# Implementation of Sallen-Key and Multi-Feedback (MFB) Architecture for Higher Order Butterworth Filters 

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#### Abstract

Filter design specifications can be manifest in all manner of necessity, whether they are high-pass, low-pass, band-pass, or stop-band filters. In addition, the parameters for such filters are defined for a very specific application. A design for a lowpass filter with a pass-band of 2 k Hz and a stop-band of 4 k will be discussed. The further properties of the filter are to allow for a 48 dB drop between the pass-band and the stopband, which results in the need to implement an $8^{\text {th }}$ order filter. This paper will compare two very different filter architectures to meet these requirements: Sallen-Key and MultiFeedback (MFB).


## I. INTRODUCTION

Filters can be constructed in a multitude of ways. Combinations of capacitors and resistors, as well as operational amplifiers and transistors, have been used effectively for many years to produce reliable and functional filters. Just as there are a number of different elements from which a filter can be made, there are a number of different configurations those elements can be arranged to produce virtually identical results. However, some configurations produce better results, based on stability, susceptibility to noise, coupling, and the number of elements involved. We look to two architecturally similar designs for a low-pass filter. Sallen-Key filters and MFB filters have both proven to be very reliable and stable filter designs [1]. Although the MFB configuration consists of more resistors, the benefits or detractions from such a design are not as readily seen. A design specification requiring a pass-band to 2 k Hz and a stop-band at 4 k Hz is implemented on the two circuits. In addition, the circuit must attenuate the signal by 48 dB in that range. This results in the need for an $8^{\text {th }}$ order filter, constructed as the cascade of 4 second order filters.

## II. METHODS

The overall methodology developed for this experiment is consistent with the conventional methodology used for testing circuits. This methodology includes, but is not limited to, PSpice simulation, MATLAB and mathematical analysis, and realization of the physical circuit to extract experimental data. The recording of the experimental data is facilitated by the use of SONY/TEKTONIC circuit analysis equipment. More specifically, we used the AFG 310 arbitrary function generator to provide the input sine
waves of varying frequencies, the PS 501-2 to provide the $\pm 9 \mathrm{~V}$ rails for the op amps, and the TDS 210 two channel digital real time oscilloscope to simultaneously record the frequency response of the circuit as well as the input signal. The individual circuits were treated identically, producing similar results.

Using the Q values for an eighth order Butterworth filter of 2.56 for stage 1, 0.899 for stage $2,0.601$ for stage 3 , and 0.509 for stage 4 , we can derive the following equations necessary to determine the capacitor values to facilitate cutoff frequency of 2 kHz for Sallen-Key implementation [1].

$$
\begin{aligned}
& C_{11}=\frac{2 Q_{1}}{\omega_{0} R}=\frac{2.56 \times 2}{(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=40.7 \mathrm{nF} \\
& C_{21}=\frac{1}{2 x Q_{1} \omega_{0} R}=\frac{1}{(2.56 \times 2)(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=1.55 \mathrm{nF} \\
& C_{12}=\frac{2 Q_{1}}{\omega_{0} R}=\frac{0.899 \times 2}{(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=14.3 \mathrm{nF} \\
& C_{22}=\frac{1}{2 x Q_{1} \omega_{0} R}=\frac{1}{(0.899 \times 2)(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=4.425 \mathrm{nF} \\
& C_{13}=\frac{2 Q_{1}}{\omega_{0} R}=\frac{0.601 \times 2}{(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=9.56 \mathrm{nF} \\
& C_{23}=\frac{1}{2 x Q_{1} \omega_{0} R}=\frac{1}{(0.601 \times 2)(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=6.6 \mathrm{nF} \\
& C_{14}=\frac{2 Q_{1}}{\omega_{0} R}=\frac{0.509 \times 2}{(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=8.1 \mathrm{nF} \\
& C_{24}=\frac{1}{2 x Q_{1} \omega_{0} R}=\frac{1}{(0.509 \times 2)(2 \pi \times 2 \mathrm{kHzx} 10 \mathrm{k} \Omega)}=7.81 \mathrm{nF}
\end{aligned}
$$

