Detection and Estimation in Non-Stationary Environments

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***Abstract*— In this paper we describe a matrix based method for estimating both a signal subspace dimension as well as signal parameters when one has an array of sensors, multiple exponential signals, and significant signal changes after a small number of snapshots. This method combines the technique of creating Hankel or Toeplitz matrices from single-channel data with methods for sensor-array processing using multiple array snapshots.**

I. Introduction

In this paper we describe a matrix based method for estimating a signal subspace dimension while controlling the probability of false alarm as well as a method for estimating parameters of that subspace. A common scenario where this method applies is when the data comes from an array of sensors, where each array snapshot consists of multiple exponential signals, but significant changes occur in the exponential signals after a small number of snapshots.

In estimation these exponentials are signals which have parameter values, such as arrival angles, which we wish to estimate. In detection these exponentials are components of interference which we temporarily treat as "signals" to be enhanced, prior to subtraction. In both cases reduced-rank approximation to a data matrix is used to improve the signal-to-noise ratio of the exponential components, prior to subsequent signal processing.

We discuss a data matrix structure, which we call the *block Hankel* structure, that combines the benefits of creating a Hankel matrix for single-channel data with the advantages of multiple channels without changing the signal subspace dimension.

Next, we present a method for estimating the dimension of the signal subspace while controlling the probability of false alarms. If we estimate that the dimension of the signal subspace is larger than the correct dimension, we say that a false alarm has occurred. This method is an extension of the method of Tufts and Shah [1] which applies to Hankel matrices. We then introduce an approximation to this matrix rank tracking method that reduces the computation significantly, while maintaining performance.

A major motivation for us is widening the applicability of the FAST algorithm [2] [3] for subspace tracking. Rank tracking, implemented using the tests of Frobenius-norm "energy" of subspace matrices is described in section III below. This is an important part of FAST. However, until now, the rank-tracking in FAST could not be applied to *block Hankel* structure matrices.

Finally, we present some results of applying this method to some simulated sonar array data which was generated by Norman Owsley. Here we estimate the number of sinusoids, their amplitudes, and their spatial frequencies for each snapshot.

II. Constructing a Block Hankel Matrix

It is well known [4] [5] [6] that a length N single-channel signal vector $\mathbf{s}_t$, which is a linear combination of k complex exponentials can be made into an $r_H \times c_H$ Hankel or Toeplitz matrix which will have rank k, if $\min(r_H, c_H) \geq k$. The vector $\mathbf{s}_t$ can be written as

$$\mathbf{s}_t = \sum_{l=1}^{k} c_{l,t}\mathbf{z}_l, \tag{1}$$

where each discrete exponential signal has the form

$$\mathbf{z}_l = [1 \;\; Z_l^1 \;\; Z_l^2 \;\; \cdots \;\; Z_l^{N-1}]^T, \tag{2}$$

in which Z_l is a complex number and the $c_{l,t}$ are the complex scale factors. The creation of a Hankel matrix is shown pictorially in figure 1. Note that $N = r_H + c_H - 1$.

When we have c different signal vectors, $\mathbf{s}_1, \mathbf{s}_2 \cdots \mathbf{s}_c$, with kc different complex scale factors $c_{l,t}$, but the same k exponentials, $\mathbf{z}_l$, we can create a *block Hankel* matrix by forming an $r_H \times c_H$ Hankel matrix from each $N \times 1$ signal vector, $\mathbf{s}_t$, then concatenating them together to form an $r_H \times c_H c$ matrix which will have rank k, if $\min(r_H, c_H c) \geq k$.

Given an $N \times c$ data matrix M, consisting of c snapshots of signal plus noise,

$$M = [\mathbf{s}_1 \;\; \mathbf{s}_2 \;\; \cdots \;\; \mathbf{s}_c] + [\mathbf{n}_1 \;\; \mathbf{n}_2 \;\; \cdots \;\; \mathbf{n}_c], \tag{3}$$

we can create an $r_H \times c_H c$ *block Hankel* matrix B. This is shown pictorially in figure 2. We often refer to a column of the original data matrix as a snapshot.

