Improved Fast Adaptive Subspace Tracking

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• Given a sequence of length $n$ column vectors $\mathbf{x}_t$, for $t = 1, 2, \cdots$, we can define the $n \times c$ matrices $M$ and $\tilde{M}$ as

$$M = \begin{bmatrix} \mathbf{x}_{t-c} & \mathbf{x}_{t-c+1} & \cdots & \mathbf{x}_{t-1} \end{bmatrix}$$

$$\tilde{M} = \begin{bmatrix} \mathbf{x}_{t-c+1} & \cdots & \mathbf{x}_{t-1} & \mathbf{x}_t \end{bmatrix}$$

• Assuming we have $\Sigma'$ – the $r$ largest singular values of $M$ and $U'$ – the corresponding left singular vectors of $M$, we would like to determine $\tilde{\Sigma}'$ – the $r$ largest singular values of $\tilde{M}$ and $\tilde{U}'$ – the corresponding left singular vectors of $\tilde{M}$ without performing a full SVD on $\tilde{M}$. 
The IFAST Algorithm

1) \( q_1 = \frac{(I - U'U'^H) x_{t-c}}{\| (I - U'U'^H) x_{t-c} \|} \)

2) \( q_2 = \frac{(I - [U' q_1][U' q_1]^H) x_t}{\| (I - [U' q_1][U' q_1]^H) x_t \|} \)

3) \( \tilde{F} = [U' Q]^H \tilde{M} \tilde{M}^H [U' Q] \)

4) \( U_F \Sigma_F U_F^H = \tilde{F} \)

5) \( \tilde{U}' = [U' Q] U_F \)
\( \tilde{\Sigma}' = \sqrt{\Sigma_F} \)

The columns of the new matrix \( \tilde{M} \), can be well approximated by projecting them onto the \( r + 2 \) dimensional subspace which has an orthogonal basis consisting of the \( r \) columns of \( U' \) Gram-Schmidt augmented by \( q_1 \) and \( q_2 \).

After projecting the columns of \( \tilde{M} \) onto this \( r + 2 \) dimensional subspace, we perform a small \( r + 2 \) dimensional SVD to obtain a good \( r \)-dimensional approximation subspace.
Efficiently Computing $\tilde{F}$

- Computing $\tilde{F}$ directly in step 3 is computationally costly because $\tilde{M}$ is an $n \times c$ matrix, and $U'$ is an $n \times r$ matrix.

- We can write $\tilde{F}$ as

$$\tilde{F} = \begin{bmatrix} \tilde{F}_c & U'^H \tilde{M} \tilde{M}'Q \\ Q^H \tilde{M} \tilde{M}'U' & Q^H \tilde{M} \tilde{M}'Q \end{bmatrix}$$

where the $r \times r$ matrix $\tilde{F}_c = U'^H \tilde{M} \tilde{M}'U'$ is

$$\tilde{F}_c = \Sigma'^2 - U'^H x_{t-c} x_{t-c}^H U' + U'^H x_t x_t^H U'$$

which is the sum of a diagonal matrix plus two rank one matrices.
• \( n = c = 64 \), complex data
• Similar plot for other matrix dimensions
• **Step 3** \( (O(32nc) \) creation of \( \tilde{F} \)) dominates for \( r < 15 \)
• **Step 4** \( (O(31r^3) \) SVD of \( \tilde{F} \)) dominates for \( r > 15 \)
• **Step 5** \( (O(8nr^2) \) rotation of \( U_F \)) similar to step 4 for \( r < 15 \)
• When the equivalent of \( \tilde{F} \) for FAST is computed similarly to how it is done in IFAST, their computation is about the same.
Accuracy

error in $\tilde{\sigma}_1^2$ in dB

error in $\tilde{\sigma}_2^2$ in dB

error in $\tilde{\sigma}_3^2$ in dB

$SNR = 1.4, 0.73, 1.4$

$r$, signal subspace rank

Normalized Frequency
IFAST Summary

Key Points:

• Accurate estimates of $r$ largest singular values and corresponding left singular vectors