Table 3.
Similarly, using the Q values for an eighth order Butterworth filter of 2.56 for stage 1 , 0.899 for stage $2,0.601$ for stage 3 , and 0.509 for stage 4 , we can derive the following equations necessary to determine the capacitor values to facilitate cutoff frequency of 2 kHz for MFB (Multiple Feedback) implementation.
$Q=\frac{\sqrt{R 2 * R 3 * C 1 * C 2}}{R 3 * C 1+R 2 * C 1+R 3 * C 1 *(-K)}$, where -K is equal to $\frac{-R 2}{R 1}$. Since all resistors are kept at the same value, $10 \mathrm{k} \Omega$, this equation reduces to
$Q=\frac{10 k \Omega \sqrt{C 1^{*} C 2}}{10 k \Omega(3 C 1)}=\frac{\sqrt{C 1^{*} C 2}}{3 C 1}$.

To find the value of the combination of $\mathrm{C} 1 * \mathrm{C} 2$, we set the cutoff frequency equal to $f_{\text {cutoff }}=\frac{1}{2 \pi \sqrt{R 1 * R 2 * C 1 * C 2}}[2]$. Since $\mathrm{R} 1=\mathrm{R} 2=10 \mathrm{k} \Omega$, this reduces to
$2000 \mathrm{~Hz}=\frac{1}{2 \pi 10 k \Omega \sqrt{C 1 * C 2}}$
$C 1 * C 2=\left(\frac{1}{2 \pi 10 k \Omega * 2000 H z}\right)^{2}$
$C 1 * C 2=6.33 E^{-17}$
$C 2=\frac{6.33 E^{-17}}{C 1}$

| For $Q=2.56$ | For $Q=0.601$ |
| :--- | :--- |
| $2.56=\frac{\sqrt{C 1 * C 2}}{3 C 1}$ | $0.601=\frac{\sqrt{C 1 * C 2}}{3 C 1}$ |
| $(2.56)^{2} *(3 C 1)^{2}=C 1 * C 2$ | $(0.601)^{2} *(3 C 1)^{2}=C 1^{*} C 2$ |
| $59.1 C 1=C 2$ | $3.25 C 1=C 2$ |
| $59.1 C 1=\frac{6.33 E^{-17}}{C 1}$ | $3.25 C 1=\frac{6.33 E^{-17}}{C 1}$ |
| $C 1^{2}=\frac{6.33 E^{-17}}{59.1}$ | $C 1^{2}=\frac{6.33 E^{-17}}{3.25}$ |
| $C 1=1 n F$ | $C 1=4.4 n F$ |
| $C 2=59.1 n F$ | $C 2=14.3 n F$ |
| $F o r Q=0.899$ | $F o r Q=0.509$ |
| $0.899=\frac{\sqrt{C 1}+C 2}{3 C 1}$ | $0.509=\frac{\sqrt{C 1 * C 2}}{3 C 1}$ |
| $(0.899)^{2} *(3 C 1)^{2}=C 1 * C 2$ | $(0.509)^{2} *(3 C 1)^{2}=C 1 * C 2$ |
| $7.27 C 1=C 2$ | $2.33 C 1=C 2$ |
| $7.27 C 1=\frac{6.33 E^{-17}}{C 1}$ | $2.33 C 1=\frac{6.33 E^{-17}}{C 1}$ |
| $C 1^{2}=\frac{6.33 E^{-17}}{7.27}$ | $C 1^{2}=\frac{6.33 E^{-17}}{2.33}$ |
| $C 1=2.9 n F$ | $C 1=5.2 n F$ |
| $C 2=21.4 n F$ | $C 2=12.1 n F$ |

Table2.

## A. Analysis and measurements of the Sallen-Key Architecture

PSpice simulations and MATLAB provide the theoretical analysis for our design. Figure 1 shows the circuit design for Sallen-Key, with the capacitor values derived in Table 1.


