Analysis of Noise Eigenvalues for Limited Snapshot Matrices

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Overview

- Show how the *block Hankel* structure can increase SNR when one has a large array of sensors, multiple exponential signals, and severe snapshot constraints
- Combine the two rank determination methods of Shah and Tufts, which apply to either unstructured or Hankel matrices, into one general method which also works for block Hankel matrices
- Present a real-time rank determination implementation, where threshold values are precalculated using only matrix dimensions, allowing quantile calculations to be done outside algorithm
- Show an application of this method on some simulated passive sonar array data

Creating Hankel Snapshots

- We have a $n \times c$ data matrix $X = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_c]$, the columns of which are c snapshots from an array of length n

$$\mathbf{x}_i = \mathbf{n}_i + \sum_{k=1}^{r} a_{i,k}\mathbf{z}_k \qquad \text{where} \qquad \mathbf{z}_k = \begin{bmatrix} z_k^0 \\ z_k^1 \\ \vdots \\ z_k^{n-1} \end{bmatrix}$$

- We now create an $n_h \times c_h$ Hankel matrix from each snapshot,

$$\hat{X}_i = \hat{N}_i + \sum_{k=1}^{r} a_{i,k}\hat{Z}_k \qquad \text{where} \qquad \hat{Z}_k = \begin{bmatrix} z_k^0 & z_k^1 & \cdots & z_k^{c_h-1} \\ z_k^1 & z_k^2 & \cdots & z_k^{c_h} \\ \vdots & \vdots & & \vdots \\ z_k^{n_h-1} & z_k^{n_h} & \cdots & z_k^{n-1} \end{bmatrix}$$

- Each matrix $\hat{Z}_k$ will be rank one, thus the signal subspace will be at most rank r, but the noise matrix $\hat{N}_i$ will be full rank.

Reasoning for Block Hankel Matrix

- Concatenate the c Hankel matrices, $\hat{X}_1, \cdots, \hat{X}_c$, that we made from our c snapshots to create the matrix $\hat{X} = [\hat{X}_1 \; \hat{X}_2 \; \cdots \; \hat{X}_c]$

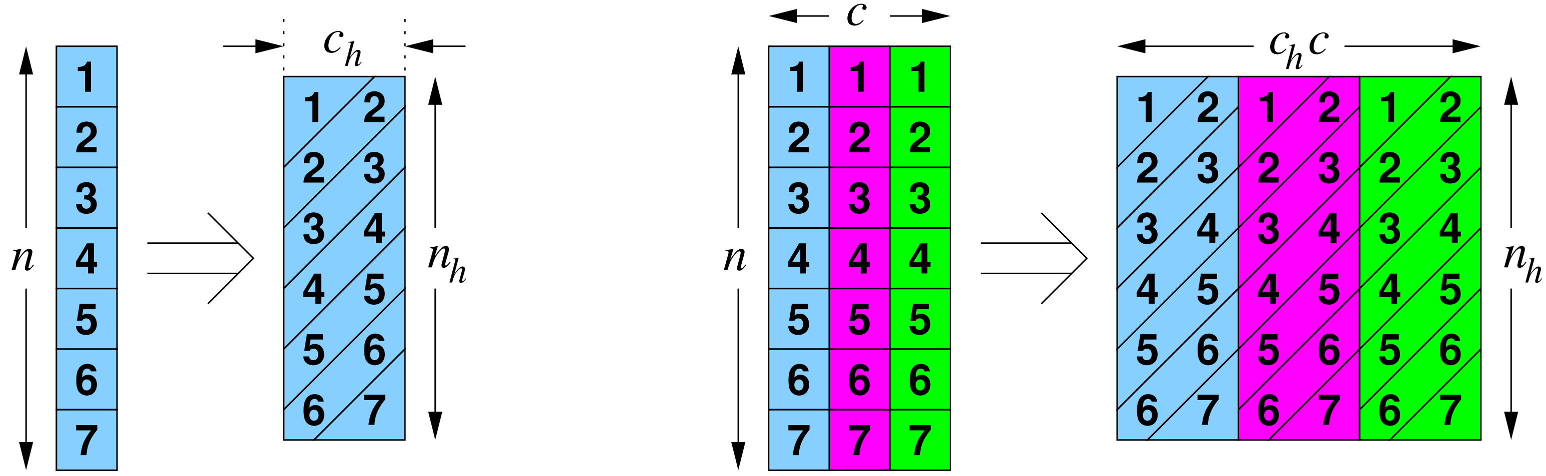
$$\hat{X} = \hat{N} + \sum_{k=1}^{r} \hat{S}_k$$

$$\hat{S}_k = \left[\; a_{1,k}\hat{Z}_k \; \middle| \; a_{2,k}\hat{Z}_k \; \middle| \; \cdots \; \middle| \; a_{c,k}\hat{Z}_k \; \right]$$

