Chebyshev Polynomials in Filter Design

The goal of this project is to use Chebyshev polynomials to design a Finite Impulse Response (FIR) filter.

Overview

We will discuss a filter design problem involving certain restrictions on a filter's frequency response function, a trigonometric polynomial. It will be shown that the constraints on this trigonometric polynomial can be transformed into constraints on an algebraic polynomial. Next, we will apply well known results from approximation theory to demonstrate that linearly warped Chebyshev polynomials satisfy our particular design specifications. The algebraic polynomial will then be transformed back to a trigonometric polynomial yielding the coefficients of the filter.

Filter Specifications

(1) Output depends only on linear combination of input values
(2) Coefficients are real and symmetric

We wish to design an FIR filter: a system with an output, y[n], that is a weighted sum of past, present, and future inputs, $x[n\text{-}k]$.

$$y[n] = \sum_{k=-L}^{L} h[k]\, x[n-k] \tag{1}$$

Suppose that we apply a complex sinusoid of frequency ω to the input of the system. That is, suppose

$$x[n] = e^{jwn}$$

The output, $y[n]$, is

$$\begin{aligned} y[n] &= \sum_{k=-L}^{L} h[k]\, e^{jw(n-k)} \\ &= e^{jwn} \sum_{k=-L}^{L} h[k]\, e^{-jwk} \end{aligned}$$

Thus, applying a complex sinusoid to the system, we find that the output is also complex sinusoid of the same frequency, ω, with amplitude

$$H(w) = \sum_{k=-L}^{L} h[k]\, e^{-jwk} \tag{2}$$

We define $H(w)$ as the *frequency response* of the system and note that the coefficients, $h[k]$, should be chosen to achieve a specified frequency response.

If we specify that the weighting factors, $h[k]$, are to have even symmetry about $k = 0$, the frequency response becomes

$$H(\omega) = h[0] + \sum_{k=1}^{L} 2h[k]\cos[k\omega] \tag{3}$$

This function is a trigonometric polynomial of degree L. It is real, periodic with period 2π, and symmetric about $\omega = 0$. Thus, a statement about the function over the interval $0 \leq \omega \leq \pi$ is sufficient specify the entire function.

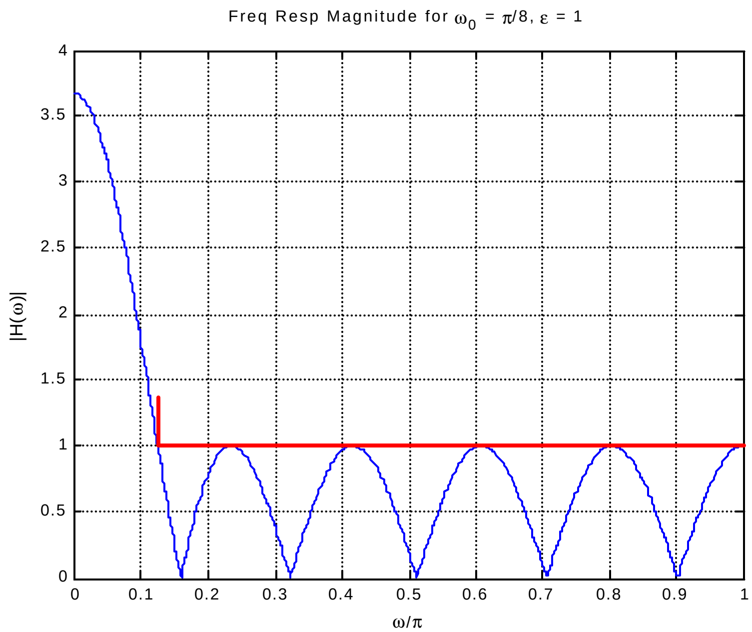
Frequency Response Specification

The frequency response of the system described above is required to have the following characteristics:

(1) $|H(\omega)| \leq 1, \quad 0 < \omega_0 \leq \omega \leq \pi$

(2) $|H(0)|$ is maximized subject to condition (1)

The filter should suppress signals with frequencies between ω_0 and π while maximally amplifying signals of zero frequency.