Figure1. Sallen-Key implementation of an 8th order Butterworth Low-Pass filter [2].
The capacitor values derived are not readily available. Every attempt was made to approximate the necessary value by putting capacitors in series or parallel, depending on the value needed as per the design. Below, figure 2, the changes in the capacitor values have been made according to the actual capacitors used. The frequency response for each design shows very good agreement and can be seen in figures 3 and 4 .


Figure2. Circuit with capacitor values as constructed.


Figure 3. Frequency response (Bode Plot) of the Sallen-Key as designed.


Figure 4. Frequency response of the Sallen-Key as built. Notice the bump at about 1.5 k due to the pole location that has been shifted as a result of the capacitor values not being exactly as designed.

## B. Analysis and measurements of the MFB architecture

Again we use PSpice simulations and MATLAB to provide the theoretical analysis for our design. Figure 5 shows the circuit design for Multi-Feedback (MFB) architecture, with the capacitor values derived in Table 2.


Figure5. MFB implementation of an8th order Butterworth Low-Pass filter.
As in the Sallen-Key implementation, the capacitor values derived are not readily available. Below, figure 6 , the changes in the capacitor values have been made according to the actual capacitors used. The frequency response for each design show very good agreement and can be seen in figures 7 and 8 .


Figure6. Sallen-Key implementation of an8th order Butterworth Low-Pass filter.


Figure 7. Frequency response (Bode Plot) of the MFB as designed.


Figure 8. Frequency response of the MFB as built. Differences can easily be seen at 6 k where the attenuation is -80 dB whereas the designed response is only down to -75 dB .

## III. RESULTS

## A. Analysis and measurements of the Sallen-Key filter

As mentioned earlier excellent results were achieved performing this circuit analysis. Much of the data obtained through measurement was in very good agreement with the PSpice model. Figure 9 and Table 3 show the results as measured using the Sallen-Key circuit. Notice that the pass-band frequency occurs at a lower frequency than the design specifications. This is most likely due to the mismatched capacitor values.


Table 3.
B. Analysis and measurements of the MFB filter

As mentioned earlier excellent results were achieved performing this circuit analysis. Much of the data obtained through measurement was in very good agreement with the PSpice model. One notable exception is the shift, once again, of the pass band to 100 Hz , very far removed from the projected 2 k pass-band. Certainly the capacitor values are affecting this to some degree; however, we must conclude also that there are inherent difficulties in constructing this type of filter. There are many benefits to this type of filter
but ultimately, we look to the data found in figure 10 and table 4 and conclude that for these purposes the Sallen-Key methodology is easier to construct.
dB Plot for MFB Implementation for 8th Order LPF

Figure 10. Frequency response of the MFB as built.

| Freq <br> $(\mathrm{Hz})$ | Vin | Vout | dB |
| ---: | ---: | ---: | ---: |
| 10 | 0.731 | 0.731 | 0 |
| 20 | 0.72 | 0.72 | 0 |
| 30 | 0.705 | 0.705 | 0 |
| 40 | 0.709 | 0.682 | -0.3372 |
| 50 | 0.711 | 0.627 | -1.0920 |
| 60 | 0.712 | 0.554 | -2.1794 |
| 70 | 0.713 | 0.494 | -3.1872 |
| 80 | 0.713 | 0.42 | -4.5968 |
| 90 | 0.713 | 0.389 | -5.2628 |
| 100 | 0.713 | 0.359 | -5.9599 |
| 120 | 0.713 | 0.309 | -7.2626 |
| 140 | 0.713 | 0.268 | -8.4990 |
| 160 | 0.713 | 0.233 | -9.7146 |
| 180 | 0.713 | 0.203 | -10.911 |
| 200 | 0.713 | 0.178 | -12.053 |
| 400 | 0.7 | 0.057 | -21.784 |
| 600 | 0.688 | 0.023 | -29.517 |
| 800 | 0.675 | 0.013 | -34.307 |
| 1000 | 0.661 | 0.0072 | -39.257 |
| 1200 | 0.648 | 0.0041 | -43.975 |
| 1400 | 0.634 | 0.0025 | -48.083 |
| 1600 | 0.619 | 0.0015 | -52.312 |
| 1800 | 0.605 | 0.0009 | -56.550 |
| 2000 | 0.591 | 0.0006 | -59.868 |

Table 4.