- Each $n_h \times c_h c$ matrix $\hat{S}_k$ will be rank one, therefore the signal subspace will be at most rank r, but the noise matrix $\hat{N}$ will be full rank.

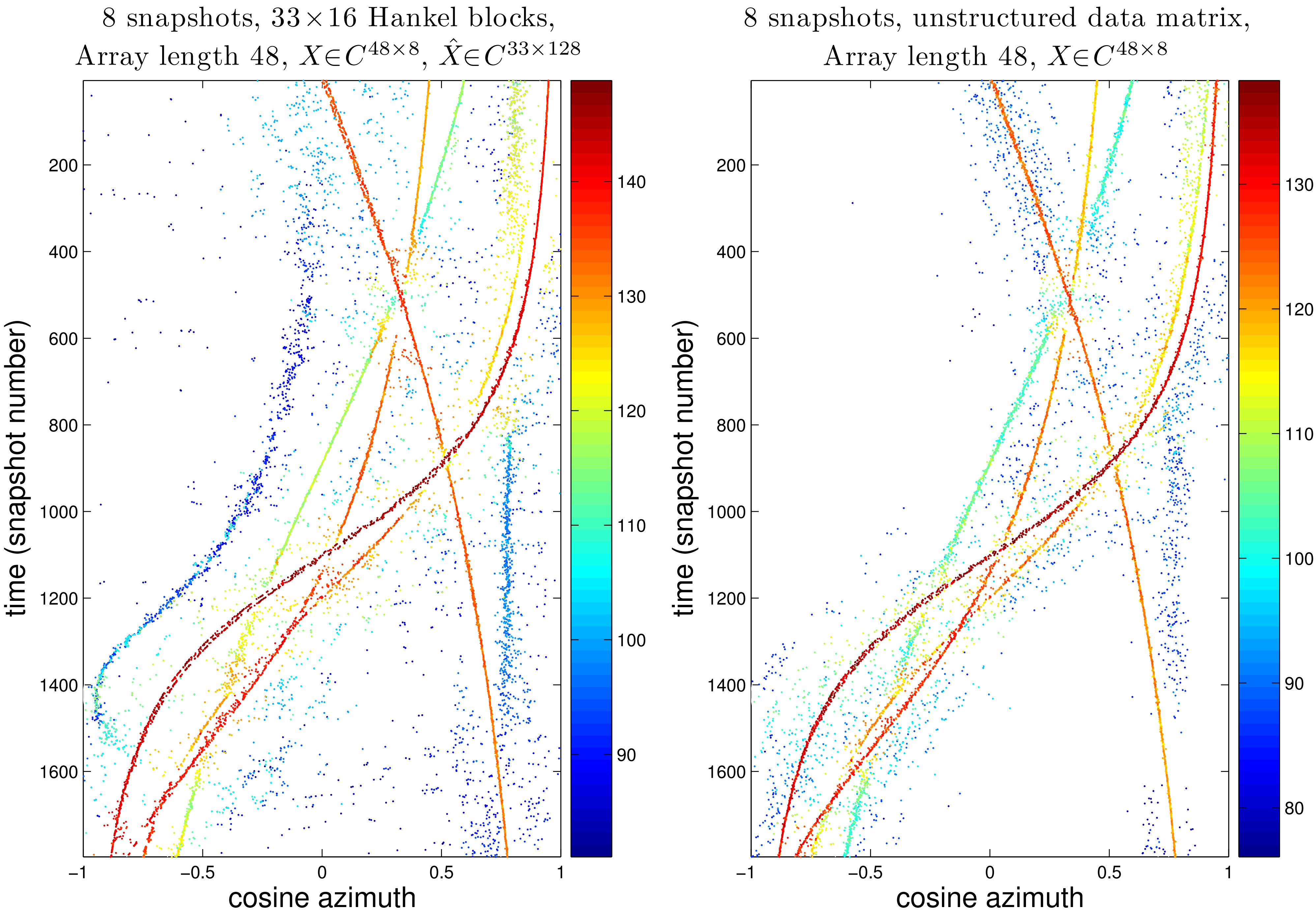
- For each element that we shorten our snapshots by, we lose one row and gain c columns in our data matrix, X.