• Computational complexity of $O(nr^2)$

• No initial SVD required, can start with a single vector and grow $M$ by making $Q$ only one column

Additional Points:

• Robust to truncating $r$ due to computational limitations

• Error in singular values proportional to error in angle for corresponding singular vector
Rectangular Window Update

- Given the SVD of $M = U \Sigma V^H$, and defining
  
  \[ a = U^H x_{t-c}, \quad b = U^H x_t \]

  we can write

  \[ \tilde{G} = U^H \tilde{M} \tilde{M}^H U = \Sigma^2 - aa^H + bb^H \]

  which is a diagonal matrix plus two rank one matrices.

- The eigenvalues of $\tilde{G}$, which are the squares of the singular values of $\tilde{M}$, are the roots of the rank-two secular equation

  \[ w(\lambda) = \left(1 - \sum_{j=1}^{n} \frac{|a_j|^2}{\sigma_j^2 - \lambda}\right) \left(1 + \sum_{j=1}^{n} \frac{|b_j|^2}{\sigma_j^2 - \lambda}\right) + \left|\sum_{j=1}^{n} \frac{a_j^* b_j}{\sigma_j^2 - \lambda}\right|^2 \]
If we separate $M$ into an $r$ dimensional principal subspace, and the orthogonal $c - r$ dimensional one

$$M = \begin{bmatrix} U' & U' \perp \end{bmatrix} \begin{bmatrix} \Sigma' & 0 \\ 0 & \Sigma \perp \end{bmatrix} \begin{bmatrix} V' & V' \perp \end{bmatrix}^H$$

The eigenvalues values of $	ilde{G}' = U'^H \tilde{M} \tilde{M}^H U'$ are the roots of the rank-two secular equation

$$w'(\lambda) = \left(1 - \sum_{j=1}^{r} \frac{|a_j|^2}{\sigma_j^2 - \lambda}\right) \left(1 + \sum_{j=1}^{r} \frac{|b_j|^2}{\sigma_j^2 - \lambda}\right) + \left|\sum_{j=1}^{r} \frac{a_j^* b_j}{\sigma_j^2 - \lambda}\right|^2$$

which differs from the full secular equation only by the upper limit of the summation.
IFAST Approximation

- Rotate the two columns of $Q$ from steps one and two of the IFAST algorithm such that $\hat{\Sigma} = Q^H M M^H Q$ is diagonal

- Assume that $U'$ is not an approximation, therefore $U'^H \tilde{M} \tilde{M}^H Q$ equals $\Sigma'^2 U'^H Q$, which must be zero

- The secular equation for $\tilde{F} = [U' \ Q]^H \tilde{M} \tilde{M}^H [U' \ Q]$, which we will call $w''(\lambda)$, is $w'(\lambda)$ with two additional terms in each summation

\[ - \sum_{j=1}^{2} \frac{|q_j^H \mathbf{x}_{t-c}|^2}{\hat{\sigma}_j - \lambda}, \quad \sum_{j=1}^{2} \frac{|q_j^H \mathbf{x}_t|^2}{\hat{\sigma}_j - \lambda}, \quad \sum_{j=1}^{2} \frac{q_j^H \mathbf{x}_t \mathbf{x}_{t-c}^H q_j}{\hat{\sigma}_j - \lambda} \]

which are equal to the first two terms of binomial expansions of the $n - r$ missing terms of $w(\lambda)$
The poles of $w(\lambda)$ are the squares of the singular values of $M$

The roots of $w(\lambda)$ are the squares of the singular values of $\tilde{M}$
Secular Function Differences

Error in secular function in $\log_{10}(|\text{difference}|)$
Analysis Summary

Key Points:

- Present the new rank-two secular equation, which is required to analyze the sliding window update eigenproblem
- Comparative analysis of full dimension secular equation with the secular equation for the IFAST algorithm
- Application of these results to show why IFAST has high accuracy

Additional Points:

- Method can be used to analyze any algorithm that can be written as a rank-two (or rank-one) modification to a diagonal matrix
- Can give estimate of error in each singular value estimate