1.0 Complex Sinusoids

1.1 The DTFT

A) Infinite Duration of *x*[*n*]

Take an arbitrary complex sinusoid with infinite duration.

$$x[n] = Ce^{j\Omega_0 n}, \quad -\infty < n < \infty, \quad |\Omega_0| \le \pi$$
$$C = Ae^{j\phi}$$

The DTFT of $x[n]$ can be shown to be an impulse at Ω_0.

$$X(e^{j\Omega}) = 2\pi C\delta(\Omega - \Omega_0)$$

Proof: Take the inverse DTFT of $X(e^{j\Omega})$.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} 2\pi C\delta(\Omega - \Omega_0)e^{j\Omega n}d\Omega$$
$$x[n] = C\int_{2\pi}\delta(\Omega - \Omega_0)e^{j\Omega_0 n}d\Omega = Ce^{j\Omega_0 n}\int_{2\pi}\delta(\Omega - \Omega_0)d\Omega$$
$$x[n] = Ce^{j\Omega_0 n}$$

B) Finite Duration of *x*[*n*]

Now assume that we have only N samples of $x[n]$.

$$x[n] = Ce^{j\Omega_0 n}, \quad 0 \le n \le N-1, \quad |\Omega_0| \le \pi$$
$$C = Ae^{j\phi}$$

The DTFT is:

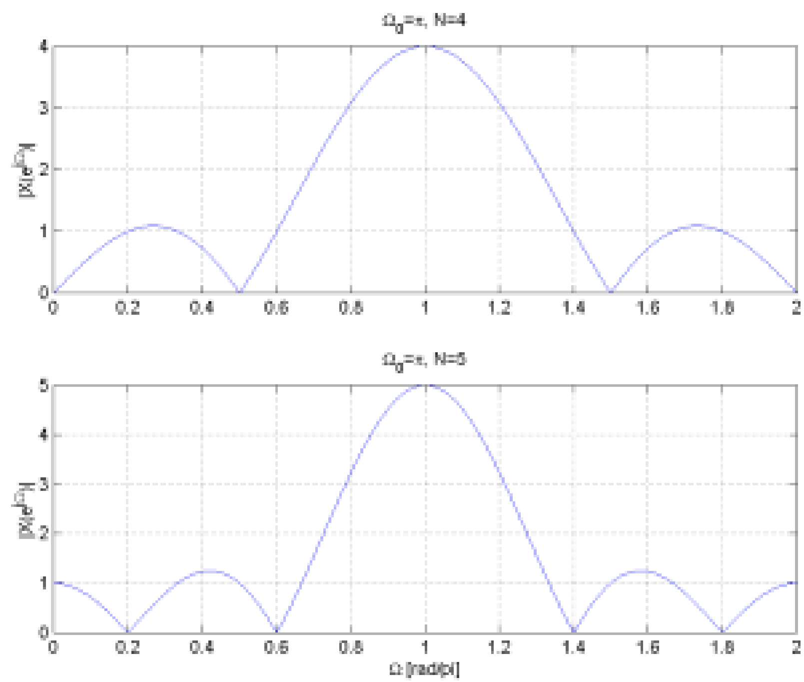
$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} Ce^{j\Omega_0 n}e^{-j\Omega n}$$
$$X(e^{j\Omega}) = C\sum_{n=0}^{N-1} e^{j(\Omega_0 - \Omega)n}$$
$$X(e^{j\Omega}) = \begin{cases} C\dfrac{1 - e^{j(\Omega_0 - \Omega)N}}{1 - e^{j(\Omega_0 - \Omega)}}, & \Omega \ne \Omega_0 \\ CN & \Omega = \Omega_0 \end{cases}$$

This expression can be rewritten as:

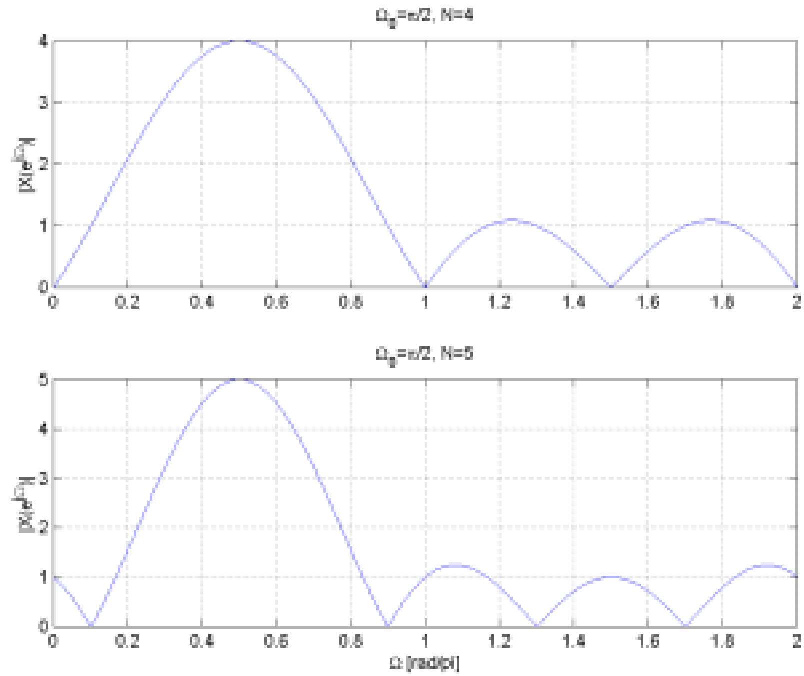
$$X(e^{j\Omega}) = \begin{cases} C\dfrac{\sin[(\Omega_0-\Omega)N/2]}{\sin[(\Omega_0-\Omega)1/2]}\cdot e^{j\left(\frac{N-1}{2}\right)(\Omega_0-\Omega)}, & \Omega \neq \Omega_0 \\ CN, & \Omega = \Omega_0 \end{cases}$$

where $$\left|X(e^{j\Omega})\right| = \begin{cases} A\dfrac{\sin[(\Omega_0-\Omega)N/2]}{\sin[(\Omega_0-\Omega)1/2]}, & \Omega \neq \Omega_0 \\ AN, & \Omega = \Omega_0 \end{cases}$$

Let's plot $\left|X(e^{j\Omega})\right|$ for various values of N and Ω_0 to get a feel for how they affect the DTFT. (Set A=1 for simplicity.)



The value of N affects the number of lobes in $\left|X(e^{j\Omega})\right|$. The side lobes have a width of $2\pi/N$ radians, while the main lobe is $4\pi/N$ radians wide. The maximum magnitude of the DTFT occurs at Ω_0 and has a height of *AN*. Changing the value of Ω_0 simply shifts peak of the DTFT to that point. Notice also that the DTFT is periodic with period 2π.



1.2 The DFT

A) Non Zero-padded DFT

The DFT is a discrete time to discrete frequency transform evaluated over a finite set of data. The formula for the DFT is as follows:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}, \qquad 0 \le k \le N-1$$

From the formula we can see the DFT is the DTFT evaluated at the points $\Omega = 2\pi k/N$ for $k = 0,1,\ldots,N-1$.

B) Zero-padded DFT

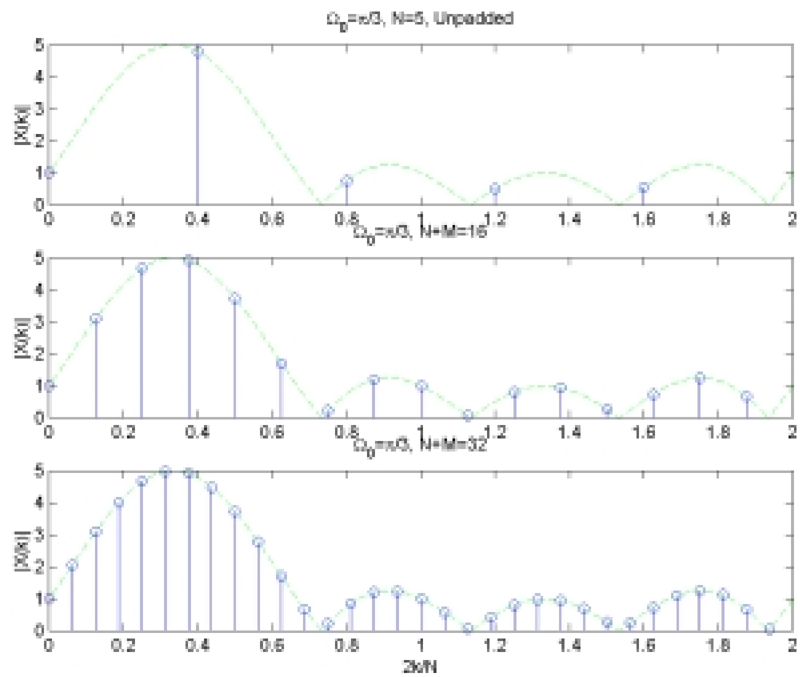
Let's see what happens when we artificially lengthen the data by appending *M* zeros. The data $x[n]$ now has N+M samples as does the DFT.

$$X(k) = \sum_{n=0}^{N+M-1} x[n] e^{-j2\pi \frac{k}{N+M} n}, \quad 0 \le k \le N+M-1$$

But since $x[n]$ is zero from k = N to k = M-1, we can rewrite the summation as

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N+M} n}, \qquad 0 \le k \le N+M-1$$

The result of zero padding is that the DFT is now the DTFT evaluated at N+M points. The following figure illustrates the DFT with and without zero padding. Notice that the DFT is simply the DTFT (the dashed line) evaluated at discrete values of Ω.