Forming a Block Hankel Matrix

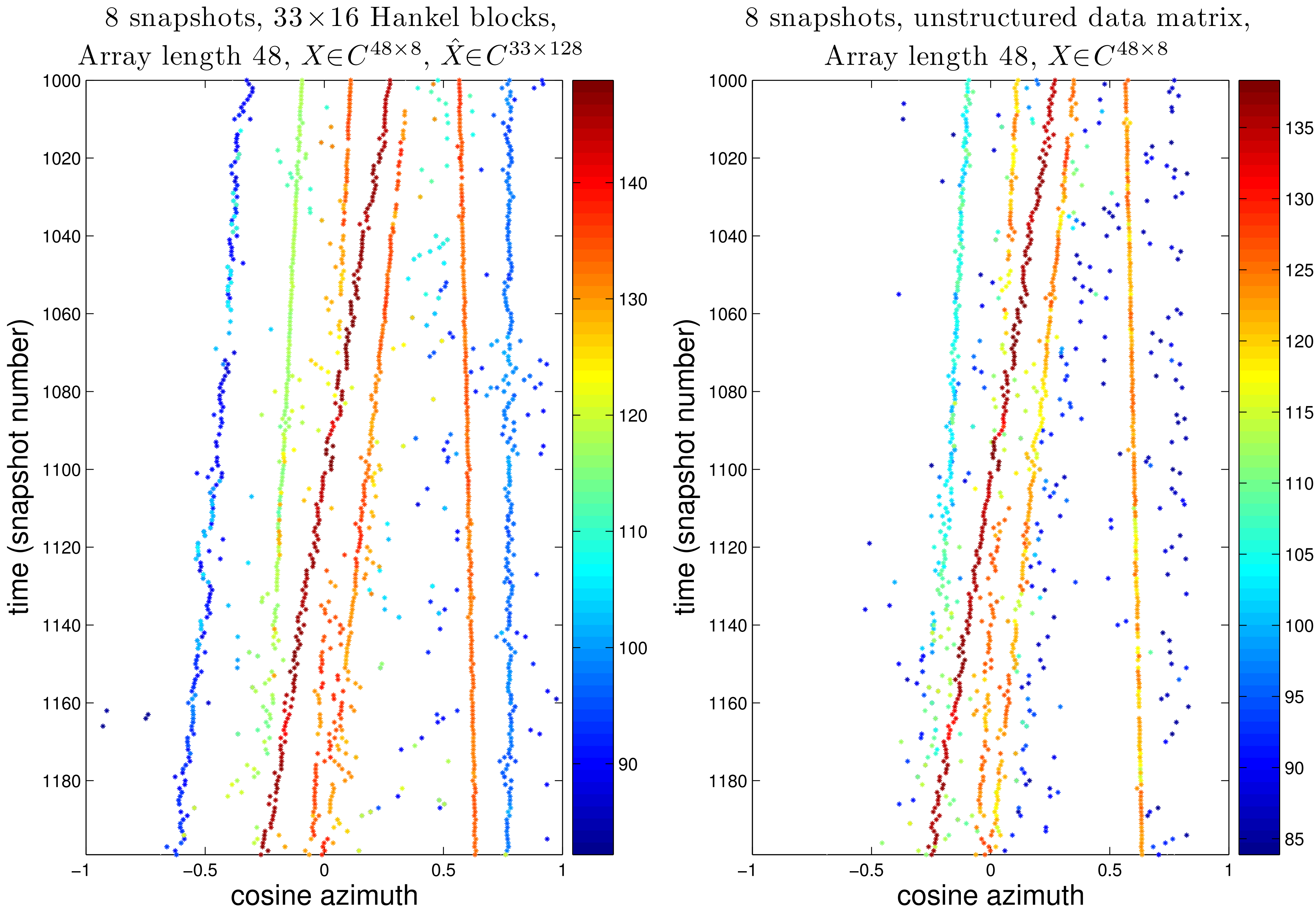


$\mathbf{x}_i \in \mathbb{C}^7 \Rightarrow \hat{X}_i \in \mathbb{C}^{6\times 2}$ $\quad$ $X \in \mathbb{C}^{7\times 3} \Rightarrow \hat{X} \in \mathbb{C}^{6\times 6}$