Transforming the Problem

We now wish to transform the design problem from the domain of trigonometric polynomials to algebraic polynomials. Recall that the function we wish to find is:

$$H(\omega) = h[0] + \sum_{k=1}^{L} 2h[k]\cos[k\omega],$$

the frequency response of the filter. In order to clean up the expression, we let

$$d_0 = h[0] \text{ and } d_k = 2h[k],\ 1 \le k \le L$$

The new expression is

$$H(w) = \sum_{k=0}^{L} d_k \cos[kw] \tag{4}$$

The coefficients of the polynomial should be chosen in such a way that the frequency response specifications are achieved. Using trigonometric identities it can be shown that each term of the form $\cos[kw]$ can be written as a sum of powers of $\cos[w]$. That is

$$\cos[kw] = \sum_{i=0}^{k} a_i (\cos[w])^i$$

Substituting these polynomials into (4) and combining terms we find that

$$H(w) = G(\cos w) = \sum_{k=0}^{L} b_k (\cos w)^k \tag{5}$$

If we let $x = \cos(w)$, then

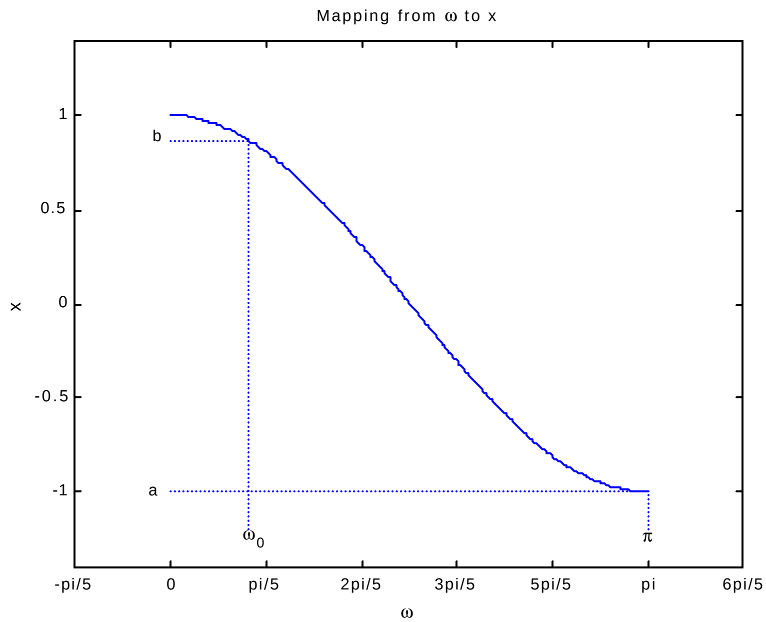
$$G(\cos w) = G(x) = \sum_{k=0}^{L} b_k x^k \tag{6}$$

The goal is to choose the coefficients, b_k, in such a way that the original design specifications are met. To this end, we must transform the design specifications to this domain.

Recall that the critical points in $H(w)$ are $w = w_0, p$. We seek the critical points in $G(x)$, $x = a$, b. Using the relation between x and ù, we find

$$\begin{aligned} a &= \cos(p) = -1 \\ b &= \cos(w_0) \end{aligned} \tag{7}$$

The mapping from the points in w to points in x is shown in the plot below.



The desired characteristics of the trigonometric polynomial, $H(w)$, are transformed to the following desired characteristics of the algebraic polynomial, *G(x)*:

(1) $|G(x)| \leq 1, \quad a \leq x \leq b < 1$

(2) $|G(1)|$ is maximized subject to condition (1) (8)

Chebyshev Polynomials

We start with the trigonometric identity

$$\cos([n+1]w) + \cos([n-1]w) = 2\cos(w)\cos(nw)$$

This can be rewritten as:

$$\cos([n+1]w) = 2\cos(w)\cos(nw) - \cos([n-1]w) \tag{9}$$

If we let $x = \cos(w)$, and the Chebyshev polynomials $T_n(x) = \cos(nw)$, we find

$$T_0(x) = \cos(0) = 1$$
$$T_1(x) = \cos(w) = x$$

Using equation (4) we have

$$\begin{aligned} T_2(x) &= \cos(2w) \\ &= 2\cos(w)\cos(w) - \cos(0) \\ &= 2x^2 - 1 \end{aligned}$$