## IV. SIGNAL PROCESSING

## A. MATLAB Graphical Analysis-Analog

We now turn our attention to further extracting information regarding the functionality of the filter. MATLAB enables the user to perform extremely complex mathematical analysis efficiently. In addition, MATLAB has many functions capable of generating exceptionally insightful graphs, many of which will be discussed.

Since this is an $8^{\text {th }}$ order system, it can be realized by the cascade of four $2^{\text {nd }}$ order filters.[3] Each of the $2^{\text {nd }}$ order filters has a characteristic equation of the form $\left(\mathrm{s}^{2}+\right.$ $2 \mathrm{Q} \omega \mathrm{s}+\omega^{2}$ ), where $\omega$ is the natural frequency, and the Q values are provided from filter tables (See appendix 1). By convolving the four characteristic equations we will generate and $8^{\text {th }}$ order equation that describes the filter [5].

```
>> LP1 =[12*0.5098*w w^2];LP2=[1 2*0.6013*w w^2];LP3 =[1 2*0.8999*w w^2];
LP4=[1 2*2.5628*W w^2];
>> LP4th=conv(LP1,LP2)
LP4th =
    1.0000
>> LP6th=conv(LP4th,LP3)
LP6th =
    1.0000
>> LP8th=conv(LP6th,LP4)
LP8th =
    1.0000
```

For simplicity, w has been set to 1 , but can easily be moved out to 2000 Hz as the specifications require. By setting $\mathrm{w}=1$, we do not have to multiply by a gain factor to maintain unity gain. With this information we can easily generate a transfer function:
$\gg \operatorname{lpf}=\mathrm{tf}(\mathrm{b}, \mathrm{LP} 8 \mathrm{th})$

Transfer function:

$$
1
$$

$s^{\wedge} 8+9.148 s^{\wedge} 7+29.84 s^{\wedge} 6+56.43 s^{\wedge} 5+68.99 s^{\wedge} 4+56.43 s^{\wedge} 3+29.84 s^{\wedge} 2+9.148 s+1$
With our transfer function it is now possible to generate our zero, pole, gain information:

```
>> [z,p,k]=tf2zp(b,LP8th)
Z \(=\)
    Empty matrix: 0-by-1
\(\mathrm{p}=\)
    -4.9224
    \(-0.8999+0.4361 \mathrm{i}\)
    \(-0.8999-0.4361 \mathrm{i}\)
    \(-0.5098+0.8603 \mathrm{i}\)
    \(-0.5098-0.8603 i\)
    \(-0.6013+0.7990 \mathrm{i}\)
    \(-0.6013-0.7990 \mathrm{i}\)
    -0.2032
\(\mathrm{k}=\)
    1
>> lpf=tf(b,LP8th)
```

Transfer function:

## 1

$s^{\wedge} 8+9.148 \mathrm{~s}^{\wedge} 7+29.84 \mathrm{~s}^{\wedge} 6+56.43 \mathrm{~s}^{\wedge} 5+68.99 \mathrm{~s}^{\wedge} 4+56.43 \mathrm{~s}^{\wedge} 3+29.84 \mathrm{~s}^{\wedge} 2+9.148 \mathrm{~s}+1$

In turn, with our transfer function it is now possible to generate our zero, pole, gain information:

```
>> [z,p,k]=tf2zp(b,LP8th)
z =
    Empty matrix: 0-by-1
p
    -4.9224
    -0.8999 + 0.4361i
    -0.8999-0.4361i
    -0.5098 + 0.8603i
    -0.5098-0.8603i
    -0.6013+0.7990i
    -0.6013-0.7990i
    -0.2032
k}
    1
```



Figure 11. All the poles are positioned in the left plane, confirming that this is indeed a low-pass filter.