n - length of array snapshot (7) n_h - length of Hankel column (6)	c - number of array snapshots (3) c_h - number of columns in a single Hankel block (2)
rank$(X) \leq 3$, rank$(\hat{X}) \leq 6$	$c_h c$ - number of columns in a block Hankel matrix (6)



8 snapshots in X, 33×16 Hankel blocks vs. unstructured



8 snapshots in X, times 1000 to 1200 only

Frobenius Norm and Singular Values

- Given an $n \times c$ matrix X, and its singular values $\sigma_1 \cdots \sigma_n$, we can write its squared Frobenius norm as

$$\|X\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{c} |x_{i,j}|^2 = \sum_{i=1}^{n} \sigma_i^2$$

- We project out the subspace corresponding to the r largest singular values of X, then take its squared Frobenius norm, to get the statistic $\hat{F}_r$

$$\hat{F}_r = \|(I - U_r U_r^H) X\|_F^2 = \sum_{i=r+1}^{n} \sigma_i^2 = \|X\|_F^2 - \sum_{i=1}^{r} \sigma_i^2$$

which is the "energy" in the hypothesized noise-only subspace

Estimating the Signal Subspace Rank

- We test the hypothesis that $\hat{F}_r$ $(= \sigma_{r+1}^2 + \sigma_{r+2}^2 + \cdots + \sigma_n^2)$ depends only on noise, by choosing a false alarm probability α, then computing the threshold values T_r, for each $r = 0, \cdots, n$

$P(\hat{F}_r > T_r | H_r) = \alpha$ Is it probable that $\hat{F}_r$ depends only on noise?

- When the energy in the hypothesized noise-only subspace is truly only from noise, $\hat{F}_r$ will be less than T_r with a probability $(1 - \alpha)$

- To determine the dimension of the signal subspace, we find the smallest value of r where both $\hat{F}_r > T_r$ and $\hat{F}_{r+1} < T_{r+1}$

- For zero-mean i.i.d. Gaussian noise and the structured matrices described previously, it is possible to determine the thresholds exactly. The distribution is a χ_n^2 mixture, but is difficult to compute for large matrix dimensions.

Estimating the Thresholds

- For the case H_0 (no signal present), the expected value, μ_f, and variance, σ_f^2, of $\hat{F}_0 = \|X\|_F^2$, are

$$\mu_f = \sigma^2 n_m c_m c \qquad \text{and} \qquad \sigma_f^2 = \sigma^4 c(dn_m^2 + 2\sum_{i=1}^{n_m-1} i^2)$$

where n_m, c_m, c, and d are matrix dimensions of the structured matrix, and σ^2 is the variance of the original unstructured noise