Recursive substitution into (4) yields all subsequent Chebyshev polynomials

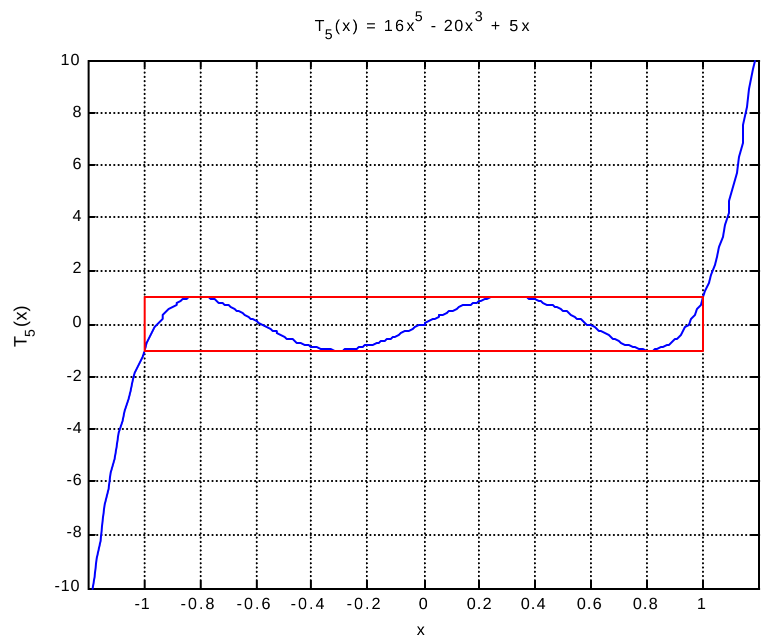
Properties of the Chebyshev Polynomial

A Chebyshev polynomial of order n, $T_n(x)$, has the following properties:

(1) $\left|T_n(x)\right| \le 1, \quad -1 \le x \le 1$

(2) for $\left|x_0\right| \ge 1, \quad \left|T_n(x_0)\right|$ is larger than any other polynomial subject to condition (1)

Property (1) follows directly from the definition of $T_n(x)$. Property (2) is a restatement of Theorem 1.10 in Rivlin (see Appendix). A 5th order Chebyshev polynomial is plotted here to show that it grows rapidly outside the interval $[-1, 1]$.



Warping the Chebyshev polynomial

Given a Chebyshev polynomial, $T_n(x)$, we wish to linearly warp it such that it has the characteristics desired of $G(x)$. That is, since $T_n(x)$ is bounded by +/- 1 for x between –1 and 1, and grows as rapidly as possible outside that interval, we want to linearly warp it so that the same properties holds for the new critical points a and b.

$T_n(x)$ Critical Points: $x = -1, 1$

$\hat{T}_n(x) = T_n(y = c_1 x + c_2)$ New Critical Points: $x = \mathrm{a}, \mathrm{b}$

We have the equations

$$-1 = c_1 a + c_2 \qquad\qquad 1 = c_1 b + c_2$$

With two equations and two unknowns, we find that

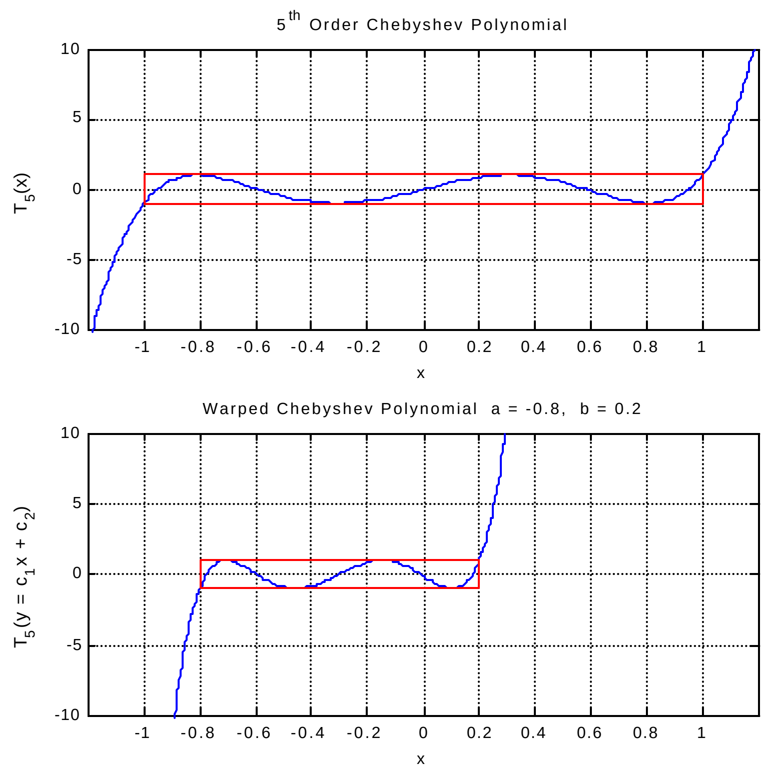
$$y = \frac{2}{b-a}x - \frac{b+a}{b-a} \tag{10}$$

The properties for the Chebyshev polynomial hold for the warped Chebyshev polynomial on the new interval (See Appendix).

