Selected Approaches to Estimation of Signal Phase

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Signal Phase of Noiseless Data

- Let g_n be a sequence of complex values defined by

$$g_n = A_n \mathrm{e}^{j\Phi_n}$$

- We want to find the phase Φ_n (the argument of the exponent)

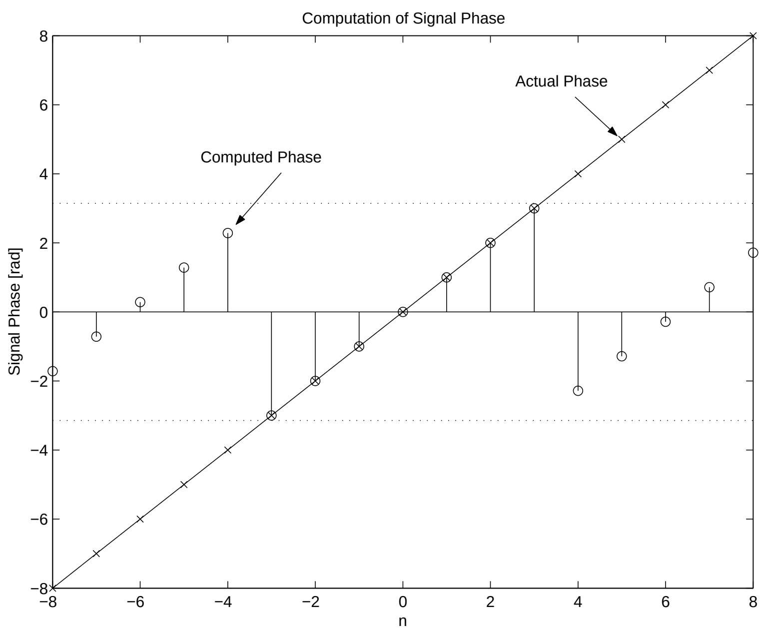
$$\Phi_n = \mathrm{arg}(g_n)$$

Phase Unwrapping

- Determination of Φ_n requires the computation of the inverse tangent

$$\widehat{\Phi}_n = \tan^{-1}\left(\frac{\mathrm{Im}\{g_n\}}{\mathrm{Re}\{g_n\}}\right)$$

- The inverse tangent function is a many-to-one function. All values of Φ_n outside the interval $(-\pi, \pi)$ are mapped back into this interval.



More Phase Unwrapping

- Technique for phase unwrapping:

  1. **Search** for jumps $> \pi$
  2. **Correct** jumps by adding a factor of $\pm 2\pi$ to all subsequent terms in the sequence

- Φ_n must obey

$$\Phi_n - \Phi_{n-1} < \pi \qquad \text{for all } n$$

otherwise unwrapping is useless

Methods for Estimating Φ_n

1. **Phase tracking**: Φ_n can be tracked using a PLL, Least Mean Squares (LMS), moments of time-frequency distribution (TFDs), or by other means.

2. **Polynomial model** for Φ_n can be assumed

$$\Phi_n = a_0 + \cdots + a_p n^p$$

3. Combine both: Compute time localized phase polynomial, shift, repeat

In this investigation analysis is restricted to item 2. We will examine the cases of linear phase ($p = 1$), and quadratic phase ($p = 2$)

Linear Phase

Phase Polynomial: $\Phi_n = 2\pi f n + \phi$

- Signal model x_n is complex exponential corrupted by complex white Gaussian noise z_n with variance σ^2

$$x_n = A\mathrm{e}^{j(2\pi f n + \phi)} + z_n$$

- Goal is to estimate the parameters of Φ_n, i.e. f and ϕ

Maximum Likelihood Solution

- The MLE of f is given by

$$\hat{f} = \arg\max_f \frac{1}{N} \left| \sum_{n=0}^{N-1} x_n e^{-j2\pi f n} \right|^2$$

This the value of f that maximizes the periodogram

- The MLE of ϕ

$$\hat{\phi} = \tan^{-1}\left[\frac{\text{Im}\left(\sum_{n=0}^{N-1} x_n e^{-j2\pi \hat{f} n}\right)}{\text{Re}\left(\sum_{n=0}^{N-1} x_n e^{-j2\pi \hat{f} n}\right)}\right]$$

Implementing the MLE

The MLE of f is typically implemented in two steps

1. **Coarse Grid Search**: The periodogram is computed at discrete values using the FFT

2. **Fine Determination** of peak location: Using maxima from last step, can use a fine grid search, Newton's method, or interpolation.

Some Suboptimal
But Computationally Efficient Methods

Start with an approximation:

- Recall original model:

$$x_n = A\,\mathrm{e}^{j(2\pi f n + \phi)} + z_n$$

  where z_n is complex Gaussian noise with variance σ^2

- The following approximation can be made at high SNR:

$$x_n \approx A\,\mathrm{e}^{j(2\pi f n + \phi + w_n)}$$

  where w_n is real white Gaussian noise with variance $\sigma^2/2A^2$

- Note that if we can get hold of the phase, we have the linear model

$$\Phi_n = 2\pi f n + \phi + w_n$$

- This leads to one method for determining f and ϕ

1. Find the phase using $\tan^{-1}$, then perform unwrapping

2. Use least squares to find the unknown parameters.

MLE on Differenced Phase (1)

- Note that if we can get hold of the differenced phase, we have

$$\begin{aligned} \Delta\Phi_n &= \Phi_{n+1} - \Phi_n \\ &= 2\pi f + \Delta w_n \end{aligned}$$

- Differenced phase is linear function of f, embedded in *colored* Gaussian noise

- $\Delta\Phi_n$ can be computed by $\angle x^*{}_n x_{n+1}$

MLE on Differenced Phase (2)

- MLE solution for f using $\Delta\Phi_n$ is

$$\hat{f} = \frac{1}{2\pi} \sum_{n=n_0}^{n_0+N-2} h_n \left(\angle {x^*}_n x_{n+1} \right)$$

- Solution is statistically identical to previous approach, but is more computationally efficient.

Comparison vs. CRLB

Simulations are for $f = 0.05$ and $f = 0.35$