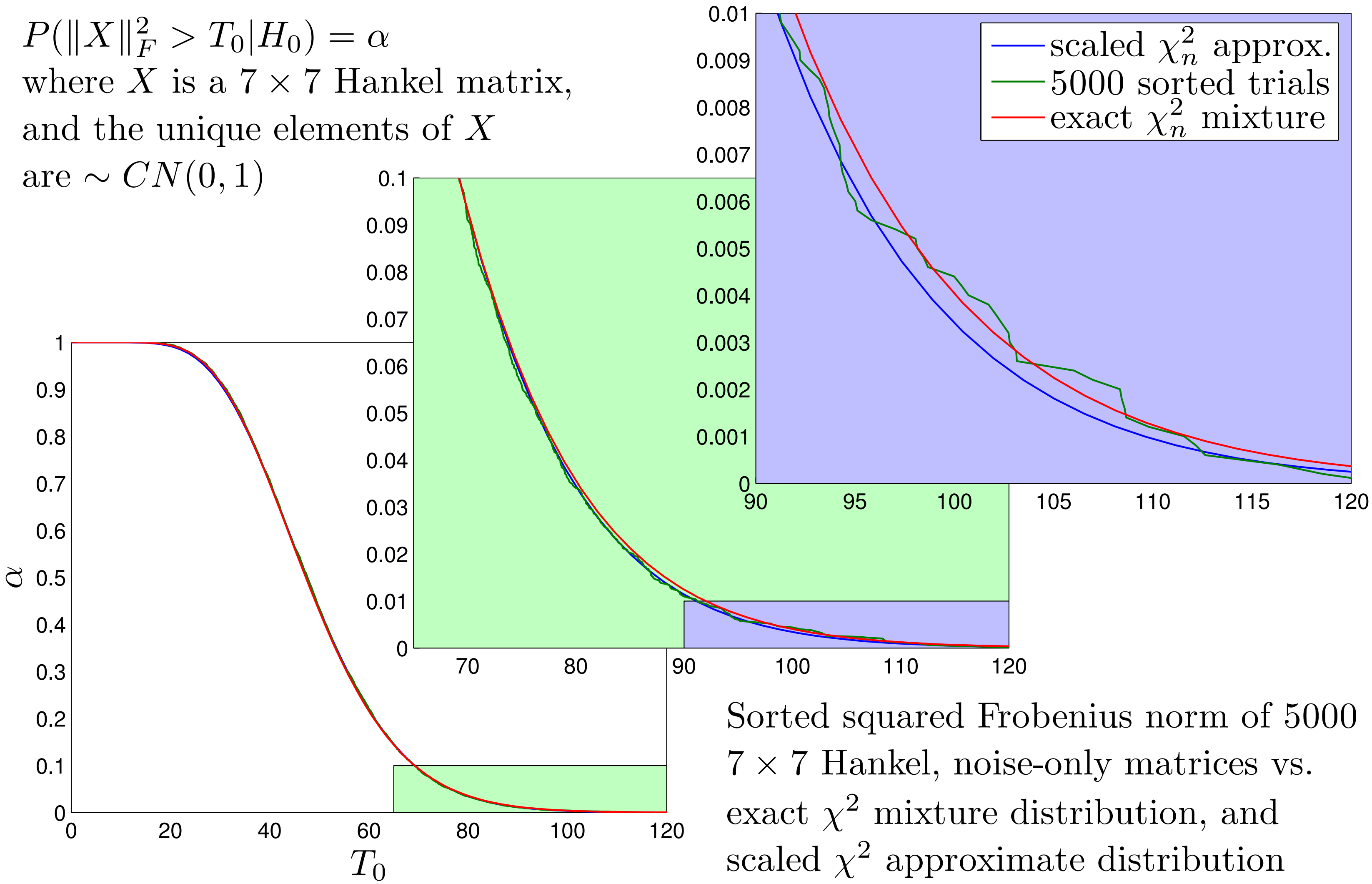
- A good approximation to the distribution of $\hat{F}_0$, is a scaled Chi-Square distribution, $s\chi_n^2$, with mean $ns = \mu_f$ and variance $2ns^2 = \sigma_f^2$, giving us

$$n = \frac{2\mu_f^2}{\sigma_f^2} \qquad \text{and} \qquad s = \frac{\sigma_f^2}{2\mu_f}$$

which are just functions of matrix dimensions and variance

False Alarm Probability vs. Threshold

$P(\|X\|_F^2 > T_0 | H_0) = \alpha$
where X is a 7×7 Hankel matrix,
and the unique elements of X
are $\sim CN(0,1)$



Sorted squared Frobenius norm of 5000 7×7 Hankel, noise-only matrices vs. exact χ^2 mixture distribution, and scaled χ^2 approximate distribution