The noise component of M will increase its dimension when we create the Hankel blocks, and should continue to

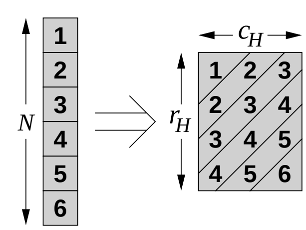

Fig. 1. Creating a Hankel matrix from a signal vector

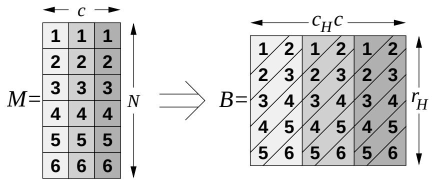

Fig. 2. Creating a Block Hankel matrix from multiple snapshots

fill the full vector space of M. As an example of how the *block Hankel* structure benefits us, if we have four snapshots of length 39 which contain eight complex exponentials, we can create four 32×8 Hankel blocks, which will give us a 32×32 *block Hankel* matrix in which the signal component is contained in an eight dimensional subspace, but the noise will span the full 32 dimensional vector space.

It should also be noted that when $Z_l = e^{jw_l}$ in (2) we can create a forward-backward matrix [7] where the backward matrix is created by conjugating and reversing the elements of M along the columns. The results in this paper are only shown for the forward matrix, but can be easily extended to contain both the forward and backward matrices.

III. Estimating the Signal Subspace Rank

To estimate the rank of the signal subspace, we take the SVD of $B = U\Sigma V^H$, and define the energy in the subspace which is orthogonal to the hypothesized signal subspace

$$S_{k+1} = \sum_{l=k+1}^{r_H} \sigma_l^2 = ||(I - U_k U_k^H)B||_F^2 \tag{4}$$

where S_{k+1} is the sum of the squares of the singular values of B less the k largest. In (4) σ_l^2 is the square of the lth largest singular value of B, and U_k is a matrix of the k leftmost columns of U.

Using the SVD of the matrix B, we ask questions based on current hypotheses about the rank of the signal subspace. The zero$^{\text{th}}$ hypothesis, H_0, is that the rank of the signal subspace, the signal portion of the matrix B, is at least zero. If the signal portion of the matrix B has exactly rank zero, then there is no signal and the matrix B consists entirely of noise values. The k^{th} hypothesis, H_k, is that the rank of the signal portion of the matrix B is at least k.

The question that we ask at the k^{th} stage (if we get that far) is "Given H_k, that we have found out that the signal rank is at least k, can we now say that H_{k+1} is true?" To do this we test whether or not the sum S_{k+1}, the energy in the orthogonal subspace, is greater than a prescribed threshold value T_k. If $s_{k+1} < T_k$, we say that the signal rank is k and stop our tests. If $S_{k+1} > T_k$ we say that H_{k+1} is true and continue our tests. The behavior of this sequence of tests is controlled by choosing each threshold value so that the associated probability of false alarm is a value α which we choose.

We choose a false alarm probability, α, and compute the threshold values, T_k, for each k

$$P(S_{k+1} > T_k | \bar{H}_{k+1}) = \alpha, \qquad 0 \le k \le r_H \tag{5}$$

where $\bar{H}_{k+1}$ is the complimentary hypothesis that H_{k+1} is not true, and the value of S_{k+1} is produced only by noise. Note that α should be the same for all k.

Finally, we find the largest k such that S_{k+1} is greater than T_k, and our rank is that k. This is a simple iterative step that is trivial to implement in practice.

These steps for estimating the rank apply for any matrix, structured or not, because the original matrix M is a *block Hankel* matrix with $r_H = N$ and $c_H = 1$, while a Hankel matrix is a *block Hankel* matrix with $c = 1$ and $c_H > 1$.

IV. Calculating the Threshold Values

The threshold values are chosen to control the probability of false alarm at each stage. Therefore the pertinent probability density is that of the noise alone in the orthogonal subspace.

A method for calculating the thresholds T_k for an unstructured matrix, such as M, is presented in [8], and a method for calculating the thresholds in the Hankel case, which is easily extended to the *Block Hankel* case, is presented in [1].

The difficulty with the method in [1] is that it requires the partial fraction expansion of a polynomial with root multiplicity of $2c$. For the case of a Hankel matrix this is not a big problem because $c = 1$, but for the *Block Hankel* case this not only requires a lot of computation, but also generally requires variable precision arithmetic.

Here we present a method to approximate the threshold values which can easily be implemented in a practical system. They are only a function of α, σ^2, and the matrix dimensions, r_H, c_H, and c. The values are compared with the results using the extension of the method in [1] as well as experimental results.