(1) $\left|\hat{T}_n(x)\right| \le 1, \quad a \le x \le b$

(2) for $x_0 < a$ or $x_0 > b$, $\left|\hat{T}_n(x_0)\right|$ is larger than any other polynomial subject to condition (1)

Below is a comparison of an unwarped and warped Chebyshev polynomial of 5th order.



Transforming back to *h*[*k*]

We are now in a position to solve the problem. To summarize:

It was shown in the section *Transforming the Problem* that the filter design specifications can be transformed from the domain of trigonometric polynomials to the domain of algebraic polynomials. Furthermore, we know that a specific algebraic polynomial, the linearly warped Chebyshev polynomial, has the properties that we are looking for. We will now show how to perform the warping, and then transform the coefficients of the warped polynomial to the domain of trigonometric polynomials.

A Chebyshev polynomial of order L can be written as

$$T_L(x) = \sum_{k=0}^{L} a_k x^k \tag{11}$$

where the a_k are the coefficients of the polynomial. If we warp the polynomial in the manner described in *Warping the Chebyshev Polynomial*, the expression becomes

$$\begin{aligned}\hat{T}_L(x) &= T_L(c_1 x + c_2) \\ &= \sum_{k=0}^{L} a_k [c_1 x + c_2]^k\end{aligned}$$

which leads to a new polynomial of the same form.

$$G(x) = \hat{T}_L(x) = \sum_{k=0}^{L} b_k x^k \tag{12}$$

This is the $G(x)$ we first described in equations (6) and (8). Recall that $x = \cos(w)$. And just as $\cos[kw]$ can be written as a sum of powers of $\cos[w]$, $(\cos w)^k$ can be expressed as

$$(\cos w)^k = \sum_{i=0}^{k} b_i \cos[iw].$$

Equation (12) can therefore be transformed to be of the form

$$H(w) = \sum_{k=0}^{L} d_k \cos(kw) \tag{13}$$

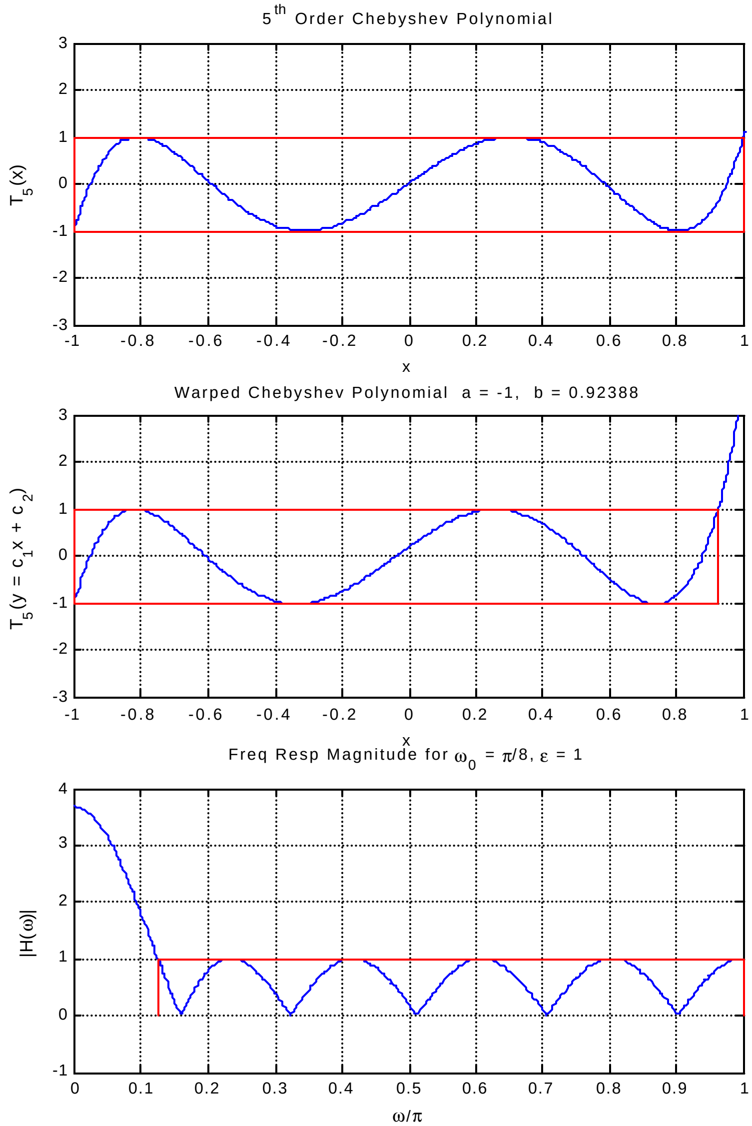
From here we can find the coefficients of the filter. Recall that