Calculating the Threshold Values

- We can calculate T_0 as

$$T_0 = sF_n^{-1}(1-\alpha)$$

where F_n^{-1} is the inverse c.d.f. of the Chi-Square distribution

- Since s contains the noise variance σ^2, we can define

$$\hat{T}_0 = \frac{T_0}{\sigma^2} = \frac{s}{\sigma^2}F_n^{-1}(1-\alpha)$$

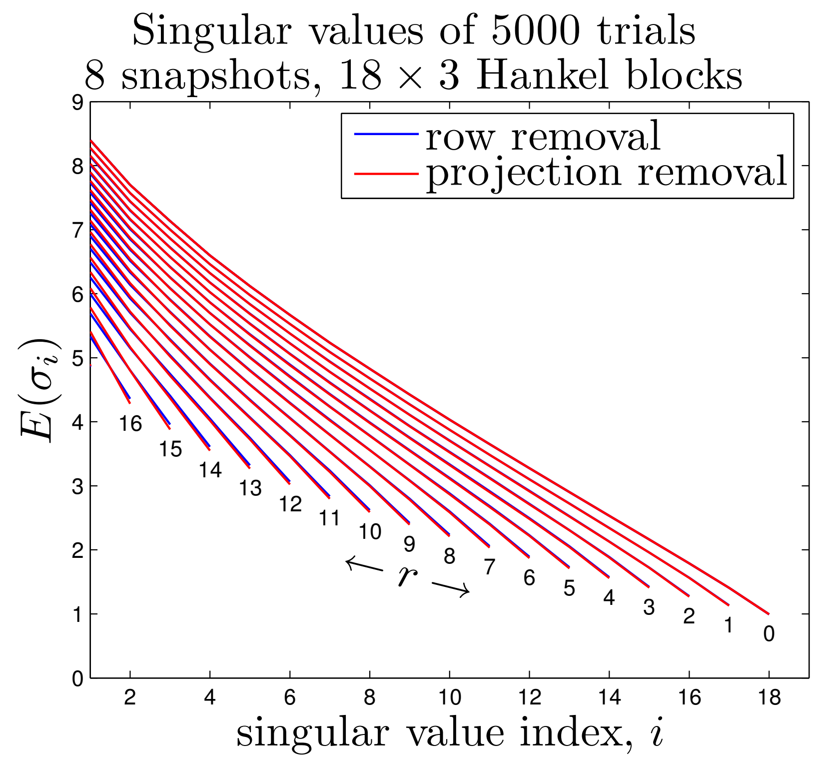
then calculate $\hat{T}_0$ independent of σ^2, since it is just a function of matrix dimensions and α

- After we have an estimate of the noise variance, σ^2, we get

$$T_0 = \hat{T}_0\sigma^2$$

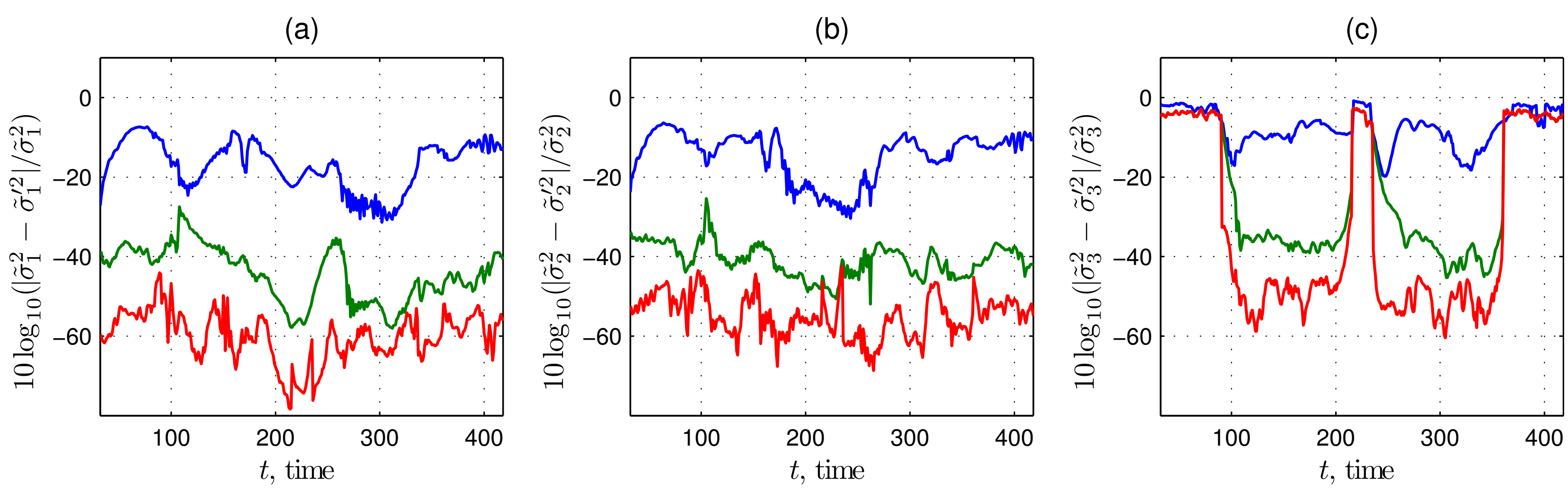
Estimating the Additional Rank Thresholds