This information makes it very easy to graph the pole, zero plot, figure 11 . We can now use this information to generate a Bode plot (figure 12) [5].


Figure 12. Bode diagram showing the attenuation and the nearly linear phase.

A fundamental analysis of many circuits includes both the impulse response and the step response. They are shown here as figures 13 and 14.


Figure 13.


Figure 14.

There are two other graphs that are not as fundamental to circuit analysis but yield a great deal of information regarding the circuit's performance. The Root Locus, rlocus, computes the Evans root locus of a SISO open-loop model.[6] The root locus gives the closed-loop pole trajectories as a function of the feedback gain ${ }^{k}$ (assuming negative feedback). Root loci are used to study the effects of varying feedback gains on closedloop pole locations. In turn, these locations provide indirect information on the time and frequency responses. Rlocus(sys) calculates and plots the root locus of the open-loop SISO model sys. [6]. The root locus plot can be seen in figure 15.


Figure 15. The root locus is shown with the percent damping weights as numerical representation for each pole location.

The other insightful graph is the Nyquist Diagram, which directly correlates the circuit stability to the region enclosed by the graph. If the point -1 lies within the smaller encircled region (see figure 16), the system is considered to be unstable. As can be easily seen, the encircled region stops at -0.4 and does not approach -1 . This is a stable system.


Figure 16. If the encircled area includes -1 on a Nyquist digram, the system is unstable. Inspection reveals that this is a very stable system.

## B. MATLAB Graphical Analysis-Digital

Although MATLAB provides exceptional analog filter analysis, it is in the discrete world that it really performs. By using a bilinear transformation to take the analog filter and make it digital, we can use similar analysis in the digital domain.

| $\gg ~[\mathrm{Zd}, \mathrm{Pd}, \mathrm{Kd}]=$ bilinear $(\mathrm{z}, \mathrm{p}, \mathrm{k}, 1)$ |
| :--- |
| $\mathrm{Zd}=$ |
| -1 |
| -1 |
| -1 |
| -1 |
| -1 |
| -1 |
| -1 |
| -1 |
|  |
| $\mathrm{Pd}=$ |
|  |
| -0.4222 |
| $0.3489+0.2028 \mathrm{i}$ |
| $0.3489-0.2028 \mathrm{i}$ |
| $0.4262+0.4889 \mathrm{i}$ |
| $0.4262-0.4889 \mathrm{i}$ |
| $0.4051+0.4316 \mathrm{i}$ |
| $0.4051-0.4316 \mathrm{i}$ |
| 0.8156 |
| $\mathrm{Kd}=$ |
| $1.4627 \mathrm{e}-004$ |

It is readily apparent that the digital version of this filter now includes 8 zeros where the analog had none. This is to allow for the movement of frequency around the unit circle and ending at $\pi$. The significance to this is, of course, the Nyquist frequency, which recognizes the inability of a digital filter to distinguish between the signal and its alias. Figure 17 shows the new pole zero plot for the digital filter.


Figure 17. Notice that the locations of the poles have moved and we are now introducing 8 zeros at negative $1, \pi$ on the frequency plot.

## V. FAILURE ANALYSIS

## A. Monte Carlo Simulations

Resistor values can vary by as much as 20 percent. Typically this value is given in the indicator bands around the resistor. For instance a resistor with a gold tolerance band has a $5 \%$ margin of error, for silver it is $10 \%$, and if no tolerance band is present, the margin of error is $20 \%$. It is important in circuit design and analysis to account for such variability. In PSpice, this is done by using Monte Carlo simulations to perform a given number of analysis runs while varying the resistor through the possible tolerance values. Figure 18 shows such an analysis using 20\% Gaussian distribution on the Sallen-Key architecture. Figure 19 shows the analysis for the MFB architecture using the same distribution. Notice the range of values over the stop band at 2 K Hz .