We now assume that the noise is distributed complex normal, $\mathbf{n}_t \sim \mathcal{CN}(0, I\sigma^2)$, with zero mean and variance σ^2. For the case H_0 (no signal present) the expected value, μ_B, and variance, σ_B^2, of the squared Frobenius norm of B are

$$\mu_B = E\left[||B||_F^2\right] = \sigma^2 r_m c_m c \tag{6}$$

and

$$\sigma_B^2 = \text{Var}\left(||B||_F^2\right) = \sigma^4 c(dr_m^2 + 2\sum_{i=1}^{r_m - 1} i^2) \tag{7}$$

where $r_m = \min(r_H, c_H)$ is the smaller dimension of a single Hankel block, $c_m = \max(r_H, c_H)$ is the longer dimension of a single block, and $d = |r_H - c_H| + 1$ is the number of full diagonals in a single block.

The distribution of $||B||_F^2$ is Chi-Square mixture, which is approximately a scaled Chi-Square with n degrees of freedom and scale factor $1/s_B$, therefore, because we know the mean and variance of any Chi-Square variable, we can say

$$n = E\left[\frac{||B||_F^2}{s_B}\right] = \frac{1}{s_B} E\left[||B||_F^2\right] \tag{8}$$

and

$$2n = \text{Var}\left(\frac{||B||_F^2}{s_B}\right) = \frac{1}{s_B^2}\text{Var}\left(||B||_F^2\right). \tag{9}$$

Combining equations 6, 7, 8, and 9, rearranging some terms, and solving for n and s_B we get

$$n = \frac{2\mu_B^2}{\sigma_B^2} = \frac{6cc_m^2}{3c_m - (r_m - 1/r_m)} \tag{10}$$

and

$$s_B = \frac{\sigma_B^2}{2\mu_B} = \sigma^2 \frac{r_m c_m c}{n} \tag{11}$$

It should be noted that n will generally not be an integer, but that is not a problem because the Chi-Square distribution can be evaluated for all real n.

For a given value of α, we can find T_0/s_B by evaluating the quantile (the inverse cumulative distribution function) of the Chi-Square distribution at $1-\alpha$.

$$\frac{T_0}{s_B} = F_n^{-1}(1-\alpha) \tag{12}$$

Since the quantile is only a function of α and n, which depends only on the matrix dimensions r_m, c_m, and c, we can calculate T_0/s_B before we know the variance of the noise, σ^2. If we then define

$$\hat{T}_0 = \sigma^2 \frac{T_0}{s_B} \tag{13}$$

which is essentially T_0 with the noise variance in s_B canceled out, then when we do get our estimate of the noise variance we can easily determine T_0 as

$$T_0 = \frac{\hat{T}_0}{\sigma^2}. \tag{14}$$

In figure 3 we show how well the Chi-Square approximation compares to the actual distribution which is a Chi-Square mixture.

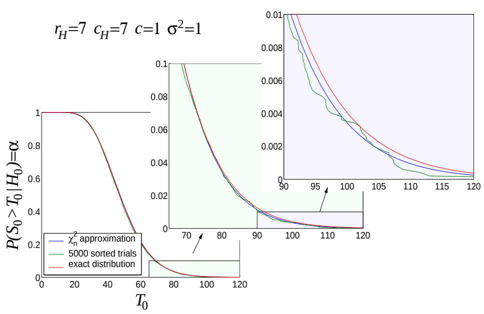

Fig. 3. False Alarm Probability vs. Threshold

V. Evaluating the Other Thresholds

Now that we have a method to calculate T_0, we need to be able to calculate the other thresholds T_k, for $k = 1 \cdots r_H$. Here we assume that if there are signals present in the data, the SNR is assumed to be above threshold [9]. That is, the probability of subspace swap is negligibly small and the signal singular vectors are independent of the noise.