Note: Sometimes we can find the peak of DTFT (and thus Ω_0) using the DFT and sometimes we cannot. In the example of above, the DTFT is not sampled at its peak. However, if we had chosen N+M = 6n for some integer n, one of the samples would have fallen on $2\pi k/(N+M) = \pi/3 = \Omega_0$. In general, for a given N and M, Ω_0 will only be found when $2\pi k/(N+M) = \Omega_0$ for some value of k.

2.0 Real Sinusoids

2.1 The DTFT

A) Infinite Duration of *x*[*n*]

Take an arbitrary sinusoid with infinite duration.

$$x[n] = A\cos(\Omega_0 n + \phi), \qquad -\infty < n < \infty, \qquad |\Omega_0| \le \pi$$
$$C = Ae^{j\phi}$$

The DTFT of $x[n]$ can be shown to be two impulses: one at $+\Omega_0$ and another at $-\Omega_0$.

$$X(e^{j\Omega}) = \pi\left[C\delta(\Omega-\Omega_0) + C^*\delta(\Omega+\Omega_0)\right]$$

B) Finite Duration of *x*[*n*]

Now assume that we have only N samples of $x[n]$.

$$x[n] = A\cos(\Omega_0 n + \phi), \qquad 0 \le n \le N-1, \qquad |\Omega_0| \le \pi$$
$$C = Ae^{j\phi}$$

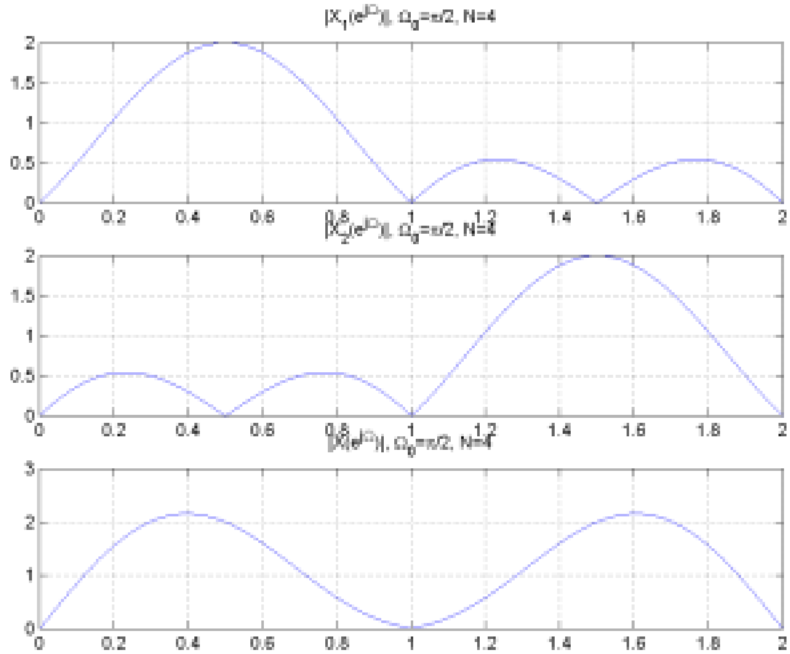
The DTFT is:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} \frac{A}{2}\left[e^{j\Omega_0 n}e^{j\phi} + e^{-j\Omega_0 n}e^{-j\phi}\right]e^{-j\Omega n}$$
$$X(e^{j\Omega}) = \frac{C}{2}\sum_{n=0}^{N-1} e^{j(\Omega_0-\Omega)n} + \frac{C^*}{2}\sum_{n=0}^{N-1} e^{-j(\Omega_0+\Omega)n}$$