Figure 18. The output voltage varies by $75 \%$ from 1 V to approximately 1.75 V .


Figure 19. The output voltage varies by almost $100 \%$ from 0.7 V to approximately 1.4 V .
By using a better resistor value, that is one with a better tolerance, we see a tighter response (Figure 20). There is not the same scattered distribution as with the $20 \%$ tolerance resistors. But what is interesting is that the output voltage is climbing to almost 2 -and-a-half volts. Of course, this difference would not be quite as extreme on a Bode plot, since the difference between them is only 7.95 dB .


Figure 20. The 5\% tolerance resistors narrow the distribution of possible output voltage values.
Another useful analysis tool is the histogram plot. It allows you to look quickly at a range of values and determine their distribution. An example can be seen in figure 21, the histogram of the Sallen-Key bandwidth samples.


Figure 21. The histogram tells us that $5 \%$ of the samples indicate the bandwidth is $355 \mathrm{~Hz}, 10 \%$ are represented at 372 Hz , etc.

## A. Worst Case Scenario

As reported earlier, resistor values can vary by as much as 20 percent. Monte Carlo simulations give us an idea of how the circuit will behave if the resistors are uniformly or Gaussian distributed. However, the possibility exists that all of the resistors will be either at the lower end of their possible tolerance values or the higher end. Even worse, there is the possibility that half will be at the absolute low end and the other half at the absolute high end, creating the greatest (or worst) case scenario. We use the PSpice Worst Case Scenario to help us predict how the circuit will behave if these conditions exist. Figure 22 illustrates the worst case scenario of the Sallen-Key architecture with 5\% Gaussian distribution of resistor values.


Figure 22.
Since this is the single worst case scenario, only one trace is output. It is, however, the expected value based on the Monte Carlo distribution of $5 \%$ Gaussian distribution resistor values. Note that the values is identical to the highest output in the Monte Carlo simulation.

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## APPENDIX A

Table 1. Butterworth Filter Table

| FILTER <br> ORDER | Stage 1 |  | Stage 2 |  | Stage 3 |  | Stage 4 |  | Stage 5 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | FSF | Q | FSF | Q | FSF | Q | FSF | Q | FSF | Q |
| 2 | 1.000 | 0.7071 |  |  |  |  |  |  |  |  |
| 3 | 1.000 | 1.0000 | 1.000 |  |  |  |  |  |  |  |
| 4 | 1.000 | 0.5412 | 1.000 | 1.3065 |  |  |  |  |  |  |
| 5 | 1.000 | 0.6180 | 1.000 | 1.6181 | 1.000 |  |  |  |  |  |
| 6 | 1.000 | 0.5177 | 1.000 | 0.7071 | 1.000 | 1.9320 |  |  |  |  |
| 7 | 1.000 | 0.5549 | 1.000 | 0.8019 | 1.000 | 2.2472 | 1.000 |  |  |  |
| 8 | 1.000 | 0.5098 | 1.000 | 0.6013 | 1.000 | 0.8999 | 1.000 | 2.5628 |  |  |
| 9 | 1.000 | 0.5321 | 1.000 | 0.6527 | 1.000 | 1.0000 | 1.000 | 2.8802 | 1.000 |  |
| 10 | 1.000 | 0.5062 | 1.000 | 0.5612 | 1.000 | 0.7071 | 1.000 | 1.1013 | 1.000 | 3.1969 |

## APPENDIX B

FLOW GRAPH
Eighth Order Low Pass Filter


Signal emerges with a total of $80 \mathrm{~dB} /$ dec. Filters are designed for unity gain.

## APPENDIX C <br> Op-Amp Specifications



## APPENDIX D

Picture of circuit and analysis


Here we see a protoboard with two circuits; the top one is Sallen-Key the bottom (being analyzed in this picture), is the MFB architecture. Notice the increase in hardware from SK to the MFB design. The two voltmeters show from left to right, Vin and Vout respectively.