We assume that to a good approximation, the mean and variance of the energy in the orthogonal subspace do not depend on the choice of signal subspace, as long as the noise does not affect this choice. Therefore, for convenience we replace replace U_k by the first k canonical vectors,

$$\hat{U}_k = [\, \mathbf{e}_1,\ \mathbf{e}_2, \cdots, \mathbf{e}_k \,] \tag{15}$$

where the kth canonical vector $\mathbf{e}_k$ is a length r_H column vector consisting of all zeros except a single one in the kth position,

$$\mathbf{e}_k = [\, \underbrace{0, \cdots, 0}_{k-1}, 1, \underbrace{0, \cdots, 0}_{r_H - k} \,]^T \tag{16}$$

we can use the method from section IV for calculating T_0 to calculate T_k by replacing r_H by $r_H - k$.

We see that when we take the product $(I - \hat{U}\hat{U}^H)B$ we zero out the first k rows of B but leave the rest of the matrix unchanged. This means that if hypothesis H_k applies, we can use our mean and variance calculations from the previous section along with our Chi-Square approximation. The mean estimate using $\hat{U}_k$ will be identical to the estimate using U_k, but the variance will not be correct because $||(I - UU^H)B||_F^2$ will actually be a Chi-Square mixture plus a Gaussian product mixture.

The reason for this approximation is to permit the thresholds to be calculated independently of the data.

VI. The Data

In this section we present some results using the techniques introduced in this paper on simulated data. We make the following assumptions about the data used in this section.

Each length 48 array snapshot $\mathbf{m}_t$, is a sum of k scaled complex sinusoids with fixed frequencies f_k, and random complex scale factors $c_{k,t} = A_k e^{j\psi_k}$, plus complex white noise $\mathbf{n}_t$

$$\mathbf{m}_t = \mathbf{n}_t + \sum_{l=1}^{k} c_{k,t}\mathbf{z}_k \tag{17}$$

where from (2), $Z_k = e^{-j2\pi f_k f_s}$ with $f_s = 0.4$, and the random components have distributions

$$\mathbf{n}_t \sim \mathcal{CN}(0, I\sigma^2) \tag{18}$$
$$A_k \sim \mathcal{N}(0, \sigma_k^2) \tag{19}$$
$$\psi_k \sim \mathcal{U}(0, 2\pi) \tag{20}$$

Because we know exactly how the simulated data was generated we also know that these assumptions are simplifications of the actual data, and do not truly reflect the far more complex model used for generating the data. In actuality, the $\mathbf{z}_k$s are not truly sinusoidal (which is why we didn't use forward-backward *block Hankel* matrices), the f_ks are slowly changing between snapshots at different rates, and the $c_{k,t}$ have a much more complicated distribution.

The steps that we use to come up with the results in this section are as follow.

- Determine c, the number of sequential snapshots to use. This will depend on the stationarity of the signal subspace.
- Determine r_H and c_H, the dimensions of the Hankel blocks. This will depend on the rank of the signal

subspace as well as other factors related to the method of parameter estimation that is used.

- Determine α, the probability of false alarm, then calculate the thresholds T_k or $\hat{T}_k$ for each k.
- Create the *block Hankel* matrix B, and take its SVD.
- Estimate k, the signal subspace rank by comparing the sums of the squares of singular values of B to the thresholds.
- Estimate the possible target azimuths.
- Find the k azimuths corresponding to the signal subspace, and determine their signal level.

To estimate the possible target azimuths, we take the polynomial roots of the r_Hth left singular vector which will be orthogonal to the signal subspace. We know that it will have zeros corresponding to the frequencies of the sinusoids in the signal subspace [5] (as well as many other zeros).

To determine which k of the $r_H - 1$ possible azimuths correspond to the k sinusoids, we beamform the k largest left singular vectors toward all of the possible azimuths, then pick the azimuth which has the largest beamformed value for a given singular vector. Once we have picked an azimuth which corresponds to a singular vector, we say the energy at that azimuth is the singular value which goes with that singular vector.

Figure 4 shows the rank estimates for 1800 snapshots using a *block Hankel* matrix with dimensions $c = 8$, $r_H = 33$, and $c_H = 16$. These are the number of azimuth estimates that are plotted in fig. 5 and fig. 8.

Figure 5 shows the cosine of azimuth estimates using eight sequential snapshots and a *block Hankel* structure with $c = 8$, $r_H = 33$, and $c_H = 16$. Fig. 7 shows the cosine of azimuth estimates using the same eight sequential snapshots and no matrix structure with $c = 8$, $r_H = 48$, and $c_H = 1$. Fig. 6 shows the cosine of azimuth estimates using 24 sequential snapshots and no matrix structure with $c = 24$, $r_H = 48$, and $c_H = 1$.

