original samples is $|(\sum_{n=0}^{N-1} x[n])/(\sum_{n=0}^{N-1} x^2[n])^{1/2}|$. This is the the GLRT, and hence the GLRT is UMPI.

Example 2: Detection of a Known Signal of Unknown Location in Gaussian Noise of Known Variance: GLRT is not UMPI, and UMPI exists.

The detection problem formulated as a hypothesis test can be written as

$$\begin{aligned}\mathcal{H}_0 &: x[n] = w[n]\\ \mathcal{H}_1 &: x[n] = A\delta[n-k] + w[n]\end{aligned}$$

where $\delta[n]$ is a discrete delta function, and $k \in \{0, 1, \ldots, N-1\}$ is unknown. As before, $n = 0, 1, \ldots, N-1$, and the $w[n]$ are samples of WGN except in this example the signal amplitude $A > 0$ and the variance σ^2 are *known*. Under $\mathcal{H}_1$, the N samples consist of $N-1$ noise samples and one signal plus noise sample. This example is a special case of the problem considered by Nicolls and de Jager [11] and can be found in [16].

As in the previous examples, we first establish the invariances of the problem. Since the location of the sample containing the signal is unknown, we should expect that the ordering of the samples is not a relevant feature of the problem. The problem is invariant under the rotation group $G = \{g : g\mathbf{x} = \mathbf{P}^l\mathbf{x},\ \text{for } l = 0, 1, \ldots, N-1\}$, where $\mathbf{P}^l$ is the right-shift permutation matrix that shifts the samples l positions. This transformation group is called a cyclic permutation in [11]. The densities of the original samples $p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_0)$ and $p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_1)$ are

$$\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right]$$

and

$$\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1} (x[n] - A\delta[n-k])^2\right].$$

The problem is easily shown to be invariant using (1).

To find the GLRT for this problem, we need the MLE of the unknown parameter k under $\mathcal{H}_1$. The equation above for $p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_1)$ is maximized when the expression of the exponent is minimized or

$$\hat{k} = \mathop{\arg\min}_{k\in[0,N-1]} \left(\sum_{n=0}^{N-1} (x[n] - A\delta[n-k])^2\right).$$

The value of k that minimizes this expression is that which maximizes $2Ax[k]$. Hence, the MLE is $\hat{k} = \arg\max_{k\in[0,N-1]} x[k]$. The GLRT is obtained by constructing the ratio $p(\mathbf{x}; \hat{k}, \mathcal{H}_1)/p(\mathbf{x}; \mathcal{H}_0)$, which is

$$\frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n] + \frac{A}{\sigma^2}\max(\mathbf{x}) - \frac{A^2}{2\sigma^2}\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right]}$$

where $\max(\mathbf{x})$ is used for $x[\hat{k}]$. This is increasing in $\max(\mathbf{x})$. Thus, the GLRT is $L_{G2}(\mathbf{x}) = \max(\mathbf{x}) > \gamma_{G2}$ where the threshold is set to satisfy the Pfa requirement. This is invariant, since $L_{G2}(\mathbf{P}^l\mathbf{x}) = L_{G2}(\mathbf{x})$ as expected.

For comparison with the optimal invariant test, we outline the derivation of the UMPI test. A maximal invariant for G is given by $\mathbf{m} = \mathbf{P}^l\mathbf{x}$, where l is selected such that the maximum sample (the probability that two or more are the same is zero) is the last element of the vector, i.e., in the $N-1$ position. The pdf of $\mathbf{m}$ is

$$\begin{aligned}p_{\mathbf{m}}(\mathbf{m}; \mathcal{H}_1) &= \sum_{l=0}^{N-1} p_{\mathbf{x}}(\mathbf{P}^{-l}\mathbf{m}; \mathcal{H}_1)\\ &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[\frac{-1}{2\sigma^2}\mathbf{m}^T\mathbf{m}\right] \exp\left[-\frac{A^2}{2\sigma^2}\right]\\ &\quad \cdot \sum_{l=0}^{N-1} \exp\left[\frac{A}{\sigma^2}\mathbf{e}_k^T\mathbf{P}^{-l}\mathbf{m}\right]\end{aligned}$$

and under $\mathcal{H}_0$ is $(2\pi\sigma^2)^{-N/2}\exp[(-1/2\sigma^2)\mathbf{m}^T\mathbf{m}]$. This signal $\delta[n-k]$ is denoted in vector form using the natural basis vector $\mathbf{e}_k$. Using these to construct the ratio and making the appropriate cancelations results in the statistic $\sum_{l=0}^{N-1}\exp[(A/\sigma^2)\mathbf{e}_k^T\mathbf{P}^{-l}\mathbf{m}]$. This can be written in terms of $\mathbf{x}$, which we will use as the UMPI test

$$L_{U2}(\mathbf{x}) = \sum_{l=0}^{N-1} \exp\left[\frac{A}{\sigma^2}x[l]\right] > \gamma_{U2}.$$

Since A/σ^2 is known, and there are no unknowns under the null hypothesis, the UMPI test exists. Observe that for this example the GLRT is not UMPI. This is a case in which the UMPI test exists and it is *not* given by the GLRT.