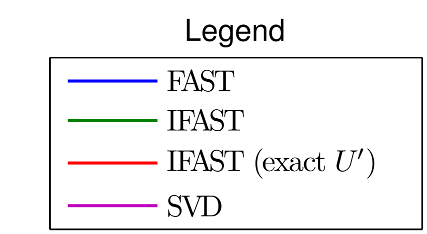
- Approximate $(I - U_r U_r^H)X$ with a noise-only matrix where the last r rows have been removed
- We are approximating U_r by the last r columns of identity matrix
- We can use $P(\hat{F}_{0_r} > T_{0_r} | H_0)$ to approximate $P(\hat{F}_r > T_r | H_r)$ where $\hat{F}_{0_r} = \| [\mathbf{e}_1 \cdots \mathbf{e}_{n-r}]^T X \|_F^2$,
- This allows us to determine all thresholds as $T_r \simeq T_{0_r}$

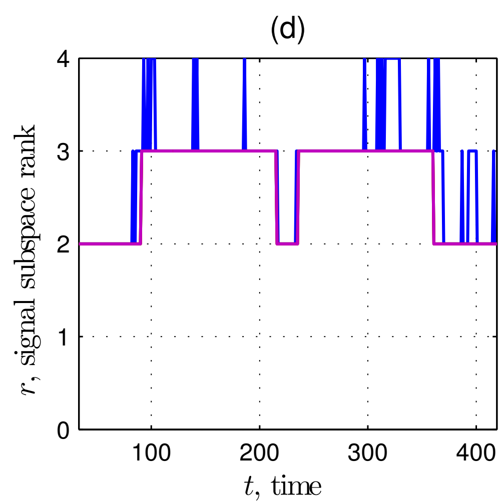


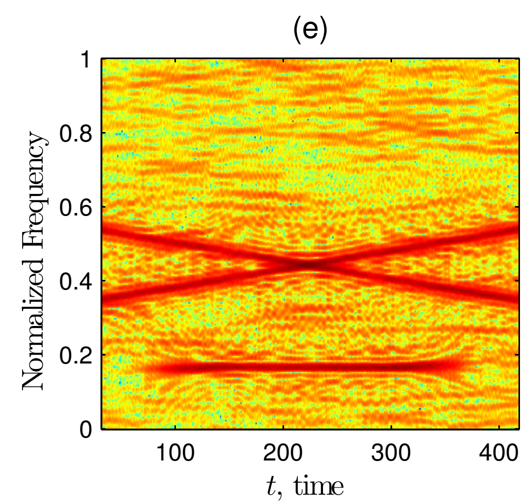
- For each value of r, we take the SVD of both $[\mathbf{e}_1 \ \cdots \ \mathbf{e}_{n-r}]^T X$ and $(I - U_r U_r^H)X$ with different random U_r, of 5000 noise-only block Hankel matrices, X, and average the result

Rank Tracking Example









SNR: 1.5, 0.8, 1.8

Sample: t=200, $\tilde{\sigma}_1'^2$	
FAST:	1210.752108327795
IFAST:	1300.793378303474
IFAST (exact U'):	1300.915147265195
SVD:	1300.916314593100

Summary of Presentation

- Block Hankel structure - Useful for limited snapshot applications with multiple exponential signals
- Improved Rank Estimation - Combined the signal rank estimation methods of Shah and Tufts into one general method, and presented a practical real-time implementation
- References

T. M. Toolan and D. W. Tufts, "Detection and estimation in non-stationary environments," in *Proc. IEEE Asilomar Conference on Signals, Systems & Computers*, Nov. 2003, pp. 797–801.

A. A. Shah and D. W. Tufts, "Determination of the dimension of a signal subspace from short data records," *IEEE Trans. Signal Processing*, no. 9, pp. 2531–2535, Sept. 1994.

D. W. Tufts and A. A. Shah, "Rank determination in time-series analysis," in *Proc. IEEE Intl. Conference on Acoustics, Speech, and Signal Proc.*, Apr. 1994, pp. IV–21–IV–24.