The two tracks of most interest are the one that is leftmost between 500 and 1600 and the one that is rightmost between 800 and 1400 in fig. 5. These two tracks are about five orders of magnitude below the stronger tracks and very near the noise level. They do not even show up fig. 7 and are not very clear in fig. 6 which uses three times the amount of data.

Figure 8 is the same as 5 but with target strength indicated by a color. The colorbar in the figure shows the strength of the target in decibels.

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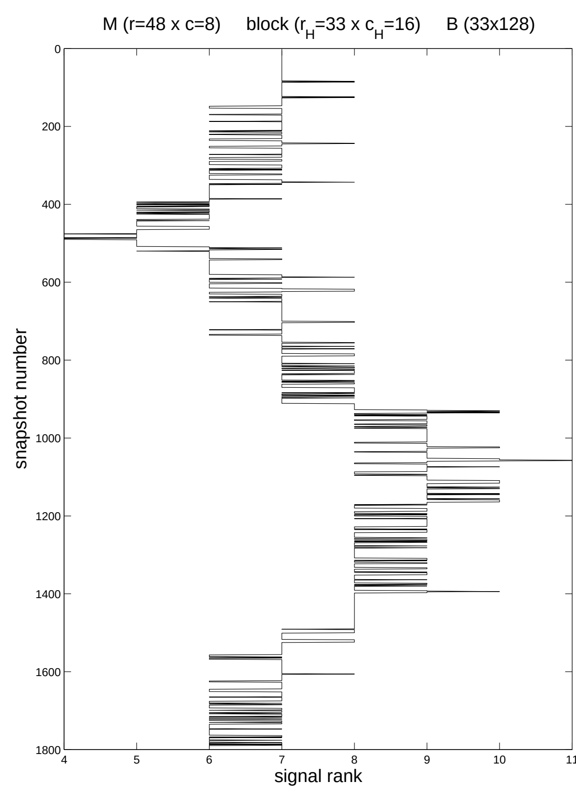

Fig. 4. Rank estimation for block Hankel matrix structure with eight sequential snapshots

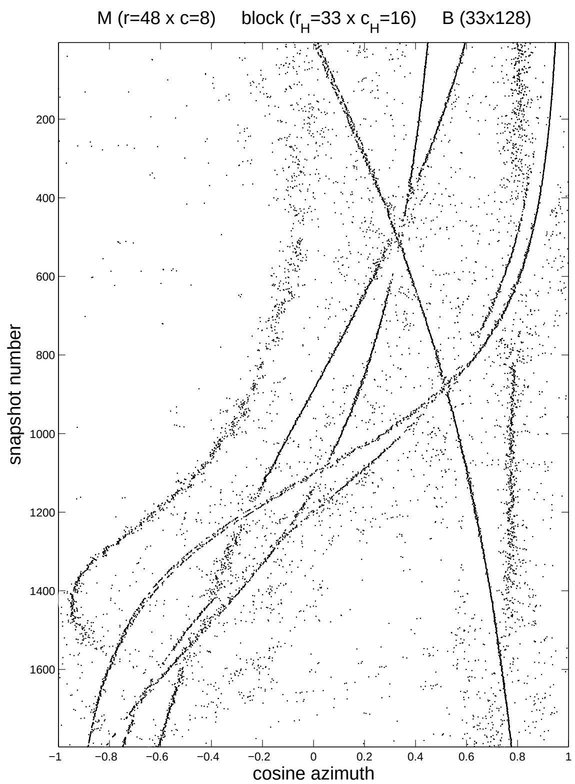

Fig. 5. Cosine of the azimuth of the k strongest sinusoids using eight sequential snapshots and block Hankel matrix structure