$$H(w) = h[0] + \sum_{k=1}^{L} 2h[k]\cos[kw] \tag{14}$$

Then,

$$h[0] = d_0 \text{ and } h[k] = \frac{d_k}{2},\ 1 \le k \le L$$

The plots on the next page show the progression from Chebyshev polynomial to warped Chebyshev polynomial, and finally to trigonometric polynomial.



5th Order Chebyshev Polynomial
$T_5(x)$
x
Warped Chebyshev Polynomial a = -1, b = 0.92388
$T_5(y = c_1 x + c_2)$
x
Freq Resp Magnitude for $\omega_0 = \pi/8$, $\varepsilon = 1$
$|H(\omega)|$
ω/π

Appendix

THEOREM 1.9.

If p_n^* is a best approximation to x^{n+1} on [-1, 1] out of P_n, then

$$x^{n+1} - p_n^* = 2^{-n} T_{n+1}(x)$$

and so

$$E_n\left(x^{n+1};[-1,1]\right) = 2^{-n}.$$

GENERALIZED THEOREM 1.9.

If p_n^* is a best approximation to x^{n+1} on $[a, b]$ out of P_n, then

$$x^{n+1} - p_n^* = c_{n+1} T_{n+1}\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right)$$

where

$$c_{n+1} = (b-a)^{n+1} 2^{-2n-1}.$$

Note that Theorem 1.9 is the special case when $a = -1$ and $b = 1$.

Quoting Rivlin: The problem resolved by (the generalized) Theorem 1.9 is equivalent to finding a polynomial of degree k with leading coefficient 1 which deviates least from 0 in absolute value on $[a, b]$. This polynomial is

$$\tilde{T}_k(x) = (b-a)^k 2^{-2k+1} T_k\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right),$$

the normalized, linearly warped Chebyshev polynomial of degree k. For every other polynomial of degree k and with leading coefficient 1 there exists a point on $[a, b]$ such that

$$|p(x)| > (b-a)^{n+1} 2^{-2n-1}$$

THEOREM 1.10.

If $p \in P_n$ and $|x_0| \geq 1$, then

$$\frac{\|T_n\|}{\|p\|} \leq \frac{|T_n(x_0)|}{|p(x_0)|}$$

Theorem 1.10 is most useful to us when $\|p\| = \|T_n\|$, i.e. when the polynomial p is bounded by +/-1 on the interval [-1, 1]. Theorem 1.10 tells us that when $p(x)$ and $T_n(x)$ are subject to the same restrictions on [-1, 1], $p(x_0)$ cannot be larger than $T_n(x_0)$ for x_0 outside the interval [-1, 1].

GENERALIZED THEOREM 1.10.

If $p \in P_n$ and $x_0 \le a$ or $x_0 \ge b$, then

$$\frac{\|T_n'\|}{\|p\|} \le \frac{|T_n'(x_0)|}{|p(x_0)|}$$

where $T_n'(x)$ is the warped Chebyshev polynomial,

$$T_n'(x) = T_k\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right).$$

Once again, when $p(x)$ and $T_n'(x)$ are subject to the same restrictions on [a, b], $p(x_0)$ cannot be larger than $T_n'(x_0)$ for x_0 outside the interval [a, b].

Jesse Hansen
hansenj@ele.uri.edu