$$\frac{\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2\sigma^2}\left(\gamma^2\left(\sum_{n=0}^{N-2} m^2[n] + 1\right) - 2A\gamma\left(\sum_{n=0}^{N-2} m[n] + 1\right) + NA^2\right)\right] |\gamma|^{N-1} d\gamma}{\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2\sigma^2}\gamma^2\left(\sum_{n=0}^{N-2} m^2[n] + 1\right)\right] |\gamma|^{N-1} d\gamma} \quad (2)$$

$$\frac{\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2\sigma^2}\left(z^2 - z2A\left(\frac{\sum_{n=0}^{N-2} m[n] + 1}{\left(\sum_{n=0}^{N-2} m^2[n] + 1\right)^{\frac{1}{2}}}\right) + NA^2\right)\right] |z|^{N-1} dz}{\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2\sigma^2}z^2\right] |z|^{N-1} dz} \quad (3)$$

$$\frac{\int_0^\infty \left(\exp\left[-\left(\sum_{n=0}^{N-2} \left| \frac{m[n]}{T(\mathbf{m})} z + \frac{A}{b} \right| + \left| \frac{z}{T(\mathbf{m})} + \frac{A}{b} \right| \right) \right] + \exp\left[-\left(\sum_{n=0}^{N-2} \left| \frac{m[n]}{T(\mathbf{m})} z - \frac{A}{b} \right| + \left| \frac{z}{T(\mathbf{m})} - \frac{A}{b} \right| \right) \right] \right) |z|^{N-1} dz}{2 \int_0^\infty \exp[-z] |z|^{N-1} dz}$$

Example 3: Detection of a DC Level of Unknown Amplitude in Laplacian Noise of Unknown Variance: GLRT is not UMPI, and UMPI does not exist.

This detection problem can be written as in the first example, except that now the noise is Laplacian. The pdfs under the hypotheses for this example are given by

$$p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_0) = \frac{1}{(2b)^N} \exp\left[\frac{-1}{b} \sum_{n=0}^{N-1} |x[n]| \right]$$
$$p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_1) = \frac{1}{(2b)^N} \exp\left[\frac{-1}{b} \sum_{n=0}^{N-1} |x[n] - A| \right].$$

As in the first example, A can be less than or greater than zero. The parameter b is greater than zero. This problem is scale invariant as in Example 1.

To find the GLRT, we first determine the MLEs of the unknown parameters under each hypothesis. These MLEs can be shown to be $\hat{A} = \text{med}(\mathbf{x})$, $\hat{b}_0 = (1/N)\sum_{n=0}^{N-1} |x[n]|$, and $\hat{b}_1 = (1/N)\sum_{n=0}^{N-1} |x[n] - \text{med}(\mathbf{x})|$, where $\text{med}(\mathbf{x})$ is the median of $\mathbf{x}$. Substituting these into $p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_0)$ and $p_{\mathbf{x}}(\mathbf{x}; \mathcal{H}_1)$ given above, constructing the ratio, and simplifying gives the GLRT

$$L_{G3}(\mathbf{x}) = \frac{\sum_{n=0}^{N-1} |x[n]|}{\sum_{n=0}^{N-1} |x[n] - \text{med}(\mathbf{x})|} > \gamma_{G3}.$$

Observe that this test statistic is scale invariant as expected, since $L_{G3}(c\mathbf{x}) = L_{G3}(\mathbf{x})$.

To find the UMPI test, we use the same maximal invariant as in the first example, and the pdf will be obtained using the same method. The pdf of $\mathbf{m}_1$ under $\mathcal{H}_1$ denoted by is $p_{\mathbf{m}_1}(\mathbf{m}_1; \mathcal{H}_1)$ is

$$\frac{1}{(2b)^N} \exp\left[\frac{-1}{b} \left(\sum_{n=0}^{N-2} |m[n]m[N-1] - A| + |m[N-1] - A| \right) \right] |m[N-1]|^{N-1}.$$

Marginalizing over $m[N-1]$ to obtain the pdf of $\mathbf{m}$, we have

$$\begin{aligned} p_{\mathbf{m}}(\mathbf{m}; \mathcal{H}_1) &= \int_{-\infty}^{\infty} \frac{1}{(2b)^N} \exp\left[\frac{-1}{b} \left(\sum_{n=0}^{N-2} |\gamma m[n] - A| + |\gamma - A| \right) \right] \cdot |\gamma|^{N-1} d\gamma \\ &= \int_0^\infty \frac{1}{(2b)^N} \left(\exp\left[\frac{-1}{b} \left(\sum_{n=0}^{N-2} |\gamma m[n] + A| + |\gamma + A| \right) \right] + \exp\left[\frac{-1}{b} \left(\sum_{n=0}^{N-2} |\gamma m[n] - A| + |\gamma - A| \right) \right] \right) \cdot |\gamma|^{N-1} d\gamma. \end{aligned}$$

Writing the integral over the domain 0 to ∞ and expanding as before, letting $z = \gamma / b(\sum_{n=0}^{N-2} |m[n]| + 1)$, and making the substitutions and appropriate cancelations, the ratio becomes as shown by the equation at the top of the page, where $T(m) = \sum_{N-1}^{n=0} |m[n]| + 1$. Since the denominator is not a function of the data, the ratio is increasing in the numerator alone, which can be considered the test statistic. Note that unlike in the integral of (3) where we could extract an increasing function of the data, revealing the independence on any unknown parameters, we cannot do so here.

Since the test statistic requires knowledge of $|A/b|$, which is unknown, it cannot be implemented, and hence the UMPI test does not exist for this example. This is a case where the GLRT is invariant, but is not UMPI, since the UMPI test does not exist.

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