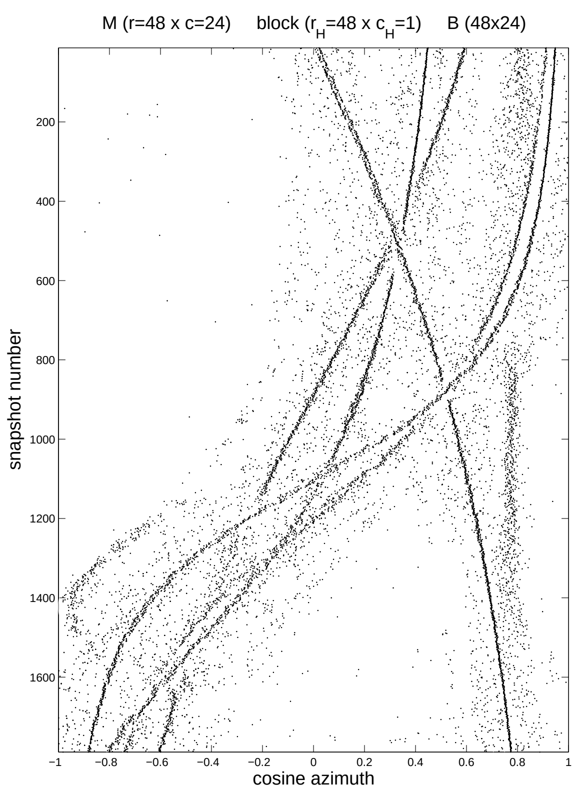

Fig. 6. Cosine of the azimuth of the k strongest sinusoids using 24 sequential snapshots and no matrix structure

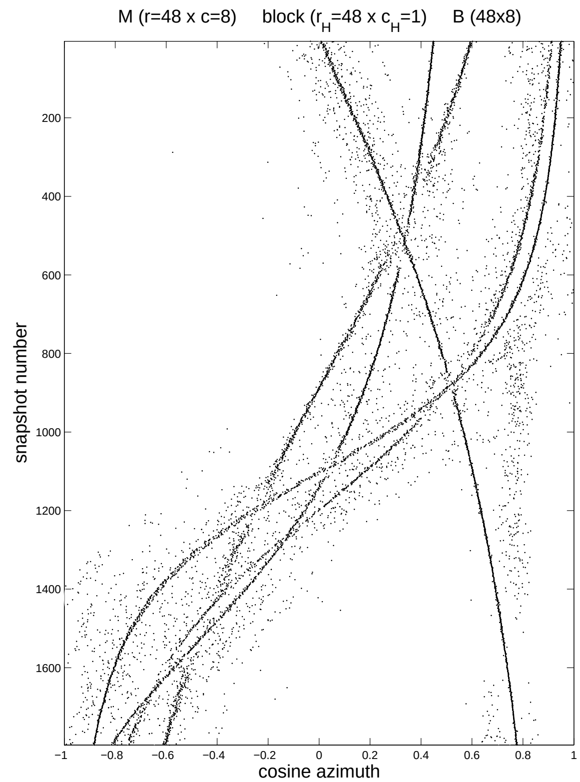

Fig. 7. Cosine of the azimuth of the k strongest sinusoids using eight sequential snapshots and no matrix structure

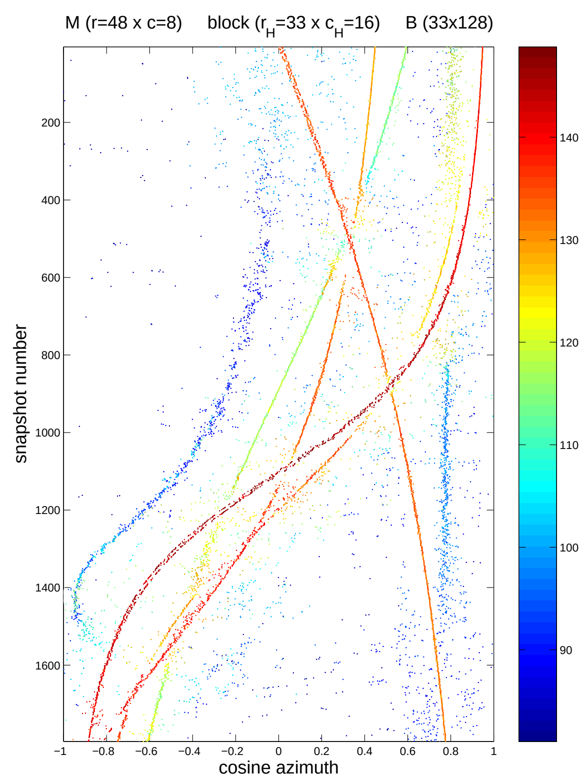

Fig. 8. Cosine of the azimuth of the k strongest sinusoids using eight sequential snapshots and block Hankel matrix structure with color indicating target strength