Break the DTFT into two parts

$$X(e^{j\Omega}) = X_1(e^{j\Omega}) + X_2(e^{j\Omega})$$
$$X_1(e^{j\Omega}) = \frac{C}{2}\sum_{n=0}^{N-1} e^{j(\Omega_0-\Omega)n}, \quad X_2(e^{j\Omega}) = \frac{C^*}{2}\sum_{n=0}^{N-1} e^{-j(\Omega_0+\Omega)n}$$
$$X_1(e^{j\Omega}) = \begin{cases} \dfrac{C}{2}\dfrac{1-e^{j(\Omega_0-\Omega)N}}{1-e^{j(\Omega_0-\Omega)}}, & \Omega \ne \Omega_0 \\[2ex] \dfrac{CN}{2} & \Omega = \Omega_0 \end{cases}$$
$$X_2(e^{j\Omega}) = \begin{cases} \dfrac{C^*}{2}\dfrac{1-e^{-j(\Omega_0+\Omega)N}}{1-e^{-j(\Omega_0+\Omega)}}, & \Omega \ne -\Omega_0 \\[2ex] \dfrac{C^*N}{2} & \Omega = -\Omega_0 \end{cases}$$

The plot below shows the three signals: $|X_1(e^{j\Omega})|$, $|X_2(e^{j\Omega})|$, and $|X(e^{j\Omega})|$. Notice that, in this example, the magnitudes of the two component signals $|X_1(e^{j\Omega})|$ and $|X_2(e^{j\Omega})|$ have peaks at $\pi/2$ and $3\pi/2$ respectively. The peak locations of either $|X_1(e^{j\Omega})|$ or $|X_2(e^{j\Omega})|$ are sufficient to determine the value of Ω_0. However, if we were to rely only on the third plot we might think that Ω_0 was approximately $2\pi/5$, which is not the case.



2.2 The DFT

Recall that the DFT is simply the DTFT sampled at discrete values. Clearly then, any frequency estimation problems occurring with the DTFT would carry through to the DFT. Therefore, DFT based frequency estimation of real sinusoids encounters two problems:

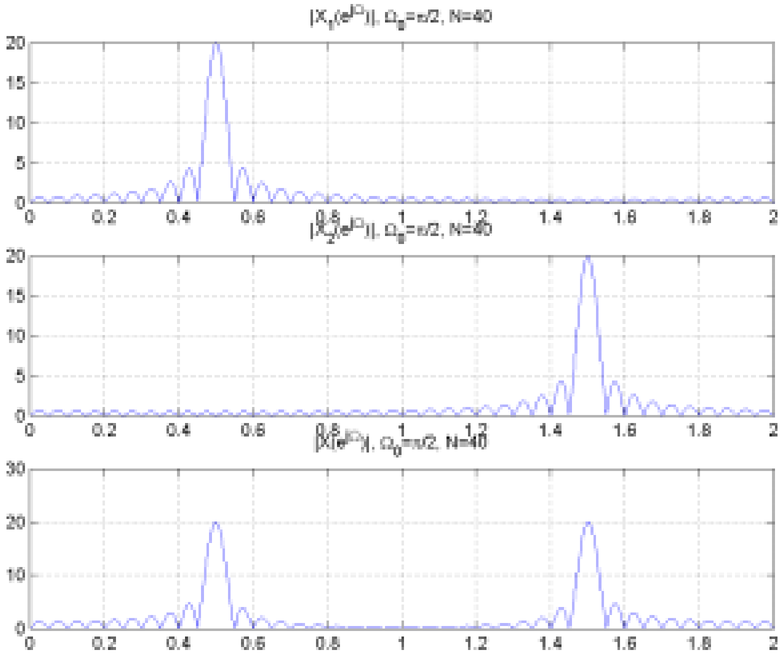
The first difficulty was just mentioned—for real sinusoids, the peak of the DTFT magnitude can be far from the value of Ω_0. The second problem was described in section regarding the DFT of complex sinusoids. Even when the DTFT can be used to correctly estimate the frequency of the signal, the DFT can encounter some error due to quantization of the frequency axis. This problem can be reduced, but not eliminated, by appending zeros to the time data before performing the DFT.

3.0 Further Thoughts

Several approaches could be adopted to reduce or circumvent the problems mentioned in the last section.

1.) One possibility might be to increase the value of N (i.e. take more data samples). This would increase the magnitude of the main lobes of the DTFT with respect to the side lobes, reducing interference between the two peaks as a result. See the plot on the following page. Notice that the peak in third plot is closer to Ω_0 than when N was 4.

2.) We might also consider various windowing options. The current approach essentially uses a rectangular window, resulting in particularly large side lobes. Using a Hamming window, for example, would reduce the size of the side lobes , thus reducing interference between sinusoidal components.



DTFT magnitude of a real sinusoid with N=40 samples